The solution follows the method in class for deriving the Black-Scholes-Merton formula with random interest rate (see Theorem 9.4.2 in Shreve).

According to Theorem 9.2 (here the role of \( \nu_t \) is played by \( 3-t \)),

\[
\tilde{W}^T_3(t) = \tilde{W}(t) - \int_0^t (3-s) \, ds = \tilde{W}(t) + (1/2)[(3-t)^2 - 9]
\]

is a Brownian motion under \( \tilde{\mathbb{P}}(3) \), for \( t \leq 3 \), and

\[
d_t \text{For}_S(t) = \text{For}_S(t)[\sigma - (3-t)] \, d\tilde{W}^T_3(t) = \text{For}_S(t)[t+\sigma-3] \, d\tilde{W}^T_3(t).
\]

Then \( \text{For}_S(3) = \frac{s_0}{B(0,3)} e^{X - \nu^2/2} \), where

\[
X = \int_0^3 [s+\sigma-3] \, d\tilde{W}^T_3(s),
\]

is, under the measure \( \tilde{\mathbb{P}}(3) \), a Gaussian random variable with mean 0 and variance

\[
\nu^2 = \int_0^3 [s+\sigma-3]^2 \, ds = (1/3)[\sigma^3 - (\sigma+3)^3].
\]

Therefore, using Theorem 2 in Lecture Notes 12, the forward price of the call at \( t = 0 \) is

\[
\frac{V(0)}{B(0,3)} = E^{T_3} [(S(3) - K)^+] = \frac{s_0}{B(0,3)} N \left( \frac{\ln(s_0/B(0,3)K) + \nu^2/2}{\nu} \right) - KN \left( \frac{\ln(s_0/B(0,3)K) - \nu^2/2}{\nu} \right)
\]

As a result,

\[
V(0) = s_0 N \left( \frac{\ln(s_0/B(0,3)K) + \nu^2/2}{\nu} \right) - KB(0,3) N \left( \frac{\ln(s_0/B(0,3)K) - \nu^2/2}{\nu} \right).
\]
2. (Shreve, 10.9). By equation (5) in Lecture Notes 12,
\[ f(t, T) = f(0, T) + \int_0^t \alpha(u, T) \, du + \int_0^t \sum_{j=1}^d \sigma_j(u, T) \, dW_j(u). \] (1)

Following the derivation of section 10.3.2, show that
\[ dt \left( - \int_t^T f(t, v) \, dv \right) = R(t) \, dt - \alpha^*(t, T) \, dt - \sum_{j=1}^d \sigma_j^*(t, T) \, dW_j(t) \]

By applying Itô’s rule to \( B(t, T) = e^{-\int_t^T f(t, v) \, dv} \), it follows that
\[ dB(t, T) = B(t, T) \left[ R(t) - \alpha^*(t, T) + \frac{1}{2} \sum_{j=1}^d (\sigma_j^*(t, T))^2 \right] \, dt - B(t, T) \sum_{j=1}^d \sigma_j^*(t, T) \, dW_j(t) \]

Write
\[ dB(t, T) = B(t, T) \left[ R(t) - \alpha^*(t, T) + \frac{1}{2} \sum_{j=1}^d (\sigma_j^*(t, T))^2 + \sum_{j=1}^d \sigma_j^*(t, T)\Theta_j(t) \right] \, dt - B(t, T) \sum_{j=1}^d \sigma_j^*(t, T) \, d\tilde{W}_j(t) \]

where \( \tilde{W}_j(t) = W_j(t) + \int_0^t \Theta_j(s) \, ds \). When we change measure to make \( \tilde{W}(t) = (\tilde{W}_1(t), \ldots, \tilde{W}_d(t)) \) a Brownian motion, the model will be risk-neutral if
\[ \alpha^*(t, T) = \frac{1}{2} \sum_{j=1}^d (\sigma_j^*(t, T))^2 + \sum_{j=1}^d \sigma_j^*(t, T)\Theta_j(t). \]

Take derivatives on both sides with respect to \( T \) to obtain,
\[ \alpha(t, T) = \sum_{j=1}^d \sigma_j^*(t, T)\sigma_j(t, T) + \sum_{j=1}^d \sigma_j(t, T)\Theta_j(t). \]

(ii) Suppose there is a solution \( \Theta(t) = (\Theta_1(t), \ldots, \Theta_d(t)) \). If \( T_1, \ldots, T_d \) is a set of distinct times then
\[ \alpha(t, T_i) = \sum_{j=1}^d \sigma_j^*(t, T_i)\sigma_j(t, T_i) + \sum_{j=1}^d \sigma_j(t, T_i)\Theta_j(t) \]
for \(1 \leq i \leq d\). If the matrix \([\sigma_j(t, T)]_{1 \leq i,j \leq d}\) is invertible, this system has a unique solution \(\Theta(t)\) for all \(t \leq \min_k T_k\).

3. a) In this case \(\sigma^*(t, T) = \int_t^T t v \, dv = (1/2)t(T^2 - t^2)\). In order that the model be arbitrage-free, there must exist a solution \(\theta(t)\) to \(\alpha(t, T) = \sigma(t, T)[\sigma^*(t, T) + \theta(t)]\), or

\[
\frac{T^3t^2}{2} + 5Tt - \frac{Tt^4}{2} = tT[(1/2)t(T^2 - t^2) + \theta(t)].
\]

Subtracting \(\frac{T^3t^2}{2} - \frac{Tt^4}{2}\) from both sides leaves \(5Tt = tT\theta(t)\). Therefore there is a solution with \(\theta(t) \equiv 5\), and so the model is arbitrage-free.

b) Let \(\sigma_1(t, T) = 1\) and \(\sigma_2(t, T) = 2T\). Then \(\sigma^*_1(t, T) = T - t\) and \(\sigma^*_2(t, T) = \int_t^T 2udu = T^2 - t^2\).

From Exercise 10.9, for the model to be arbitrage-free there must be a solution \(\theta_1(t), \theta_2(t)\), independent of \(T\), to

\[
\alpha(t, T) = T - t - 2Tt^2 = [T - t + \theta_1(t)] + 2T[T^2 - t^2 + \theta_2(t)].
\]

By cancellation of terms common to both sides, \((\theta_1(t), \theta_2(t))\) must solve

\[
0 = \theta_1(t) + 2T\theta_2(t) + 2T^3
\]

for all \(0 \leq t \leq T \leq \bar{T}\). If this were true, then taking partial derivatives with respect to \(T\) on both sides implies \(2\theta_2(t) + 6T^2 = 0\). But this contradicts the condition that \(\theta_2(t)\) is independent of \(T\), and hence there can be no solution of the required form. Therefore, we conclude that the given model is not arbitrage-free.

4. (Shreve, Exercise 10.11) The value at \(t = 0\) of a payment of \(\delta K\) at \(T\) is \(\delta K B(0, T)\). The value at \(t = 0\) of a series of payments of \(\delta K\) at time \(T_1, \ldots, T_{n+1}\) is thus

\[
\delta K \sum_{j=1}^{n+1} B(0, T_j).
\]

By Theorem 10.4.1, the value at \(t = 0\) of a payment of amount \(\delta L(T_{j-1}, T_j)\) at \(T_j\) is \(\delta B(0, T_j) L(0, T_{j-1})\)—see equation (10.4.5). The value of a contract at \(t = 0\) promising fixed legs in return for paying floating legs is therefore

\[
\delta K \sum_{j=1}^{n+1} B(0, T_j) - \delta \sum_{j=1}^{n+1} B(0, T_j) L(0, T_{j-1})
\]

(2)

5. For the one-factor Vasicek model, \(dR(t) = (a - bR(t)) \, dt + \sigma R(t) \, d\bar{W}(t)\), the results of section 10.3.5 show that

\[
d[D(t)B(t, T)] = -\sigma^*(t, T)[D(t)B(t, T)] \, d\bar{W}(t),
\]

3
where \( \sigma^*(t, T) = \frac{\sigma}{b} \left(1 - e^{-b(T-t)}\right) \). By (10.4.9) and (10.4.15),

\[
dL(t, T) = \frac{1 + \delta L(t, T)}{\delta L(t, T)} \left[ \sigma^*(t, T + \delta) - \sigma^*(t, T) \right] L(t, T) \, d\tilde{W}^{T+\delta}(t) = \left[1 + \delta L(t, T)\right] \frac{\sigma e^{-b(T-t)}(1 - e^{-b\delta})}{\delta b} \, d\tilde{W}^{T+\delta}(t).
\]

Let \( Y(t) = 1 + \delta L(t, T) \). Then it follows that

\[
dY(t) = \delta dL(t, T) = Y(t) \beta(t, T) \, d\tilde{W}_T + \delta(t).
\]

Let \( B = \int_0^T \beta(u, T) \, d\tilde{W}^{T+\delta}(u) \). This is a normal random variable with mean 0 and variance \( \int_0^T \beta^2(u, T) \, du \). Then

\[
L(T, T) = \delta^{-1}[Y(T) - 1] = \delta^{-1} \left[(1 + \delta L(0, T))e^{B-(1/2)\int_0^T \beta^2(u, T) \, du} - 1\right].
\]

Let \( V(0) \) denote the price of a caplet at strike \( K \) for \([T, T+\delta]\). The \( T+\delta \)-forward price is thus

\[
\frac{V(0)}{B(0, T+\delta)} = \tilde{E}^{T+\delta} \left[(L(t, T) - K)^+\right] = \tilde{E}^{T+\delta} \left[\left(\delta^{-1}(1 + \delta L(0, T))e^{B-(1/2)\int_0^T \beta^2(u, T) \, du} - \delta^{-1} - K\right)^+\right]
\]

The Black-Scholes formula tells us how to price this. It is the same as the price of a call at strike \( \delta^{-1} + K \), when \( \sigma^2 T = \int_0^T \beta^2(u, T) \, du \), \( r = 0 \) and the initial price is \( \delta^{-1} + L(0, T) \). This is

\[
(\delta^{-1} + L(0, T))N(\bar{d}_+) - (\delta^{-1} + K)N(\bar{d}_-),
\]

where

\[
d_\pm = \frac{1}{\sqrt{\int_0^T \beta^2(u, T) \, du}} \left[\log \frac{1 + \delta L(0, T)}{1 + \delta K} \pm \frac{1}{2} \int_0^T \beta^2(u, T) \, du \right].
\]

We could compute \( \int_0^T \beta^2(u, T) \, du \) explicitly, but have not done so here. Finally,

\[
V(0) = B(0, T + \delta) \left[(\delta^{-1} + L(0, T))N(\bar{d}_+) - (\delta^{-1} + K)N(\bar{d}_-)\right].
\]