1 Sums of continuous RVs

1.1 Introduction

We want to obtain a density for the random variable $Z = X + Y$, where $X, Y$ are independent continuous RVs with densities $f_X, f_Y$ respectively.

1.2 The general formula

We have

$$F_{X+Y}(z) = P(X + Y \leq z)$$

$$= \int\int_{x+y \leq z} f_{XY}(x, y)dxdy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{z-y} f_X(x)f_Y(y)dxdy$$

$$= \int_{-\infty}^{\infty} f_Y(y)\int_{-\infty}^{z-y} f_X(x)dxdy$$

$$= \int_{-\infty}^{\infty} F_X(z-y)f_Y(y)dy.$$

Therefore

$$f_Z(z) = \frac{d}{dz} \int_{-\infty}^{\infty} F_X(z-y)f_Y(y)dy$$

$$= \int_{-\infty}^{\infty} f_X(z-y)f_Y(y)dy. \quad (1)$$

We say $f_Z(z)$ is the convolution of $f_X, f_Y$. Note that we also have

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x)f_Y(z-x)dx.$$
1.3  Examples

1.3.1  Sum of Uniforms

Example 1.1. Let $X$ and $Y$ be independent Uniform[0,1] RVs. Find the density of $Z = X + Y$.

Ans: Before we even attempt the solution, you should note that $Z$ is takes values in the range $[0, 2]$. Applying the above formula gives

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z - y) f_Y(y) dy$$

$$= \int_{0}^{1} 1_{0 \leq z - y \leq 1} f_Y(y) dy.$$

We utilize the fact that $0 \leq z \leq 2$ here. The inequality $0 \leq z - y \leq 1$ implies that $z - 1 \leq y \leq z$.

So we need to compare $z - 1$ with 0 and $z$ with 1, as those are the lower and upper limits of our original integral.

If $0 \leq z \leq 1$ then

$$f_Z(z) = \int_{0}^{z} 1dy = z.$$

If $1 \leq z \leq 2$ then

$$f_Z(z) = \int_{z-1}^{1} 1dy = 1 - (z - 1) = 2 - z.$$

In summary, the sum of 2 independent Uniforms [0,1] has density

$$f_Z(z) = z; 0 \leq z \leq 1$$

$$= 2 - z; 1 \leq z \leq 2.$$

1.3.2  Sum of Gammas

Proposition 1.2. Let $X, Y$ be independent Gamma with parameters $(\alpha, \lambda)$ and $(\beta, \lambda)$, respectively. Then $X + Y$ has distribution Gamma $(\alpha + \beta, \lambda)$.

Proof. Recall that the density of Gamma$(\alpha, \lambda)$ is

$$f_X(x) = \frac{\lambda e^{-\lambda x} (\lambda x)^{\alpha - 1}}{\Gamma(\alpha)}, x \geq 0.$$
Thus by the formula (1)

\[ f_Z(z) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_{-\infty}^{\infty} \lambda e^{-\lambda(z-y)}(\lambda(z-y))^{\alpha-1} \lambda e^{-\lambda y}(\lambda y)^{\beta-1} dy. \]

The requirement that \( x = z - y \geq 0 \) reduces the above integral to

\[ f_Z(z) = Ke^{-\lambda z} \int_{0}^{z} (z-y)^{\alpha-1} y^{\beta-1} dy, \]

where we absorb all constants related to \( \alpha, \beta, \lambda \) into \( K \). The reason is only the form of the density is important. The constant has to work out right for the integral of the density to be 1, since we know a priori that \( f_Z(z) \) is a density.

Now

\[
\int_{0}^{z} (z-y)^{\alpha-1} y^{\beta-1} dy = z^{\alpha-1} \int_{0}^{z} (1 - \frac{y}{z})^{\alpha-1} y^{\beta-1} dy = z^{\alpha+\beta-1} \int_{0}^{1} (1 - u)^{\alpha-1} u^{\beta-1} du,
\]

by the substitution \( u = \frac{y}{z} \), \( du = \frac{1}{z} dy \). Since \( \int_{0}^{1} (1 - u)^{\alpha-1} u^{\beta-1} du \) is just another constant, it can be absorbed into \( K \) to give us

\[ f_Z(z) = Ke^{-\lambda z} z^{\alpha+\beta-1}, \]

which is the right form for a Gamma \((\alpha + \beta, \lambda)\). The proof is finished.

**Corollary 1.3.** Let \( X_1, X_2, \cdots, X_n \) be independent Exponential \((\lambda)\). Then \( Y = X_1 + X_2 + \cdots + X_n \) has Gamma \((n, \lambda)\) distribution.

**Proof.** Since Exponential\((\lambda) = \text{Gamma}(1, \lambda)\), by the above Proposition, we have \( X_1 + X_2 \) has distribution Gamma \((2, \lambda)\). But \( X_1 + X_2 \) are independent of \( X_3 \), thus \( X_1 + X_2 + X_3 \) has distribution Gamma \((3, \lambda)\). Continue in this fashion we have the result.

### 1.3.3 Sum of Normals

**Proposition 1.4.** Let \( X_i \) be independent Normal with parameters \((\mu_i, \sigma_i^2), i = 1, 2 \) respectively. Then \( X + Y \) has distribution Normal \((\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)\).

**Proof.** Since we have showed that \( X_i - \mu_i \) have Normal\((0, \sigma_i^2)\) distribution, WLOG we can assume \( \mu_1 = \mu_2 = 0 \). By a similar reasoning, we can assume \( \sigma_2 = 1 \).
Now since only form matters but not constants, we have

\[ f_X(z - y)f_Y(y) = Ke^{-\frac{(z-y)^2}{2\sigma^2}} e^{-\frac{y^2}{2}}. \]

Completing the square we have

\[
(z - y)^2 + \sigma^2 y^2 = (1 + \sigma^2) y^2 - 2zy + z^2
\]

\[
= \left( \sqrt{1 + \sigma^2} y - \frac{z}{\sqrt{1 + \sigma^2}} \right)^2 + \left( 1 - \frac{1}{1 + \sigma^2} \right) z^2
\]

\[
= (1 + \sigma^2) \left( y - \frac{z}{1 + \sigma^2} \right)^2 + \frac{\sigma^2}{1 + \sigma^2} z^2.
\]

Therefore,

\[
f_X(z - y) f_Y(y) = Ke^{-\frac{1}{2\sigma^2} \left( y - \frac{z}{1 + \sigma^2} \right)^2} e^{-\frac{z^2}{2(1+\sigma^2)}}.
\]

Integrating the above in \(y\) just leaves us with

\[ f_Z(z) = Ke^{-\frac{z^2}{2(1+\sigma^2)}}. \]

This is the form of a Normal\((0, 1 + \sigma^2)\). The proof is complete.

**Corollary 1.5.** Let \(X_1, X_2, \ldots, X_n\) be independent Normal \((\mu_i, \sigma_i^2)\). Then \(Y = X_1 + X_2 + \cdots + X_n\) has Normal \((\sum \mu_i, \sum \sigma_i^2)\) distribution.

### 1.3.4 The Chi-square distribution

**Definition 1.6.** Let \(Y = Z^2\), where \(Z\) has standard normal distribution. Then we say \(Y\) has the Chi-square distribution with 1 degree of freedom.

Let \(Y_1, Y_2, \ldots, Y_n\) be independent Chi-square distribution with one degree of freedom. Then we say \(Y = \sum_{i=1}^n Y_i\) has the Chi-square distribution with \(n\) degrees of freedom.

We have:

\[ P(Y \leq y) = P(-\sqrt{y} \leq Z \leq \sqrt{y}) = 2P(0 \leq Z \leq \sqrt{y}). \]

Therefore,

\[
f_Y(y) = \frac{1}{\sqrt{y}} f_Z(\sqrt{y}) = \frac{y^{1/2-1}e^{-y/2}}{\sqrt{2\pi}}.
\]

That is, a Chi-square RV with one degree of freedom is just a Gamma\((1/2, 1/2)\) RV. From our result for summing independent Gamma RVs, a Chi-square RV with \(n\) degrees of freedom is just a Gamma\((n/2, 1/2)\) RV.
2 Sums of discrete RVs

2.1 The formula
Let $X, Y$ be independent discrete RVs. Let $Z = X + Y$. Then

$$P(Z = k) = P(X + Y = k) = \sum_{i,j:i+j=k} P(X = i, Y = j)$$

$$= \sum_{i,j:i+j=k} P(X = i)P(Y = j)$$

$$= \sum_i P(X = i)P(Y = k - i)$$

$$= \sum_j P(X = k - j)P(Y = j).$$

2.2 Sum of Poissons
Proposition 2.1. Let $X_i$ be independent Poisson with parameters $\lambda_i$, $i = 1, 2$ respectively. Then $X + Y$ has distribution Poisson $\lambda_1 + \lambda_2$.

Proof. Let $Z = X + Y$. Note that $Z$ takes values $0, 1, 2 \cdots$. Then

$$P(Z = k) = \sum_i P(X = i)P(Y = k - i)$$

$$= \sum_i e^{-\lambda_1} \frac{\lambda_1^i}{i!} e^{-\lambda_2} \frac{\lambda_2^{k-i}}{k-i!}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{k!} \sum_{i=0}^k \binom{k}{i} \lambda_1^i \lambda_2^{k-i}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{k!} (\lambda_1 + \lambda_2)^k \sum_{i=0}^k \binom{k}{i} \left( \frac{\lambda_1}{\lambda_1 + \lambda_2} \right)^i \left( \frac{\lambda_2}{\lambda_1 + \lambda_2} \right)^{k-i}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{k!} (\lambda_1 + \lambda_2)^k.$$

Corollary 2.2. Let $X_1, X_2, \cdots, X_n$ be independent Poisson $\lambda_i$. Then $Y = X_1 + X_2 + \cdots + X_n$ has Poisson $\sum_i \lambda_i$ distribution.

2.3 Sum of Binomials
Proposition 2.3. Let $X_i$ be independent Binomial with parameters $\nu_i$, $i = 1, 2$ respectively and success probability $p$. Then $X + Y$ has distribution Binomial $(\nu_1 + \nu_2, p)$. 5
Proof. Let $Z = X + Y$. Note that $Z$ takes values $0, 1, 2, \cdots, n_1 + n_2$. Then

$$P(Z = k) = \sum_i P(X = i)P(Y = k - i)$$

$$= \sum_{i=0}^{k} \binom{n_1}{i}p^i(1-p)^{n_1-i}\binom{n_2}{k-i}p^{k-i}(1-p)^{n_2-(k-i)}$$

$$= p^k(1-p)^{n_1+n_2-k} \sum_{i=0}^{k} \binom{n_1}{i}\binom{n_2}{k-i}.$$ 

We claim that

$$\sum_{i=0}^{k} \binom{n_1}{i}\binom{n_2}{k-i} = \binom{n_1+n_2}{k},$$

via the following argument: to pick $k$ objects out of $n_1 + n_2$ total, we just have to pick $i$ objects out of the first $n_1$, $k - i$ objects out of $n_2$, where $i = 0, 1, \cdots, k$.

2.4 Sum of Negative Binomial

**Proposition 2.4.** Let $X_i$ be independent Negative Binomial with parameters $r_i, i = 1, 2$ respectively and success probability $p$. Then $X + Y$ has distribution Negative Binomial $(r_1 + r_2, p)$.

**Proof.** Let $Z = X + Y$. Note that $Z$ takes values $r_1 + r_2, r_1 + r_2 + 1, r_1 + r_2 + 2, \cdots$. Then for $k \geq r_1 + r_2$

$$P(Z = k) = \sum_i P(X = i)P(Y = k - i)$$

$$= \sum_{i=r_1}^{k-r_2} \binom{i-1}{r_1-1}p^{r_1}(1-p)^{i-r_1}\binom{k-i-1}{r_2-1}p^{r_2}(1-p)^{k-i-r_2}$$

$$= p^{r_1+r_2}(1-p)^{k-(r_1+r_2)} \sum_{i=r_1}^{k-r_2} \binom{i-1}{r_1-1}\binom{k-i-1}{r_2-1}$$

$$= p^{r_1+r_2}(1-p)^{k-(r_1+r_2)} \binom{k-1}{r_1+r_2},$$

where we explain the combinatorics identity as followed: $\binom{k-1}{r_1+r_2}$ is the number of ways we can write a sum of $r_1 + r_2$ positive integer summands, adding up to $k$. But that is equivalent to considering the number of ways we can write the first $r_1$ summands adding up to $i$ and the last $r_2$ summands adding up to $k - i$, where $i$ can be $r_1$ up to $k - r_2$ (since the least the last $r_2$ summand can sum up to be is $r_2$).