1. Let \( \tau \) and \( \rho \) be stopping times with respect to a filtration \( \{ \mathcal{F}(t); t \geq 0 \} \).

a) Show that \( \tau \wedge \rho \) (\( \equiv \min\{\tau, \rho\} \)) is a stopping time.

b) Show that \( \tau \vee \rho \) (\( \equiv \max\{\tau, \rho\} \)) is a stopping time.

2. Let \( X = \{X(t); t \geq 0\} \) be a stochastic process whose sample paths are all continuous, and assume \( X(0)(\omega) = 0 \) for all \( \omega \). Let \( \{\mathcal{F}^X(t)\}_{t \geq 0} \) be the filtration generated by \( X \).

In each case below, determine whether the random time must necessarily be an \( \{\mathcal{F}^X(t); t \geq 0\} \)-stopping time. Justify your answer briefly in each case; you may use the informal rule of thumb and/or general results about stopping times, as in the Lecture Notes for lecture 4.

(a) \( T_1 = \inf\{t; X^2(t) \geq 1\}; \)

(b) \( T_2 = \inf\{t; \int_0^t X^2(s) \, ds > 1\}. \)

(c) \( T_3 = \sup\{t; t \leq 1 \text{ and } X(t) = 0\}; \)

(d) Suppose that \( |X(t)|(\omega) > 0 \) for all \( \omega \in \Omega \) and all \( t > 0 \), and reconsider \( T_2 = \inf\{t; \int_0^t X^2(s) \, ds > 1\}. \)

(e) \( T_5 = \inf\{t; X(t) \geq X(t + 1)\}. \)

3. Let \( \tau \) be a stopping time with respect to a filtration, \( \{\mathcal{F}(t); t \geq 0\} \).

a) Let \( n \) be any positive integer. Define a discrete approximation \( \tau^{(n)} \) to \( \tau \) by setting \( \tau^{(n)}(\omega) = \frac{k}{n} \) if \( \frac{k-1}{n} < \tau \leq \frac{k}{n} \). This approximates \( \tau \) from above. Show that \( \tau^{(n)} \) is an \( \{\mathcal{F}(t); t \geq 0\} \)-stopping time.
b) Let \( n \) be any positive integer and define a discrete approximation to \( \tau \) from below by \( \tau_n(\omega) = \frac{k-1}{n} \) if \( \frac{k-1}{n} < \tau \leq \frac{k}{n} \). Is \( \tau_n \) in general an \( \{\mathcal{F}(t); t \geq 0\}\)-stopping time? Explain.

4. (Optional Stopping) Let \( \{X_n\} \) be a martingale with respect to the filtration \( \{\mathcal{F}_n\} \); thus (i) \( X_n \) is \( \mathcal{F}_n \)-measurable for each \( n \), (ii) \( E[|X_n|] < \infty \) for each \( n \), and (iii) \( E[X_{n+1}|\mathcal{F}_n] = X_n \) for each \( n \). Let \( \tau \) be a stopping time with respect to \( \{\mathcal{F}_n\} \). Show that the stopped process \( X_{n \wedge \tau} \) is also a martingale with respect to \( \{\mathcal{F}_n\} \).

Hint: Write \( X_{n \wedge \tau} = \sum_{k=0}^{n} X_k 1_{\{\tau = k\}} + X_n 1_{\{\tau > n\}} \). Observe that \( \{\tau > n\} \) is \( \mathcal{F}_n \)-measurable (why?).