1. Let $\tau$ and $\rho$ be stopping times with respect to a filtration $\{F(t); t \geq 0\}$.
   a) Show that $\tau \wedge \rho (= \min\{\tau, \rho\})$ is a stopping time.

   \[ \{\tau \wedge \rho > t\} = \{\tau > t\} \cap \{\rho > t\} \in F(t), \]

   since both $\tau$ and $\rho$ are stopping times. It follows that

   \[ \{\tau \wedge \rho \leq t\} = \{\tau \wedge \rho > t\}^c \in F(t), \]

   and $\tau \wedge \rho$ is a stopping time.

   b) Show that $\tau \vee \rho (= \max\{\tau, \rho\})$ is a stopping time.

   Similar to part a:

   \[ \{\tau \vee \rho \leq t\} = \{\tau \leq t\} \cap \{\rho \leq t\} \in F(t). \]

   Thus $\tau \vee \rho$ is a stopping time.

2. Let $X = \{X(t); t \geq 0\}$ be a stochastic process whose sample paths are all continuous, and assume $X(0)(\omega) = 0$ for all $\omega$. Let $\{F^X(t)\}_{t \geq 0}$ be the filtration generated by $X$.

   (a) $T_1 = \inf\{t; X^2(t) \geq 1\};$

   Ans: Note that $T_1 = \inf\{t; X^2(t) = 1\}$ since $X(0) = 0$ therefore it is a stopping time.

   (b) $T_2 = \inf\{t; \int_0^t X^2(s) \, ds > 1\}.$

   Ans: It is very tempting to quote the example in the lecture note to say $T_2$ may not be a stopping time. But the situation is slightly different here (see
also part d). Denote \( Y_t = \int_0^t X^2(s) \, ds \) then \( Y_t \) is differentiable in \( t \), moreover, \( Y'(t) = X^2(t) \geq 0 \). So it cannot be the case that \( Y(t) \) just touches 1 at time \( t \) and bounces back. BUT it is possible that \( Y(t) \) touches 1 at time \( t \) and stays at 1 for a short interval, say \([t, t+\epsilon)\), instead of crossing 1 at \( t \). In the first scenario, \( T_2 = t + \epsilon \) and second, \( T_2 = t \). However, up to time \( t \), the sample path of \( Y(t) \) is exactly the same, so we cannot determine the event \( \{T_2 \leq t\} \) based on observation of \( Y_s, 0 \leq s \leq t \). So \( T_2 \) may not be a stopping time.

(c) \( T_3 = \sup\{t; t \leq 1 \ \text{and} \ X(t) = 0\} \);

May not be a stopping time, same as example given in the note.

(d) Suppose that \( |X(t)|(\omega) > 0 \) for all \( \omega \in \Omega \) and all \( t > 0 \), and reconsider \( T_2 = \inf\{t; \int_0^t X^2(s) \, ds > 1\} \).

In this case \( T_2 \) IS a stopping time, since the scenario of staying at 1 on \([t, t+\epsilon)\) is no longer possible (which would imply \( Y'(s) = X^2(s) = 0, s \in [t, t+\epsilon) \) contradicting the hypothesis that \( |X(s)| \neq 0 \)). So \( T_2 = \inf\{t \geq 0 : Y(t) = 1\} \) which is a stopping time.

(e) \( T_5 = \inf\{t; X(t) \geq X(t+1)\} \).

\( T_5 \) may not be a stopping time since if we denote \( Y(t) = X(t) - X(t+1) \) then we can only say \( Y(t) \) is \( \mathcal{F}(t+1) \) measurable. Thus \( \{T_5 \leq t\} = \{Y(s) \geq 0, 0 \leq s \leq t\} \) which may not be in \( \mathcal{F}(t) \).

3. Let \( \tau \) be a stopping time with respect to a filtration, \( \{\mathcal{F}(t); t \geq 0\} \).

a) Let \( n \) be any positive integer. Define a discrete approximation \( \tau^{(n)} \) to \( \tau \) by setting \( \tau^{(n)}(\omega) = \frac{k}{n} \) if \( \frac{k-1}{n} < \tau \leq \frac{k}{n} \). This approximates \( \tau \) from above. Show that \( \tau^{(n)} \) is an \( \{\mathcal{F}(t); t \geq 0\} \)-stopping time.

Ans:

\[
\{\tau_n \leq t\} = \bigcup_{k=1}^{\infty} \{\tau_n \leq t, \frac{k-1}{n} < \tau \leq \frac{k}{n}\} \\
= \bigcup_{k=1}^{\infty} \left\{\frac{k}{n} \leq t, \frac{k-1}{n} < \tau \leq \frac{k}{n}\right\}.
\]

Note that for each \( k \), the event \( \left\{\frac{k}{n} \leq t, \frac{k-1}{n} < \tau \leq \frac{k}{n}\right\} \in \mathcal{F}(t) \). The reason is the event

\[
\left\{\frac{k-1}{n} < \tau \leq \frac{k}{n}\right\} = \left\{\frac{k-1}{n} < \tau\right\} \cap \left\{\tau \leq \frac{k}{n}\right\} \in \mathcal{F}\left(\frac{k}{n}\right).
\]
And the event \( \{ \frac{k}{n} \leq t \} \) is either \( \emptyset \) or \( \Omega \). If it is empty then \( \{ \frac{k}{n} \leq t, \frac{k-1}{n} < \tau \leq \frac{k}{n} \} = \emptyset \) and is in \( \mathcal{F}(t) \). If it is \( \Omega \) then it must be the case that \( \frac{k}{n} \leq t \) and so \( \mathcal{F}(\frac{k}{n}) \subseteq \mathcal{F}(t) \) and we reach the same conclusion.

Hence \( \{ \tau_n \leq t \} \) being countable union of event in \( \mathcal{F}(t) \) is in \( \mathcal{F}(t) \) and \( \tau_n \) is a stopping time.

b) Let \( n \) be any positive integer and define a discrete approximation to \( \tau \) from below by \( \tau_n(\omega) = \frac{k}{n} \) if \( \frac{k-1}{n} < \tau \leq \frac{k}{n} \). Is \( \tau_n \) in general an \( \{ \mathcal{F}(t); t \geq 0 \} \)-stopping time? Explain.

Ans: \( \tau_n \) may not be a stopping time. Arguing similar to the above, we get

\[
\{ \tau_n \leq t \} = \bigcup_{k=1}^{\infty} \{ \tau_n \leq t, \frac{k-1}{n} < \tau \leq \frac{k}{n} \} = \bigcup_{k=1}^{\infty} \{ \frac{k-1}{n} \leq t, \frac{k-1}{n} < \tau \leq \frac{k}{n} \}.
\]

Note that now we deal with the event \( \{ \frac{k-1}{n} \leq t \} \) while it is still the case that the event \( \{ \frac{k-1}{n} < \tau \leq \frac{k}{n} \} \subseteq \mathcal{F}(\frac{k}{n}) \). But if \( \frac{k-1}{n} \leq t \) it does not necessarily follow that \( \frac{k}{n} \leq t \) and thus the event \( \{ \frac{k-1}{n} < \tau \leq \frac{k}{n} \} \) may not be in \( \mathcal{F}(t) \).

4. (Optional Stopping) Let \( \{ X_n \} \) be a martingale with respect to the filtration \( \{ \mathcal{F}_n \} \); thus (i) \( X_n \) is \( \mathcal{F}_n \)-measurable for each \( n \), (ii) \( E[|X_n|] < \infty \) for each \( n \), and (iii) \( E[X_{n+1}|\mathcal{F}_n] = X_n \) for each \( n \). Let \( \tau \) be a stopping time with respect to \( \{ \mathcal{F}_n \} \). Show that the stopped process \( X_{n\land\tau} \) is also a martingale with respect to \( \{ \mathcal{F}_n \} \).

Ans: We only need to show for all \( n \)

\[
E(X_{n\land\tau}|\mathcal{F}(n-1)) = X_{n-1\land\tau}.
\]

Note that

\[
X_{n\land\tau} = \sum_{k=0}^{n-1} X_k 1_{\{\tau = k\}} + X_n 1_{\{\tau \geq n\}}.
\]

and all events \( \{ \tau = k \}, k = 0, \ldots, n-1 \) and \( \{ \tau \geq n \} \) is in \( \mathcal{F}(n-1) \). Only the last one needs explanation. Note that \( \{ \tau \geq n \} = \{ \tau < n \}^c = \left( \bigcup_{k=1}^{n-1} \{ \tau = k \} \right)^c \) so it is in \( \mathcal{F}(n-1) \). Thus

\[
E(X_{n\land\tau}|\mathcal{F}(n-1)) = \sum_{k=0}^{n-1} X_k 1_{\{\tau = k\}} + 1_{\{\tau \geq n\}} E(X_n|\mathcal{F}(n-1))
\]

\[
= \sum_{k=0}^{n-1} X_k 1_{\{\tau = k\}} + 1_{\{\tau \geq n\}} X_{n-1}
\]

\[
= \sum_{k=0}^{n-2} X_k 1_{\{\tau = k\}} + 1_{\{\tau \geq n-1\}} X_{n-1} - X_{n-1\land\tau}.
\]

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