Homework 1 (Due 02/10/2016)

Math 622

February 5, 2016

1. Let $0 < a < b$. Let $G$ be a càdlàg function of bounded variation. In the following, the notation $\int H(s)dG(s)$ will mean $\int_{(0,\infty)} H(s)dG(s)$ as in the lecture Note 1.

(i) Use the definition in Section 8.3.B Lecture 1 note to show that $\int_{(a,b]}\int dG(s) = G(b) - G(a)$.

(ii) Show that $\lim_{n \to \infty} \int_{1} \int_{(a,b]}(t) = 1_{(a,b]}(t)$ and $\lim_{n \to \infty} \int_{1} \int_{(a+\frac{1}{n},b]}(t) = 1_{(a,b]}(t)$.

(iii) Show that $\lim_{n \to \infty} \int_{1} \int_{(a,b]}(t) = \lim_{n \to \infty} \int_{1} \int_{(a+\frac{1}{n},b]}(t) = \int_{1} \int_{(a,b]}(t) = \int_{1} \int_{(a,b]}(t)$.

(iv) Is it true that $\lim_{n \to \infty} \int_{1} \int_{(a,b-\frac{1}{n})}(t) = \int_{1} \int_{(a,b]}(t)$?

(v) Evaluate $\int_{1} \int_{(a,b]}(t)$, $\int_{1} \int_{(a,b)}(t)$, $\int_{1} \int_{(a,b]}(t)$ (Hint: Approximate these integrands with left continuous functions, and use the Dominated Convergence Theorem).

2. Let $G(t) = \begin{cases} 2t & \text{if } 0 \leq t < 1; \\ t^2 - 3 & \text{if } 1 \leq t < 2; \\ t + 1 & \text{if } 2 \leq t. \end{cases}$

Evaluate $\int_{1}^{3} s dG(s)$.
3. Let $0 < t_1 < t_2$ and $a_1, a_2 \in \mathbb{R}$. Define
\[
G(t) = \begin{cases} 
0, & 0 \leq t < t_1; \\
a_1, & t_1 \leq t < t_2; \\
a_1 + a_2, & t_2 \leq t.
\end{cases}
\]

(i) Let $\sigma > 0$. Solve for $Z(t)$, where $Z(t)$ satisfies
\[
Z(t) = 1 + \int_0^t \sigma Z(s-)dG(s).
\]

(ii) Now let $\sigma(s)$ be a function of $s$. Solve for $Z(t)$, where $Z(t)$ satisfies
\[
Z(t) = 1 + \int_0^t Z(s)ds + \int_0^t \sigma(s)Z(s-)dG(s).
\]

4. (i) Let $X(t)$ be a Levy process and $\mathcal{F}(t)$ be a filtration for $X(t)$. Let $\mu t = \mathbb{E}(X(t))$ and $\sigma^2 t = \text{Var}(X(t))$. Show that $(X(t) - \mu t)^2 - \sigma^2 t$ is a martingale w.r.t. $\mathcal{F}(t)$.

(ii) Let $N(t)$ be a Poisson process and $\mathcal{F}(t)$ be a filtration for $N(t)$. Show $\exp\left(iuN(t) - \lambda t(e^{iu} - 1)\right)$ is a martingale w.r.t. $\mathcal{F}(t)$.

(iii) Show that the Geometric Poisson process discussed in Example 9.1 of Lecture note 1 is a martingale (w.r.t its own filtration), without using Shreve’s Theorem 11.4.5.

5. Let $X(t)$ be a Levy process and $\mathcal{F}(t)$ a filtration for $X(t)$. Use Lemma 2.3.4 and Definition 2.3.6 in Shreve to show that $X(t)$ is a Markov process.

6. (i) Let $J$ be a counting process, that is $J(0) = 0$, $J$ has finitely many jumps on any finite intervals and $\Delta J(t) = 1$ at any jump point of $J$. Show that
\[
\int_0^t J(u)dJ(u) = \frac{J(t)(J(t) + 1)}{2},
\]
\[
\int_0^t J(u-)dJ(u) = \frac{J(t)(J(t) - 1)}{2}.
\]

Let $N(t)$ be a Poisson process with rate $\lambda$ and $\mathcal{F}(t)$ a filtration for $N(t)$.

(ii) Find an explicit formula for
\[
X(t) := \int_0^t (N(s) - N(s-))d(N(s) - \lambda s),
\]
and conclude that $X(t)$ is not a martingale (w.r.t $\mathcal{F}(t)$). (Hint: Using the fact that if $f(t) = 0$ at all but finitely many points $t$, then $f(s) = 0$ so that $\int_0^t f(s)ds = \int_0^t f(s-)ds = 0$, it should be almost immediate to guess what $X(t)$ is).

(iii) Show that

$$Y(t) := \int_0^t N(s-)d(N(s) - \lambda s),$$

is a martingale (w.r.t $\mathcal{F}(t)$).

Hint: Recall that $\int_0^t N(s-)d(N(s) - \lambda s) = \int_0^t N(s-)dN(s) - \int_0^t \lambda N(s-)ds$ and part (i) of this problem. You can also use the fact that

$$\mathbb{E}\left(\int_0^t N(u)du|\mathcal{F}(s)\right) = \int_0^s N(u)du + \int_s^t \mathbb{E}(N(u)|\mathcal{F}(s))du.$$

(iv) Show that

$$Z(t) := \int_0^t N(s)d(N(s) - \lambda s)$$

is not a martingale w.r.t $\mathcal{F}(t)$.

7. Extra credit (5pts).

Let $f(t)$ be defined on $[0, \infty)$. Fix $T > 0$. The total variation of $f$ on $[0, T]$, denoted as $TV_f(T)$ is defined as the smallest (finite) number such that for all partitions $0 = t_0 < t_1 < t_2 < \ldots < t_n = T$

$$\sum_{i=0}^{n-1} |f(t_{i+1}) - f(t_i)| \leq TV_f(T).$$

If there is no such number, we define $TV_f(T) = \infty$.

We also say $f$ is a function of bounded variation (on $[0, \infty)$) if $TV_f(T) < \infty$ for all $T > 0$.

(i) Let $A$ be an increasing function on $[0, \infty)$. Show that for all $T > 0$, $TV_A(T) = A(T) - A(0)$. Thus any increasing function is of bounded variation.

(ii) Let $A_1, A_2$ be increasing functions on $[0, \infty)$. Show that $TV_{A_1 - A_2}(T) \leq TV_{A_1}(T) + TV_{A_2}(T)$. Thus the difference between two increasing functions is of bounded variation. This is the reason for definition 4.1 in Lecture note 1.

(iii) Let $G(t)$ be a function of bounded variation. Show that for any partition $0 = t_0 < t_1 < t_2 < \ldots < t_n = T$,

$$\sum_{i=0}^{n-1}(G(t_{i+1}) - G(t_i))^2 \leq \max_i |G(t_{i+1}) - G(t_i)|TV_G(T).$$
(iv) We say a function $f$ is uniformly continuous on $[0, T]$ if there exists a non-negative function $\rho, \lim_{t \to 0} \rho(t) = 0 = \rho(0)$ and for all $0 \leq t, s < T$, $|f(t) - f(s)| \leq \rho(|t - s|)$. Use the fact that a continuous function on $[0, T]$ is uniformly continuous to show that if $G$ is continuous, $G$ is of bounded variation then its quadratic variation $[G, G](T) = 0$ for any $T > 0$ (See Sheve’s Definition 3.4.1)

(v) Show that the sample paths of Brownian motion is not of bounded variation with probability 1.