Wave Functions in Thermal Equilibrium

GAP Measures and Canonical Typicality

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Literature

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Literature

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  - R.T. et al. (2007)
    Typicality of the GAP Measure. In preparation
  - R.T. et al. (2003-6)
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- Also:
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    Smoothness of Wave Functions in Thermal Equilibrium.
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  - R.T. et al. (2007)
    Elementary Proof for Asymptotics of Large Haar-Distributed Unitary Matrices. *Lett. Math. Phys.* **82**, 51–59

The theme of this talk

Claim

A quantum system in thermal equilibrium at temperature $1/\beta$ is described by a random state vector ψ with distribution $GAP(\beta)$.

$GAP(\beta)$ is a novel measure on Hilbert space $\mathscr{H}$

Classical Claim

A classical system in thermal equilibrium at temperature $1/\beta$ is described by a random phase point (q,p) with distribution $\frac{1}{Z}e^{-\beta H(q,p)}$, with $H(q,p)$ the Hamiltonian function.

Slogan

$GAP(\beta)$ is the canonical distribution of wave functions.

Applications

In many cases, we don't know the wave function:

- Photons from the sun or a star
- Photons from a lamp
- Electrons boiled off a piece of metal

The wave function is random, but with which distribution?

When the particle comes from a system in thermal equilibrium, the answer is $GAP(\beta)$.

GAP Measures

Measure and Density Matrix

Hilbert space $\mathscr{H}$,
unit sphere $\mathbb{S}(\mathscr{H}) = \{\psi \in \mathscr{H} : \|\psi\| = 1\}$ with Borel σ-algebra
probability measure μ
associated density matrix (DM)

$$\rho_\mu = \int_{\mathbb{S}(\mathscr{H})} \mu(d\psi)\, |\psi\rangle\langle\psi|$$

($=$ covariance matrix of μ, since w.l.o.g. $\mathbb{E}_\mu \psi = 0$.)

Many-to-one: $\rho_\mu = \rho_{\mu'} \not\Rightarrow \mu = \mu'$

What the density matrix is good for

For any experiment on the system, the probability of the outcome associated with projection P is $\mathrm{tr}(P\,\rho_\mu)$. Thus, if $\rho_\mu = \rho_{\mu'}$ then μ, μ' are empirically indistinguishable.

What measures are good for

- *Typicality statements* hold for most wave fcts (relative to μ),

$$\mu\{\omega : p(\omega)\} > 1 - \varepsilon$$

  Ex [D. Page 1993] For most $\psi \in \mathbb{S}(\mathscr{H}_1 \otimes \mathscr{H}_2)$, $\rho_1 = \mathrm{tr}_2\,|\psi\rangle\langle\psi|$ has near-maximal $-\mathrm{tr}(\rho_1 \log \rho_1)$, as dim $\mathscr{H}_2 \to \infty$.
  Ex [L. Boltzmann] For most microstates (q, p) given the macrostate, the (Boltzmann) entropy increases.
- If we attribute wave functions to systems, their distribution is of interest.

$GAP(\beta) = GAP(\rho_\beta)$

For every DM ρ on $\mathscr{H}$, there is a probability measure $GAP(\rho)$ on $\mathbb{S}(\mathscr{H})$.

For thermal equilibrium:

$$\rho_\beta = \frac{1}{Z} e^{-\beta H}$$

canonical density matrix
$H =$ Hamiltonian operator, $Z = \operatorname{tr} e^{-\beta H}$ normalizing constant

GAP Measures: Definition in 3 Steps

G AUSSIAN

A DJUSTED

P ROJECTED

GAP Measures: Definition in 3 Steps

GAUSSIAN: Start with
$G(\rho)$ = Gaussian measure on $\mathscr{H}$ with covariance ρ,
i.e., $\mathbb{E}_{G(\rho)}\langle\phi|\psi\rangle\langle\psi|\chi\rangle = \langle\phi|\rho|\chi\rangle \quad \forall \phi, \chi \in \mathscr{H}$.

Construction

If $\rho = \sum_n p_n |n\rangle\langle n|$ spectral decomposition
then let $\mathrm{Re}\, Z_n$, $\mathrm{Im}\, Z_n$ be independent Gaussian random variables with mean 0 and variance $p_n/2$; set $\psi = \sum_n Z_n |n\rangle$.

Ex $\mathscr{H} = \mathbb{C}^k$: $\dfrac{dG(\rho)}{d\lambda}(\psi) = \dfrac{1}{\pi^k \det \rho} e^{-\langle\psi|\rho^{-1}|\psi\rangle}$

GAP Measures: Definition in 3 Steps

GAUSSIAN: Start with
$G(\rho)$ = Gaussian measure on $\mathscr{H}$ with covariance ρ

ADJUSTED: To obtain the measure $GA(\rho)$ on $\mathscr{H}$, multiply by a density function $\psi \mapsto \|\psi\|^2$:
$GA(\rho)(d\psi) = \|\psi\|^2 \, G(\rho)(d\psi)$

GAP Measures: Definition in 3 Steps

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$GA(\rho)(d\psi) = \|\psi\|^2 \, G(\rho)(d\psi)$

PROJECTED to the unit sphere $\mathbb{S}(\mathscr{H})$: $\psi^{GAP} = \dfrac{\psi^{GA}}{\|\psi^{GA}\|}$
or $GAP(\rho)(B) = GA(\rho)(\mathbb{R}^+ B)$ for $B \subseteq \mathbb{S}(\mathscr{H})$.

The adjustment factor compensates the change in covariance due to projection to $\mathbb{S}(\mathscr{H})$, thus $\rho_{GAP(\rho)} = \rho$.

GAP Measures: Properties

- the right density matrix

$$\rho_{GAP(\rho)} = \mathbb{E}_{GAP(\rho)}|\psi\rangle\langle\psi| = \mathbb{E}_{GA(\rho)}\frac{|\psi\rangle\langle\psi|}{\|\psi\|^2} = \mathbb{E}_{G(\rho)}|\psi\rangle\langle\psi| = \rho$$

- covariant

$$U_* \, GAP(\rho) = GAP(U\rho U^{-1})$$

  for every unitary U on $\mathscr{H}$
  $\Rightarrow$ stationary under every unitary evolution that preserves ρ

- hereditary
  "If a system has temperature $1/\beta$ then also every subsystem"
  "GAP of a product density matrix has GAP marginal"
  If $\psi \in \mathbb{S}(\mathscr{H}_1 \otimes \mathscr{H}_2)$ has distribution $GAP(\rho_1 \otimes \rho_2)$ then, for any ONB $\{b_i\}$ of $\mathscr{H}_2$, the conditional wave fct ψ_1 has distribution $GAP(\rho_1)$.

Contrast with Another Measure

$EIG(\rho)$: another measure on $\mathbb{S}(\mathscr{H})$ with density matrix ρ. [von Neumann 1932]
Supose $\rho = \sum_n p_n |n\rangle\langle n|$ is non-degenerate.
Choose random N with distribution p_n and set

$$\psi = e^{i\Theta}|N\rangle$$

with uniform random phase Θ.

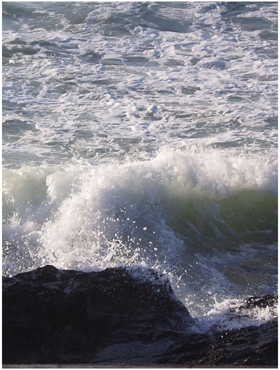
E. Schrödinger (1952)

"To ascribe to every system always one of its sharp energy values is an indefensible attitude."

Unpleasant features of $EIG(\rho)$:

- highly concentrated (on eigenvectors!)
- defines a rather eccentric sense of "typical wave fct"
- no continuous extension to degenerate ρ possible
- thus, $EIG(\rho_\beta)$ is unstable against perturbations of H.

What does a *GAP*-distributed ψ look like?



or

?

Most fcts in $L^2(\mathcal{Q})$, $\mathcal{Q} \subseteq \mathbb{R}^n$, are not differentiable, and $GAP(\rho)$ is very spread-out $\Rightarrow$ might expect that $GAP(\rho)$-typical wave fcts are not differentiable.

But that's not true: Instead, for relevant H, $GAP(\rho_\beta)\big(C^\infty(\mathcal{Q})\big) = 1$
even analytic: $GAP(\rho_\beta)\big(C^\omega(\mathcal{Q})\big) = 1$

Example

$H = -\Delta$ on $\mathcal{Q} =$ a box. For every measure μ on $\mathbb{S}(\mathscr{H})$ with $\rho_\mu = \rho_\beta$, the Fourier coefficients of ψ almost surely decay exponentially $\Rightarrow \psi$ analytic.

Smoothness Theorem 1 [RT, N. Zanghì 2005]

If ρ has C^∞ eigenfunctions φ_n with eigenvalues p_n and

$$\sum_n \left\|\nabla^\ell \varphi_n\right\|_\infty \sqrt{p_n} < \infty$$

then $GAP(\rho)(C^\infty) = 1$.

Proof: Show that $\psi(q) = \sum_n \langle \varphi_n | \psi \rangle \, \varphi_n(q)$ converges uniformly, and so do the derivatives. □

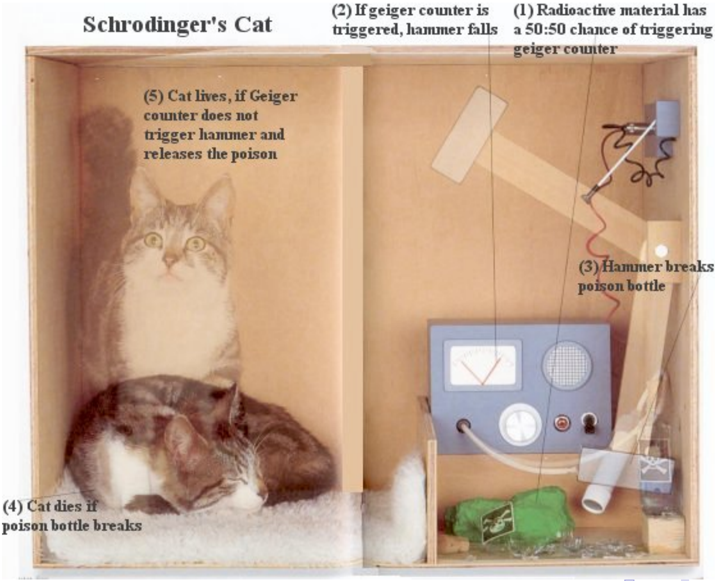
Smoothness Theorem 2 [RT, NZ 2005]

Let $tr \exp(-\beta_0 H) < \infty$ and $\beta > \beta_0$. For every measure μ on $\mathbb{S}(\mathscr{H})$ with $\rho_\mu = \rho_\beta$ and all $\ell = 1, 2, 3, \ldots$,

$$\mu\big(\text{domain}(H^\ell)\big) = 1\,.$$

Ex: $\exists$ Bohmian trajectories for GAP-typical ψ.

Schrödinger's Cat


Schrodinger's Cat
(2) If geiger counter is triggered, hammer falls
(1) Radioactive material has a 50:50 chance of triggering geiger counter
(5) Cat lives, if Geiger counter does not trigger hammer and releases the poison
(3) Hammer breaks poison bottle
(4) Cat dies if poison bottle breaks

A variant of Schrödinger's cat: measurement problem

Consider a quantum measurement of the observable $A = \sum_n \alpha_n |n\rangle\langle n|$.

$$|n\rangle \otimes \phi_0 \xrightarrow{t} |n\rangle \otimes \phi_n$$

($\phi_0 =$ ready state of apparatus, $\phi_n =$ state displaying result α_n)

$$\Rightarrow \sum_n c_n |n\rangle \otimes \phi_0 \xrightarrow{t} \sum_n c_n |n\rangle \otimes \phi_n$$

But one would believe that a measurement has an actual, random outcome n_0, so that one can ascribe the "collapsed state" $|n_0\rangle$ to the system and, more importantly, the state ϕ_{n_0} to the apparatus.

More abstractly: Consider system 1 entangled with system 2. Then $1+2$ has a wave function $\psi \in \mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2$, but system 1 alone does not. System 1 has a reduced density matrix

$$\rho_1 = \mathrm{tr}_2 |\psi\rangle\langle\psi|$$

However, at least in some cases one would believe that system 1 has an actual, random wave function ψ_1.

Precise solutions of the measurement problem

- Bohmian mechanics [Bohm 1952]
- Spontaneous collapse [GRW = Ghirardi, Rimini, Weber 1986]
- many worlds [Everett 1957]
- ...

Conditional wave function

Bohmian mechanics

A version of QM with particle trajectories

$$\text{state at time } t = \big(Q(t), \psi(t)\big) = (\text{configuration, wave fct})$$

Probability distribution of $Q(t)$ is $|\psi(t)|^2$. Here, $Q(t) = (Q_1(t), Q_2(t))$.

Def *conditional wave function* [Dürr, Goldstein, Zanghì 1992]

$$\psi_1(q_1, t) = \frac{1}{\mathcal{N}} \psi(q_1, Q_2(t), t)$$

Many worlds

In each "world" the system possesses a (conditional) wave function.

General definition of conditional wave function

Def *conditional wave function* [RT et al. 2006]
Let $\psi \in \mathscr{H}_1 \otimes \mathscr{H}_2$ with $\|\psi\| = 1$, choose orthonormal basis $\{b_j\}$ of $\mathscr{H}_2$.

$$\psi_1 = \frac{1}{\mathcal{N}} \langle b_J | \psi \rangle_2 \in \mathscr{H}_1$$

b_J = random basis vector with $\mathbb{P}(J = j) = \big\| \langle b_j | \psi \rangle_2 \big\|^2$

- We attribute ψ_1 to system 1.
- ψ_1 is random, even though ψ is not
  (in BM, $Q(t)$ is random,
  in GRW ψ collapses stochastically to something like $\psi_1 \otimes \psi_2$,
  in MW ψ_1 depends on the "world")
- ψ determines $\mathbb{P}_1^{\psi}$, the distribution of ψ_1
- $\rho_{\mathbb{P}_1^{\psi}} = \rho_1$
  "the density matrix of ψ_1 is the reduced density matrix", i.e.,

$$\begin{array}{ccc} \psi & \longrightarrow & \mathbb{P}_1^{\psi} \\ \downarrow & & \downarrow \\ |\psi\rangle\langle\psi| & \xrightarrow{\mathrm{tr}_2} & \rho_1 \end{array}$$

- Among measures μ_1 on $\mathbb{S}(\mathscr{H}_1)$ with $\rho_{\mu_1} = \rho_1$, not all are equally reasonable:

$$\begin{array}{c} 50\% \, |\text{dead}\rangle \\ 50\% \, |\text{alive}\rangle \end{array} \quad \text{vs.} \quad \begin{array}{c} 50\% \, 2^{-1/2}(|\text{dead}\rangle + |\text{alive}\rangle) \\ 50\% \, 2^{-1/2}(|\text{dead}\rangle - |\text{alive}\rangle) \end{array}$$

GAP is the Average Distribution

System s, heat bath b (large), $\mathscr{H} = \mathscr{H}_s \otimes \mathscr{H}_b$
For every $\psi \in \mathscr{H}_s \otimes \mathscr{H}_b$, cond. wf $\psi_s = \psi_1$ is random. If ψ itself is random, then ψ_s is doubly random.

Microcanonical distribution: Pick energy interval $[E, E + \delta E]$ containing many energy eigenvalues, but so that δE is small on the macroscopic scale.
Let $\mathscr{H}_{E,\delta E}$ be the corresponding (spectral) subspace of $\mathscr{H} = \mathscr{H}_s \otimes \mathscr{H}_b$,
microcanonical DM $\rho_{E,\delta E} = (\dim \mathscr{H}_{E,\delta E})^{-1} P_{E,\delta E}$,
microcanonical distribution $u_{E,\delta E}$ = uniform on $\mathbb{S}(\mathscr{H}_{E,\delta E})$ (= normalized surface area) [Bloch, Schrödinger].

Claim

If ψ is random with distribution $u_{E,\delta E}$, then the (marginal) distribution of ψ_s is

$$\mathbb{P}(\psi_s \in \cdot) = \int u_{E,\delta E}(d\psi)\, \mathbb{P}_1^{\psi} \approx GAP(\rho_\beta).$$

"If ψ is microcanonical then ψ_s is canonical."

Outline of argument:

by equivalence of ensembles,

$$\rho_{E,\delta E} \approx \rho_\beta^{s+b} \text{ on } \mathscr{H}_s \otimes \mathscr{H}_b \text{ for suitable } \beta$$

by continuity of $\rho \mapsto GAP(\rho)$,

$$u_{E,\delta E} = GAP(\rho_{E,\delta E}) \approx GAP(\rho_\beta^{s+b}) \text{ on } \mathscr{H}_s \otimes \mathscr{H}_b$$

neglecting interaction,

$$\rho_\beta^{s+b} = \rho_\beta^s \otimes \rho_\beta^b$$

by hereditarity,

$$\text{distr}(\psi) = GAP(\rho_\beta^s \otimes \rho_\beta^b) \Rightarrow \text{distr}(\psi_s) = GAP(\rho_\beta^s)$$

Thus,

$$\text{distr}(\psi) = u_{E,\delta E} \Rightarrow \text{distr}(\psi_s) \approx GAP(\rho_\beta^s)$$

Role of Interaction

Paradox? First we consider a system coupled to a heat bath, then neglect the interaction.

Interaction is relevant...

- for creating typical wave functions:
  it helps evolve atypical wave functions into typical ones.
- to the system's canonical density matrix
  $\rho_{can} := \mathrm{tr}_b\, \rho_{E,\delta E}$,
  as interaction causes deviation from exponential form.

...but can be neglected

- once $s + b$ has a μ-typical wave function, interaction is irrelevant to the distribution of the conditional wave function.
- In the limit of negligible interaction,
  $\rho_{can} = \frac{1}{Z} e^{-\beta H}$

GAP is the Typical Distribution

What does "typical" mean?

$f(\omega) \approx y$ is typical for $\omega \in \Omega$

$\Leftrightarrow$ y is the typical value of f on Ω

$\Leftrightarrow$ $f(\omega) \approx y$ for most $\omega \in \Omega$

$\Leftrightarrow$ $\mu\{\omega \in \Omega : |f(\omega) - y| > \delta\} < \varepsilon$

$\Leftrightarrow$ f is nearly constant on Ω (in fact, $y \approx \mathbb{E}f$)

$\Leftrightarrow$ f has small variance $\mathbb{E}(f - \mathbb{E}f)^2$

$\Leftrightarrow$ more precisely, consider
sequence $(\Omega_n, \mu_n)_{n\in\mathbb{N}}$ of probability spaces, $f_n : \Omega_n \to Y$
$\forall \delta > 0 :$ $\mu_n\{\omega \in \Omega_n : |f_n(\omega) - y| < \delta\} \to 1$

Slightly different: convergence in probability
(here, $\Omega_n = \Omega, \mu_n = \mu$, only f_n varies,
convergence to f_∞ rather than constant y; otherwise the same)

Ex: In classical statistical mechanics, thermodynamic functions are often nearly constant on the energy surface.

GAP is typical

Claim

For most ψ from the microcanonical ensemble of $s+b$, $\mathbb{P}_1^{\psi} \approx GAP(\rho_\beta)$.

This claim corresponds to "$f(\omega) \approx y$ is typical" with
$\omega = \psi \in \mathscr{H}_s \otimes \mathscr{H}_b$ from the microcanonical ensemble,
$f(\omega) = \mathbb{P}_1^{\psi}$ is the probability distribution of ψ_1,
$y = GAP(\rho_\beta)$
"The distribution of ψ_1 is nearly constant"

Classically, randomness in yields randomness out. In quantum mechanics, we can have randomness out, without randomness in.

Classically: fixed phase point $\big((q_s, q_b), (p_s, p_b)\big)$ yields fixed (q_s, p_s)
Quantum: fixed $\psi \in \mathscr{H}_s \otimes \mathscr{H}_b$, random ψ_s

General typicality of GAP

Consider $\mathscr{H}_1$ fixed, and fixed DM ρ_1 on $\mathscr{H}_1$,
but $\mathscr{H}_2 = \mathscr{H}_2^{(n)}$, $n \in \mathbb{N}$, $\dim \mathscr{H}_2^{(n)} \to \infty$ as $n \to \infty$;
let $\mathcal{R}_n = \{\psi \in \mathbb{S}(\mathscr{H}_1 \otimes \mathscr{H}_2^{(n)}) : \mathrm{tr}_2 |\psi\rangle\langle\psi| = \rho_1\}$,
let u_n be the uniform probability distribution on $\mathcal{R}_n$;
for any orthonormal basis $B = \{b_1, \ldots, b_{d(n)}\}$ of $\mathscr{H}_2^{(n)}$,
consider $\mathbb{P}_1^{(n,\psi,B)}$ = distribution of ψ_1, given $\psi \in \mathscr{H}_2^{(n)}$ and B.

General Typicality Theorem [RT et al., 2007]

For every $\delta > 0$ and every bounded continuous test fct $\varphi : \mathbb{S}(\mathscr{H}_1) \to \mathbb{R}$,

$$u_n\left\{\psi \in \mathcal{R}_n \ : \ \left|\mathbb{P}_1^{(n,\psi,B)}(\varphi) - GAP(\rho_1)(\varphi)\right| < \delta\right\} \to 1$$

as $n \to \infty$, uniformly in B, using notation $\mathbb{P}(\varphi) = \int \mathbb{P}(d\psi)\, \varphi(\psi) = \mathbb{E}_{\mathbb{P}}\varphi$.

General typicality of GAP

In other words:

For typical ψ with $\mathrm{tr}_2 |\psi\rangle\langle\psi| = \rho_1$, ψ_1 has distribution $\approx GAP(\rho_1)$.

Proof: Based on

- Schmidt decomposition
  $\psi = \sum_i c_i |\chi_i\rangle_1 \otimes |\phi_i\rangle_2 = \sum_i \sqrt{p_i}\, |i\rangle_1 \otimes |\phi_i\rangle_2$, ONS $\{\phi_i\}$
- For a random unitary matrix $(U_{ij}) \in U(m)$ with uniform (Haar) distribution, the upper left $k \times k$ submatrix converges in distribution, after multiplying by a normalization factor $\sqrt{m}$ and as $m \to \infty$, to a matrix of independent complex Gaussian random variables with mean 0 and variance 1 [Petz and Réffy 2004; RT and CM 2007]

Typical basis

We need

- either typical wf ψ, any basis B
- or any wf ψ, typical basis B

General Typicality Theorem 2 [RT et al., 2007]

For every $\delta > 0$ and every bounded continuous test fct $\varphi : \mathbb{S}(\mathscr{H}_1) \to \mathbb{R}$,

$$u_{ONB}^{(n)} \left\{ B \in ONB(\mathscr{H}_2^{(n)}) \; : \; \left| \mathbb{P}_1^{(n,\psi,B)}(\varphi) - GAP(\rho_1)(\varphi) \right| < \delta \right\} \to 1$$

as $n \to \infty$, uniformly in $\psi \in \mathcal{R}_n$.

(Notation $\mathbb{P}(\varphi) = \int \mathbb{P}(d\psi)\, \varphi(\psi) = \mathbb{E}_{\mathbb{P}} \varphi$)
$u_{ONB}^{(n)} =$ uniform on $ONB(\mathscr{H}_2^{(n)})$ ($\leftrightarrow$ Haar measure)

Ingredients for typicality of $GAP(\rho_\beta)$ in the microcanonical ensemble:

- general typicality of $GAP(\rho)$ given ρ
- canonical typicality

Canonical Typicality

System s, heat bath b, coupling negligible:

$$H = H_s \otimes I_b + I_s \otimes H_b \,.$$

Known: ("average" statement)

$$\mathrm{tr}_b \, \rho_{E,\delta E} \approx \rho_\beta$$

$\nearrow$ partial trace $\uparrow$ microcan. DM $\nwarrow$ canonical DM

in the thermodynamic limit $N_b \to \infty, E/N_b \to e < \infty$

Novel: ("almost always" statement) canonical typicality

$$u_{E,\delta E}\Big\{\psi : \mathrm{tr}_b|\psi\rangle\langle\psi| \approx \rho_\beta\Big\} \to 1$$

$\uparrow$ microcanonical measure on $\mathbb{S}(\mathscr{H}_s \otimes \mathscr{H}_b)$

in the thermodynamic limit

For most ψ of $s + b$ from the microcanonical ensemble, the reduced density matrix of s is canonical

GAP typicality ↔ canonical typicality

To show that $\mathbb{P}_1^{\psi} \approx GAP(\rho_\beta)$, one needs to show canonical typicality first.

But note that canonical typicality is also a consequence:
by definition,

$$\rho_{\mathbb{P}_1^{\psi}} = \mathbb{E}|\psi_1\rangle\langle\psi_1| = \sum_j \big\|\langle b_j|\psi\rangle_2\big\|^2 \frac{\langle b_j|\psi\rangle\langle\psi|b_j\rangle}{\big\|\langle b_j|\psi\rangle_2\big\|^2} = \mathrm{tr}_b\,|\psi\rangle\langle\psi|;$$

as we know,

$$\rho_{GAP(\rho_\beta)} = \rho_\beta\,.$$

Thus,

$$\mathrm{tr}_b\,|\psi\rangle\langle\psi| \approx \rho_\beta\,.$$

Derivation of canonical typicality

Known part:

$$H_s = \sum_n E_n |n\rangle_s \langle n|$$

$$\rho_{E,\delta E} = (\dim \mathscr{H}_{E,\delta E})^{-1} \sum_n P_{\mathscr{H}_{b,E-E_n,\delta E}} \otimes |n\rangle_s \langle n|$$

$$\mathrm{tr}_b\, \rho_{E,\delta E} = (\dim \mathscr{H}_{E,\delta E})^{-1} \sum_n \left(\dim \mathscr{H}_{b,E-E_n,\delta E}\right) |n\rangle_s \langle n| \approx \rho_\beta$$

since $\dim \mathscr{H}_{b,E-E_n,\delta E} \approx e^{S(E-E_n)} \propto \exp\left(-\frac{\partial S}{\partial E} E_n\right) = e^{-\beta E_n}$

Novel part: $u_{E,\delta E} \approx G(\rho_{E,\delta E})$ ("All nonzero components are essentially i.i.d. Gaussian"). Let $\Psi \sim G(\rho_{E,\delta E})$. Then

$$\Psi = \sum_n |n\rangle_s \otimes |\Phi_n\rangle_b \text{ with } \Phi_n = \sum_{E-E_n \leq E_{b,m} \leq E+\delta E-E_n} X_{nm}|m\rangle_b$$

Here, X_{nm} are i.i.d. complex Gaussian random variables with mean 0 and variance $\mathbb{E}|X_{nm}|^2 = (\dim \mathscr{H}_{E,\delta E})^{-1}$. Then

$$\mathrm{tr}_b\, |\Psi\rangle\langle\Psi| = \sum_{n,n'} \langle\Phi_n|\Phi_{n'}\rangle_b\, |n\rangle_s\langle n'| \approx \sum_n p_n\, |n\rangle_s\langle n|$$

$$\text{with } p_n = \sum_{E-E_n \leq E_{b,m} \leq E+\delta E-E_n} |X_{nm}|^2 \approx \frac{\dim \mathscr{H}_{b,E-E_n,\delta E}}{\dim \mathscr{H}_{E,\delta E}}$$

because $\langle\Phi_n|\Phi_{n'}\rangle \approx 0$ for $n' \neq n$:

Φ_n = 00000000000000000000XXXXXXXXXXXXXXXXXXXXXXX000000000
$\Phi_{n'}$ = 00000000000XXXXXXXXXXXXXXXXXXXXXXX000000000000000000

Theorem on Canonical Typicality [Popescu, Short, Winter 2005]

Let $\mathscr{H}_R \subseteq \mathscr{H}_s \otimes \mathscr{H}_b$ arbitrary subspace (e.g., $\mathscr{H}_R = \mathscr{H}_{E,\delta E}$ microcanonical), u_R the uniform distribution on $\mathbb{S}(\mathscr{H}_R)$,

$$\rho_R = \frac{1}{\dim \mathscr{H}_R} P_{\mathscr{H}_R} \text{ and } \varepsilon > 0.$$

Then

$$u_R\Big\{\psi : \Big\|\mathrm{tr}_b|\psi\rangle\langle\psi| - \mathrm{tr}_b\,\rho_R\Big\|_1 \geq \eta\Big\} \leq \eta'$$

where η, η' are small when $\dim \mathscr{H}_s \ll 1/\mathrm{tr}(\mathrm{tr}_s \rho_R)^2$ (system $\ll$ bath) and $\varepsilon \ll 1 \ll \varepsilon^2 \dim \mathscr{H}_R$ (many states allowed).

$$\|M\|_1 = \mathrm{tr}\,|M| = \mathrm{tr}\sqrt{M^* M}$$

GAP Typicality

Putting the facts together

$\mathscr{H}_1$ fixed, while $\mathscr{H}_2 = \mathscr{H}_2^{(n)}$, $n \in \mathbb{N}$, $\dim \mathscr{H}_2^{(n)} \to \infty$ as $n \to \infty$;
let $\mathscr{H}_R^{(n)} \subseteq \mathscr{H}_1 \otimes \mathscr{H}_2^{(n)}$ subspace, $\dim \mathscr{H}_R^{(n)} \to \infty$ as $n \to \infty$
(e.g., $\mathscr{H}_R^{(n)} = \mathscr{H}_{E,\delta E}$ microcanonical). Let $\rho_R^{(n)} = (\dim \mathscr{H}_R^{(n)})^{-1} P_{\mathscr{H}_R^{(n)}}$
let $u_R^{(n)}$ be the uniform probability distribution on $\mathbb{S}(\mathscr{H}_R^{(n)})$
let $\rho_{can}^{(n)} = \mathrm{tr}_2\, \rho_R^{(n)}$
let $u_{ONB}^{(n)}$ be the uniform probability distribution on $ONB(\mathscr{H}_2^{(n)})$.

Theorem *in spe* [RT et al., 2007]

For every $\delta > 0$ and every bounded continuous test fct $\varphi : \mathbb{S}(\mathscr{H}_1) \to \mathbb{R}$,

$$u_R^{(n)} \otimes u_{ONB}^{(n)} \left\{ (\psi, B) \;:\; \left| \mathbb{P}_1^{(n,\psi,B)}(\varphi) - GAP(\rho_{can}^{(n)})(\varphi) \right| < \delta \right\} \to 1$$

as $n \to \infty$.

(Notation $\mathbb{P}(\varphi) = \int \mathbb{P}(d\psi)\, \varphi(\psi) = \mathbb{E}_{\mathbb{P}} \varphi$.)

Thank you for your attention