1. Suppose the risk-neutral model for a risky asset $S(t), t \leq 2,$ and a zero-coupon bond $B(t, 2)$ maturing at $T = 2$ is
\[
\begin{align*}
    dS(t) &= R(t)S(t) \, dt + [5 - t]S(t) \, d\tilde{W}(t), \quad t \leq 2; \\
    dB(t, 2) &= R(t)B(t, 2) \, dt + (2 - t)B(t, 2) \, d\tilde{W}(t), \quad t \leq 2.
\end{align*}
\] (1)

Let $S^{(2)}(t) = \frac{S(t)}{B(t, 2)}$ be the price of the risky asset denominated in the zero-coupon bond price. This is the $\{T = 2\}$-forward price of $S(t),$ which is denoted by For$_S(t, 2)$ in Shreve, but the notation $S^{(2)}$ is simpler to write!

Let $\tilde{P}^{(2)}$ denote the risk-neutral measure for the numéraire $B(t, 2)$.

(a) Show how to define $\tilde{W}^{(2)}$ so that it is a Brownian motion under $\tilde{P}^{(2)}$. Write down a stochastic differential equation for $S^{(2)}$ under the measure $\tilde{P}^{(2)}$ using $\tilde{W}^{(2)}$.

**Ans:** The change of measure to the $\tilde{P}^{(2)}$ measure is defined by
\[
\frac{d\tilde{P}^{(2)}}{dP} = \frac{D(2)B(2, 2)}{B(0, 2)} = e^{\int_0^2 (2-s) d\tilde{W}(s) - \frac{1}{2} \int_0^2 (2-s)^2 \, ds}.
\] Therefore, by Girsanov’s theorem,
\[
\tilde{W}^{(2)}(t) = \tilde{W}(t) - \int_0^t (2 - s) \, ds
\] is a martingale under $\tilde{P}^{(2)}$.

By Theorem 9.2,
\[
dS^{(2)}(t) = S^{(2)}(t)[(5 - t) - (2 - t)] \, d\tilde{W}^{(2)}(t) = 3S^{(2)}(t) \, d\tilde{W}^{(2)}(t).
\]

(b) Find an explicit formula for $V(0) = \tilde{E}[D(2)(S(2) - K)^+ | S(0) = s_0],$ in terms of $s_0$ and $B(0, 2)$.
Ans:

\[
\frac{V(0)}{B(0, 2)} = V(0) = \bar{E}(2)\left[\frac{(S(2) - K)^+}{B(2, 2)}\right] = \hat{E}(2)\left[\frac{(S(2) - K)^+}{S(0) = s_0} = \hat{E}(2)\left[\frac{(S(2) - K)^+}{S(0) = s_0/B(0, 2)}\right],
\]

because \( B(2, 2) = 1 \) and hence \( S(2) = S(0) \). But from part (a), \( S(2) \) follows a standard Black-Scholes model with \( r = 0 \) and \( \sigma = 3 \). Thus by using the Black-Scholes formula,

\[
V(0) = B(0, 2) \left[ \frac{s_0}{B(0, 2)} N(d_+(s_0/B(0, 2), K)) - K N(d_-(s_0/B(0, 2), K)) \right],
\]

where

\[
d_\pm(x, K) = \frac{1}{3\sqrt{2}} \left[ \ln \frac{x}{K} \pm 9 \right].
\]

2. In section 9.4.3, Shreve derives a formula for the price of a call option when the risk-free rate is random by assuming that the \( T \)-forward price of the underlying has constant volatility \( \sigma \). This model is written as equation (9.4.8).

Assume that under the original risk neutral model, the zero-coupon bond price satisfies equation (9.4.4). Derive the stochastic differential equation for \( S(t) \) valid under the original risk-neutral measure implied by the model (9.4.8), and determine how the volatility of \( S(t) \) is related to the volatility of the zero-coupon bond price. (You might ponder whether this is a realistic scenario or not!)

(This is another exercise on Theorem 9.2.2 — you have to use it in reverse.)

Ans: Suppose \( dS(t) = R(t) S(t) \, dt + \sigma(t) S(t) \, d\bar{W}(t) \) under the original risk-neutral measure. According to Theorem 9.2, if the zero-coupon bond price satisfies (9.4.4), namely \( d[D(t)B(t, T)] = -D(t)B(t, T) \sigma^*(t, T) \, d\bar{W}(t) \), then \( -\sigma^*(t, T) \) is playing the role of \( \nu \) in Theorem 9.2 and

\[
d\text{For}_S(t, T) = \text{For}_S(t, T)\left[\sigma(t) + \sigma^*\right] d\bar{W}^T(t).
\]

Here \( \bar{W}^T \) is the Brownian motion under the \( T \)-forward measure as define in section 9.4; \( \bar{W}^T(t) = \bar{W}(t) + \int_0^t \sigma^*(u, T) \, du \), for \( 0 \leq t \leq T \). But we are requiring that

\[
d\text{For}_S(t, T) = \sigma_T \text{For}_S(t, T) \, d\bar{W}^T(t),
\]

for some constant \( \sigma \). (We have put a subscript \( T \) on \( \sigma \) because the model depends on \( T \).) It follows that, \( \bar{\sigma}(t) = \sigma_T - \sigma^*(t, T) \), that is

\[
dS(t) = R(t) S(t) \, dt + [\sigma_T - \sigma^*(t, T)] S(t) \, d\bar{W}(t).
\]
This says that the volatility of the asset price is a constant translate of the volatility of the zero-coupon bond price. This seems very unrealistic. For example, if true, it should for different $T$, which would then imply that the volatilities of zero-coupon bonds of different maturities are all translates of one another, which also seems unlikely.

3. Shreve, Exercise 9.3. (Yet another exercise using Theorems 9.2.1 and 9.2.2.)

Ans: (i) By Itô’s rule,

$$d\frac{1}{N(t)} = (\nu^2 - r)\frac{1}{N(t)} \, dt - \nu\frac{1}{N(t)} \, dW_3(t).$$

Thus $dS(t)\, d[1/N(t)] = -\sigma\nu[S(t)/N(t)] \, d\tilde{W}_1(t) \, d\tilde{W}_3(t) = -\sigma\nu p S^{(N)}(t) \, dt$. Then, using Itô’s rule,

$$dS^{(N)}(t) = \frac{1}{N(t)} dS(t) + S(t) d\frac{1}{N(t)} + dS(t) d[1/N(t)]$$

$$= (\nu^2 - \sigma\nu p) S^{(N)}(t) \, dt + \sigma S^{(N)}(t) \, d\tilde{W}_1(t) - \nu S^{(N)}(t) \, d\tilde{W}_3(t).$$

Define $\gamma = \sqrt{\sigma^2 - 2\rho\sigma\nu + \nu^2}$ and

$$\tilde{W}_4(t) = \frac{1}{\gamma} \left[ \sigma d\tilde{W}_1(t) - \nu d\tilde{W}_3(t) \right].$$

Then $(d\tilde{W}_4(t))^2 = \frac{1}{\gamma} \left[ \sigma^2(d\tilde{W}_1(t))^2 - 2\sigma\nu d\tilde{W}_1(t)d\tilde{W}_3(t) + \nu^2(d\tilde{W}_3(t))^2 \right] = \frac{\sigma^2 - 2\rho\sigma\nu + \nu^2}{\gamma^2} \, dt = dt$. By Lévy’s theorem $\tilde{W}_4$ is a Brownian motion. Using this in the equation given above for $S^{(N)}(t)$ leads to

$$dS^{(N)}(t) = (\nu^2 - \sigma\nu p) S^{(N)}(t) \, dt + \gamma S^{(N)}(t) \, d\tilde{W}_4(t).$$

(ii) Define $\tilde{W}_2 = \frac{1}{\sqrt{1 - \rho^2}} \left[ \tilde{W}_3 - \rho \tilde{W}_1(t) \right]$. Then one can verify easily that $(d\tilde{W}_2(t))^2 = dt$, so that $\tilde{W}_2(t)$ is a Brownian motion, and $d\tilde{W}_1(t) \, d\tilde{W}_2(t) = 0$, so that $\tilde{W}_1$ and $\tilde{W}_2$ are independent. Then $d\tilde{W}_3(t) = \rho \, d\tilde{W}_1(t) + \sqrt{1 - \rho^2} \, d\tilde{W}_2(t)$, and

$$dN(t) = rN(t) \, dt + \nu N(t) \left[ \rho \, d\tilde{W}_2(t) + \sqrt{1 - \rho^2} \, d\tilde{W}_2(t) \right]$$

Let $\nu = (\nu p, \nu \sqrt{1 - \rho^2})$ for all $t$, and let $\tilde{W}(t) = (\tilde{W}_1(t), \tilde{W}_2(t))$. Then it follows from equation (3) that

$$d(D(t) N(t)) = D(t) N(t) \, \nu \cdot d\tilde{W}(t).$$
Similarly $d(D(t)S(t)) = D(t)S(t)(\sigma, 0) \cdot d\tilde{W}(t)$. Thus, by equation (9.2.9) in Theorem 9.2.2,

$$
\begin{align*}
\frac{dS^{(N)}(t)}{S^{(N)}(t)} &= \frac{dS(t)}{S(t)} - \frac{\sigma_d(t)}{\sigma_d(t)} \cdot d\tilde{W}(t) \\
&= S^{(N)}(t) \left[ (\sigma, 0) - (\nu \rho, \sqrt{1 - \rho^2}) \right] \cdot d\tilde{W}(t)
\end{align*}
$$

Let $v_1 = \sigma - \nu \rho$ and $v_2 = -\nu \sqrt{1 - \rho^2}$.

Then

$$
v_1^2 + v_2^2 = \sigma^2 - 2\nu \sigma \rho + \nu^2 \rho^2 + \nu^2 (1 - \rho^2) = \sigma^2 - 2\rho \sigma \nu + \nu^2.
$$


**Ans:**

(i) The price $S$ satisfies $dS(t) = rS(t)\, dt + \sigma_1 S(t)\, d\tilde{W}_1(t)$, and the solution to this is $S(t) = S(0) \exp \{ \sigma_1 \tilde{W}_1(t) + (r - \frac{1}{2} \sigma_1^2) t \}$.

(ii) The equation for $Q$ is $dQ(t) = (r - r^r)Q(t)\, dt + \sigma_2 \left[ \rho d\tilde{W}_1(t) + \sqrt{1 - \rho^2} d\tilde{W}_2 \right]$. We know (see equations (11), (12), and (13) in Lecture Notes 7) that this solution to this is

$$
Q(t) = Q(0) \exp \left\{ \int_0^t \nu \cdot d\tilde{W}(t) + \int_0^t (r - r^f - \frac{1}{2} \|\nu\|^2) \, du \right\},
$$

where $\nu = (\sigma_2 \rho, \sigma_2 \sqrt{1 - \rho^2})$. One easily computes that $\|\nu\|^2 = \sigma_2^2$. Because $\nu$ is constant,

$$
Q(t) = Q(0) \exp \left\{ \sigma_2 \rho \tilde{W}_1(t) + \sigma_2 \sqrt{1 - \rho^2} \tilde{W}_2(t) + (r - r^f - \frac{1}{2} \sigma_2^2) t \right\}.
$$

(iii) Define $\tilde{W}_4(t) = \frac{\sigma_1 - \sigma_2 \rho}{\sigma_4} \tilde{W}_1(t) - \frac{\sigma_2 \sqrt{1 - \rho^2}}{\sigma_4} \tilde{W}_2(t)$, where $\sigma_4^2 = \sigma_1^2 - 2\rho \sigma_1 \sigma_2 + \sigma_2^2$. It is easily verified that $[d\tilde{W}_4(t)]^2 = dt$ and hence that $\tilde{W}_4$ is a Brownian motion. Clearly,

$$
\sigma_4 \, d\tilde{W}_4(t) = (\sigma_1 - \sigma_2 \rho) \, d\tilde{W}_1(t) - \sigma_2 \sqrt{1 - \rho^2} \, d\tilde{W}_2(t).
$$

From parts (i) and (ii),

$$
\frac{S(t)}{Q(t)} = \frac{S(0)}{Q(0)} \exp \left\{ (\sigma_1 - \sigma_2 \rho) \, d\tilde{W}_1(t) - \sigma_2 \sqrt{1 - \rho^2} \, d\tilde{W}_2(t) + (r^f + \frac{1}{2} \sigma_4^2) t \right\}
$$

Define $a$ so that

$$
r - a - \frac{1}{2} \sigma_4^2 = -r^f + \frac{1}{2} (\sigma_2^2 - \sigma_1^2).
$$
A calculation shows that $a = r - r^f - \sigma_2^2 + \rho \sigma_1 \sigma_2$. The purpose of this definition is to write

$$\frac{S(t)}{Q(t)} = \frac{S(0)}{Q(0)} \exp\{\sigma_4 \tilde{W}_4(t) + (r - a - \frac{1}{2}\sigma_4^2)t\}.$$ 

This is the solution to the Black-Scholes price equation with initial price $S(0)/Q(0)$, driving Brownian motion $\tilde{W}_4$, volatility $\sigma_4$, and interest rate $r - a$.

(iv) Let $C(T - t, x, K, r, \sigma)$ be the price of a call option with strike $K$, interest rate $r$, and volatility $\sigma$. The formula for $C$, with different notation, is given in (4.5.22). As just pointed out $S(t)/Q(t)$ has the form of a Black-Scholes price. The value of a quanto at strike $K$ is

$$V(t) = e^{-r(T-t)} \tilde{E} \left[ \left( \frac{S(T)}{Q(T)} - K \right)^+ \mid \mathcal{F}(t) \right]$$

Adjust this to $V(t) = e^{-at} e^{-(r-a)t} \tilde{E} \left[ \left( \frac{S(T)}{Q(T)} - K \right)^+ \mid \mathcal{F}(t) \right]$. Aside from the factor $e^{-at}$ this is the price of a call with risk-free rate $r - a$ and volatility $\sigma_4$. Hence,

$$V(t) = e^{-at} C(T - t, \frac{S(t)}{Q(t)}, K, r - a, \sigma_4),$$

and substituting this into (4.5.22) gives the formula shown in the problem statement.