1. Compute \( \phi_X(t) := E\left( \exp(itX) \right) \), where \( \exp(x) := e^x \), \( i \) is the imaginary number: \( i^2 = -1 \) and \( X \) has \( N(\mu, \sigma^2) \) distribution. Recall that the density of \( N(\mu, \sigma^2) \) is

\[
f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right).
\]

\( \phi(t) \) is called the characteristic function of \( X \).

Ans:
Using the fact that for \( X \) having distribution \( N(0, \sigma^2) \), \( E(e^X) = e^{\frac{1}{2}\sigma^2} \).

Note that

\[
e^{itX} = e^{it(X-\mu)+it\mu} = e^{it\mu}e^{it(X-\mu)},
\]

where \( it(X - \mu) \) has "distribution" \( N(0, i^2t^2\sigma^2) \) (The \( i^2 \) is treated purely as a symbol here, nevertheless the calculation is still correct).

Apply the above result, we have

\[
e^{itX} = e^{it\mu + \frac{1}{2}i^2t^2\sigma^2} = e^{it\mu - \frac{1}{2}t^2\sigma^2}.
\]

2. a) Suppose \( \mu = 0 \). Compute \( \phi_X^{(4)}(t) \): the 4th derivative of \( \phi_X \) with respect to \( t \).

Ans: Use some online differentiating solver (Wolfram Alpha comes to mind) we get

\[
\phi_X^{(4)}(t) = \sigma^4 e^{-\frac{1}{2}t^2\sigma^2}(\sigma^4 t^4 - 6\sigma^2 t^2 + 3).
\]

b) Use the following fact:

\[
E[X^k] = (-i)^k \phi_X^{(k)}(0)
\]

to compute \( E[(B_t)^4] \) where \( B \) is a Brownian motion.
Ans: Recall that $B_t$ has distribution $N(0, t)$, replacing $t = 0$ and $\sigma^2 = t$ into the expression of part a, we get $E[(B_t)^4] = 3t^2$.

3. Let $0 \leq s \leq t \leq T$ and $B$ a Brownian motion. Compute the followings:
   a) $E(B^2_s B_t)$
      Ans:
      \[
      E(B^2_s B_t) = E(B^2_s (B_s + B_t - B_s)) = E(B^3_s) + E(B^2_s (B_t - B_s))
      \]
      \[
      = E(B^3_s) + E(B^2_s)E(B_t - B_s) = 0 + s \times 0 = 0.
      \]
   b) $E(B^2_s B_s)$
      Ans:
      \[
      E(B_s B^2_t) = E(B_s (B_s + B_t - B_s)^2) = E(B_s (B^2_s + 2B_s (B_t - B_s) + (B_t - B_s)^2))
      \]
      \[
      = E(B^3_s) + E(2B^2_s (B_t - B_s)) + E(B_s (B_t - B_s)^2)
      \]
      \[
      = E(B^3_s) + E(2B^2_s)E(B_t - B_s) + E(B_s)E((B_t - B_s)^2)
      \]
      \[
      = 0 + s \times 0 + 0(t - s) = 0.
      \]
   c) $E(\exp(\sigma B_t - \frac{1}{2}\sigma^2 t))$, where $\sigma$ is a constant.
      Ans: Since $\sigma B_t$ has distribution $N(0, \sigma^2 t)$, $E(e^{\sigma B_t}) = e^{\frac{1}{2}\sigma^2 t}$. Thus $E(\exp(\sigma B_t - \frac{1}{2}\sigma^2 t)) = 1$.
   d) $E(\exp(\int_0^t \sin(s)dB_s))$.
      Ans: Recall that $\int_0^t \sin(s)dB_s$ has distribution $N(0, \int_0^t \sin^2(s)ds)$ since $\sin(s)$ is deterministic. Therefore
      \[
      E(\exp(\int_0^t \sin(s)dB_s)) = \exp\left(\frac{1}{2} \int_0^t \sin^2(s)ds\right)
      \]

4. Use Ito formula to compute the following
   a) $d\sin(B_t)$
      \[
      d\sin(B_t) = \cos(B_t)dB_t - \frac{1}{2} \sin(B_t)dt.
      \]
   b) $d\exp(B_t)$
      \[
      d\exp(B_t) = \exp(B_t)dB_t + \frac{1}{2} \exp(B_t)dt.
      \]
5. Let \(0 = t_0 < t_1 < t_2 < \ldots < t_n = T\). Show that

\[
E\left[ \sum_{i=0}^{n-1} (B_{t_{i+1}} - B_{t_i})^2 - T \right] = 2 \sum_{i=0}^{n-1} (t_{i+1} - t_i)^2.
\]

Short Ans:

Since

\[
E\left[ \sum_{i=0}^{n-1} (B_{t_{i+1}} - B_{t_i})^2 \right] = T,
\]

\[
E\left[ \sum_{i=0}^{n-1} (B_{t_{i+1}} - B_{t_i})^2 - T \right] = Var\left( \sum_{i=0}^{n-1} (B_{t_{i+1}} - B_{t_i})^2 \right)
\]

\[
= \sum_{i=0}^{n-1} Var((B_{t_{i+1}} - B_{t_i})^2)
\]

\[
= \sum_{i=0}^{n-1} E((B_{t_{i+1}} - B_{t_i})^4) - E^2((B_{t_{i+1}} - B_{t_i})^2)
\]

\[
= \sum_{i=0}^{n-1} 3(t_{i+1} - t_i)^2 - (t_{i+1} - t_i)^2
\]

\[
= 2 \sum_{i=0}^{n-1} (t_{i+1} - t_i)^2.
\]

Long Ans: Note that

\[
\left[ \sum_{i=0}^{n-1} (B_{t_{i+1}} - B_{t_i})^2 - T \right]^2 = \left[ \sum_{i=0}^{n-1} (B_{t_{i+1}} - B_{t_i})^2 \right]^2 - 2\left( \sum_{i=0}^{n-1} (B_{t_{i+1}} - B_{t_i})^2 \right)T + T^2.
\]

And

\[
E\left(2\sum_{i=0}^{n-1} (B_{t_{i+1}} - B_{t_i})^2 T \right) = 2T \left( \sum_{i=0}^{n-1} (t_{i+1} - t_i) \right) = 2T^2.
\]

Hence

\[
E\left[ -2\sum_{i=0}^{n-1} (B_{t_{i+1}} - B_{t_i})^2 T + T^2 \right] = -T^2.
\]

Now

\[
\left[ \sum_{i=0}^{n-1} (B_{t_{i+1}} - B_{t_i})^2 \right]^2 = \left[ \sum_{i=0}^{n-1} (B_{t_{i+1}} - B_{t_i})^2 \right] \left[ \sum_{j=0}^{n-1} (B_{t_{j+1}} - B_{t_j})^2 \right]
\]

\[
= \sum_{i=0}^{n-1} (B_{t_{i+1}} - B_{t_i})^4 + \sum_{i \neq j, i, j=0}^{n-1} (B_{t_{i+1}} - B_{t_i})(B_{t_{j+1}} - B_{t_j}).
\]
Apply Problem 2, noting that $B_{t_{i+1}} - B_t$ has distribution $N(0, t_{i+1} - t_i)$ we have

$$E\left[ \sum_{i=0}^{n-1} (B_{t_{i+1}} - B_t)^4 \right] = 3 \sum_{i=0}^{n-1} (t_{i+1} - t_i)^2.$$  

Also if $i \neq j$, $B_{t_{i+1}} - B_t$ and $B_{t_{j+1}} - B_t$ are independent. So

$$E\left[ \sum_{i \neq j, i,j=0}^{n-1} (B_{t_{i+1}} - B_t)(B_{t_{j+1}} - B_t) \right] = \sum_{i \neq j, i,j=0}^{n-1} (t_{i+1} - t_i)(t_{j+1} - t_j).$$  

The last thing to do is to simplify $\sum_{i \neq j, i,j=0}^{n-1} (t_{i+1} - t_i)(t_{j+1} - t_j)$.

We have

$$\sum_{i \neq j, i,j=0}^{n-1} (t_{i+1} - t_i)(t_{j+1} - t_j) = \sum_{i=0}^{n-1} \sum_{j=0, j \neq i}^{n-1} (t_{i+1} - t_i)(t_{j+1} - t_j)$$

$$= \sum_{i=0}^{n-1} [(t_{i+1} - t_i)( \sum_{j=0, j \neq i}^{n-1} (t_{j+1} - t_j)] = \sum_{i=0}^{n-1} [(t_{i+1} - t_i)(T - (t_{i+1} - t_i)]$$

$$= T \sum_{i=0}^{n-1} (t_{i+1} - t_i) - \sum_{i=0}^{n-1} (t_{i+1} - t_i)^2 = T^2 - \sum_{i=0}^{n-1} (t_{i+1} - t_i)^2.$$  

Putting all these results together, we get the desired calculation.

6. Suppose $S_t$ satisfies

$$dS_t = \sin(S_t)t^2dt + \exp(\sqrt{St} - t)dB_t.$$  

Compute

a) $d\log(S_t)$

$$d\log(S_t) = \frac{1}{S_t}dS_t - \frac{1}{2S_t^2}[\exp(\sqrt{St} - t)]^2dt$$

$$= \frac{1}{S_t}\left[ \sin(S_t)t^2dt + \exp(\sqrt{St} - t)dB_t \right] - \frac{1}{2S_t^2}[\exp(\sqrt{St} - t)]^2dt$$

$$= \frac{1}{S_t}\exp(\sqrt{St} - t)dB_t$$

$$+ \left[ \frac{1}{S_t}(\sin(S_t)t^2) - \frac{1}{2S_t^2}[\exp(2\sqrt{St} - 2t)] \right] dt.$$  

b) $d \exp(S_t^2)$

\[
d \exp(S_t^2) = 2S_t \exp(S_t^2) dS_t + \frac{1}{2} \left(2 \exp(S_t^2) + 4S_t^2 \exp(S_t^2)\right) [\exp(\sqrt{S_t} - t)]^2 dt
\]

\[
= 2S_t \exp(S_t^2) \left[ \sin(S_t)t^2 dt + \exp(\sqrt{S_t} - t) dB_t \right]
\]

\[
+ \frac{1}{2} \left(2 \exp(S_t^2) + 4S_t^2 \exp(S_t^2)\right) [\exp(\sqrt{S_t} - t)]^2 dt
\]

\[
= \ldots
\]

c) $d \sqrt{S_t}$

\[
d \sqrt{S_t} = \frac{1}{2\sqrt{S_t}} dS_t - \frac{1}{4S_t^2} [\exp(\sqrt{S_t} - t)]^2 dt
\]

\[
= \frac{1}{2\sqrt{S_t}} \left[ \sin(S_t)t^2 dt + \exp(\sqrt{S_t} - t) dB_t \right]
\]

\[
- \frac{1}{4S_t^2} [\exp(\sqrt{S_t} - t)]^2 dt
\]

\[
= \ldots
\]

Please note that this exercise is only for you to symbolically practice Ito’s formula. There may not exist a process $S_t$ that satisfies the above dynamics, or even if it does exist, to make sense in the expression a, b and c.

7. (Extra credit - 5 pts) a) Look up precisely in what “weak sense” the convergence in the definition of Ito integral is. Explain in a few words your understanding of that mode of convergence.

b) Find 3 properties of Brownian motion we have not discussed in the lecture that you find interesting.