1 Text book problem

Page 133 Problem 1: a, b, c Problem 2 a, b, c

2 Additional problem

1. Verify that the following Ito’s integral is a martingale using Ito’s formula:
   a. $\int_0^t B_s^2 dB_s$.  
   Ans:  
   Apply Ito’s formula to $B_t^3$:  
   
   $$dB_t^3 = B_t^3 dB_t + 3B_t dt.$$  
   Therefore, $X_t := \int_0^t B_s^2 dB_s = \frac{1}{3}[B_t^3 - 3 \int_0^t B_s dr].$  
   We need to show for $s < t, E(X_t|B_s) = X_s.$ Now  
   $$E(B_t^3|B_s) = E((B_s + (B_t - B_s))^3|B_s)$$  
   $$= E(B_s^3 + 3B_s(B_t - B_s)^2 + 3B_s^2(B_t - B_s) + (B_t - B_s)^3|B_s)$$  
   $$= B_s^3 + 3B_s(t - s).$$  
   $$E(\int_0^t B_r dr|B_s) = E(\int_0^s B_r dr + \int_s^t B_r dr|B_s)$$  
   $$= \int_0^s B_r dr + \int_s^t E(B_r|B_s) dr$$  
   $$= \int_0^s B_r dr + \int_s^t B_s dr$$  
   $$= \int_0^s B_r dr + B_s(t - s).$$
Plug in we can see that indeed $E(X_t|B_s) = X_s$. b. $\int_0^t e^{B_r} dB_r$.

Ans: See class notes. c. $\int_0^t B_s^2 dB_r$.

Ans: Apply Ito’s formula to $B_t^4$:

$$dB_t^4 = 4B_t^3 dB_t + 6B_t^2 dt.$$ 

Therefore, $X_t := \int_0^t B_s^3 dB_r = \frac{1}{4}[B_t^4 - 6 \int_0^t B_r dB_r]$.

We need to show for $s < t$, $E(X_t|B_s) = X_s$. Now

$$E(B_t^4|B_s) = E((B_s + (B_t - B_s))^4|B_s)$$

$$= E(B_s^4 + 4B_s(B_t - B_s)^3 + 6B_s^2(B_t - B_s)^2 + 4B_s^3(B_t - B_s) + (B_t - B_s)^4|B_s)$$

$$= B_s^4 + 6B_s^2(t-s) + 3(t-s)^2.$$ 

$$E(\int_0^t B_r^2 dB_r|B_s) = E(\int_0^s B_r^2 dB_r + \int_s^t B_r^2 dB_r|B_s)$$

$$= \int_0^s B_r^2 dB_r + \int_s^t E(B_r^2|B_s) dB_r$$

$$= \int_0^s B_r^2 dB_r + \int_s^t [B_r^2 + (r-s)] dB_r$$

$$= \int_0^s B_r^2 dB_r + B_s^2(t-s) + \frac{(t-s)^2}{2}.$$ 

Again, plug in we can see that indeed $E(X_t|B_s) = X_s$.

2. Let $s \leq t$. Compute the following conditional expectations:

a. $E(B_t^3|B_s)$.

Ans: $B_s^3 + (t-s)$. Also see class notes. b. $E(e^{B_t}|B_s)$. Ans: $e^{B_s}e^{\frac{r-s}{2}}$. Also see class notes. c. $E(\int_0^t B_r dB_r|B_s)$.

Ans: $\int_0^s B_r dB_r$, since it is a martingale. Also see class notes.

3. Let $S_t$ be a geometric Brownian motion:

$$dS_t = rS_t dt + \sigma S_t dB_t$$

$$S_0 = 10.$$ 

Suppose $r = .05, \sigma = 0.2, T = 1$. Compute the numerical value of $E([(S_T)^2 - 10]^+)$. 

Ans: From class notes:

$$V_0 = e^{(r+\sigma^2)T} [S_0^2 N(\bar{d}_1) - e^{-rT} K N(\bar{d}_2)].$$ 

2
(There is a typo in the old note where we wrote \( V_0 = e^{rT+\sigma^2 S_0^2 N(\bar{d}_1)} - e^{-rT} K N(\bar{d}_2) \) without the \([\cdots]\).)

We have \( \bar{K} \approx 9.13, \bar{d}_1 = 6.08, \bar{d}_2 = 5.68. \) Plug in we get \( V_0 \approx 99.9. \)

4. Let \( S_t \) be a geometric Brownian motion:
\[
\begin{align*}
\frac{dS_t}{S_t} &= rS_t dt + \sigma S_t dB_t \\
S_0 &= 1000.
\end{align*}
\]

Suppose \( r = .05, \sigma = 0.2, T = 1, K = 1100. \)

Follow the procedure described in class, compute the Euro call on \( S_T \) with strike price \( K \) using:

a. The Black-Scholes formula.
We have \( d_1 \approx -.12, d_2 \approx -.32. \) So Black-Scholes formula gives \( V_0 \approx 60.4. \)

b. The Binomial approximation to the Black-Scholes model with 5 steps.
We follow the procedure in the classnote, except that now \( r \) is not 0. Everything is the same up to the line: The evolution equation for \( S_k \) becomes
\[
S_{k+1} = S_k(1 + r(t_{k+1} - t_k) + \sigma \sqrt{t_{k+1} - t_k} Y_k).
\]

Therefore we have \( X_k = 1 + r(t_{k+1} - t_k) + \sigma \sqrt{t_{k+1} - t_k} Y_k. \) Plug in:
\[
\begin{align*}
X_k &= 1.0994 \text{ with probability } \frac{1}{2} \\
&= 0.9206 \text{ with probability } \frac{1}{2}.
\end{align*}
\]

Note that the risk neutral still stays as \( \frac{1}{2} \) here. It is because if we use \( u = 1.0994, d = 0.9206, \) then
\[
p = \frac{1 + r \times 0.2 - d}{u - d} \approx 0.5,
\]
where \( 1 + r \times 0.2 \) can be thought of as approximating \( \exp(r \times 0.2). \) Now that we have all the necessary ingredients. The only three nodes that give positive pay off are: 5 ups: \( S_5 \approx 1606, 4 \text{ ups 1 down: } S_5 \approx 1345, 3 \text{ ups 2 downs: } S_5 \approx 1126. \) So the price of the Euro-Call using Binomial approximation in 5 steps is:
\[
e^{-0.05} \times (506 + 5 \times 245 + 10 \times 26) \times \frac{1}{2^5} \approx 59.16.
\]
c. The Binomial approximation to the Black-Scholes model with 10 steps. Similar to the above, except that the time step now is 0.1. So

\[
X_k = 1.0682 \text{ with probability } \frac{1}{2}
\]

\[
= 0.9418 \text{ with probability } \frac{1}{2}.
\]

The nodes that give positive pay off are: 10 ups: \( S_{10} = 1935 \), 9 ups 1 down: \( S_{10} = 1706 \), 8 ups 2 downs: \( S_{10} = 1504 \), 7 ups 3 downs: \( S_{10} = 1325 \), 6 ups 4 downs: \( S_{10} = 1168 \). So the price of the Euro-Call using Binomial approximation in 10 steps is (We need the Pascal Triangle line 10 here):

\[
e^{-0.05} \times \left( 835 + 10 \times 606 + 404 \times 45 + 225 \times 120 + 68 \times 210 \right) \frac{1}{2^{10}} \approx 61.63.
\]

Note: All answers in a, b, c are close. Answers in b and c are meant to be approximation of answer a. Answer in c is meant to be a better approximation than answer in b. Indeed you can check the error is slightly smaller, but not a significant improvement. This possibly means the convergence rate of this approximation scheme is very slow.