1. Assume that $S$ satisfies the risk-neutral, Black-Scholes model,

$$dS_t = rS_t dt + \sigma S_t d\tilde{W}(t).$$

This problem is about analyzing the average strike call with payoff

$$V_T = (S_T - \frac{1}{T} \int_0^T S_u du)^+. $$

Let $Y_t = \int_0^t S_u du$.

a) Show that the price of the can be written as $V(t) = v(t, S_t, Y_t)$ and give an expression for $v(t, x, y)$ of the form $v(t, x, y) = \tilde{E}[L(t, x, y)]$ where $L$ is a random variable, and give an explicit formula for $L(t, x, y)$ in terms of $t, x, y$ and the process $\{W(u) - W(t); u \geq t\}$.

b) Find a p.d.e. for $v$. Be sure to specify the domain of $(t, x, y)$-space where the equation holds. Derive or write down all boundary and terminal conditions. You will have to specify a boundary condition that fixes the value of $v$, either as $y \to \infty$ or $y \to -\infty$.

2. Extra Credit (5 points) Let $C(t)$ be the price of a European call with strike $K$ expiring at time $T$. Let $A(t)$ be the price of the corresponding American call. No particular model is put on the price process, except that we assume no dividends are paid on the asset. Use a no-arbitrage argument to show that $C(t) > (S_t - K)^+$ for all $t < T$, as long as $S_T$ can fall to either side of $K$ with positive probability. Use this result to show that $A(t) = C(t)$ for all $t$ and that early exercise is never optimal. (It is also helpful to keep this observation in mind: If $A(t) > (S_t - K)^+$ then it is not optimal to exercise the American option at time $t$ since trading the option itself gives higher pay off than exercising the option).