1 Textbook:

Section 5.4 (page 92) Problem 1 a - d
Ans: 1 a) 11.84 b) 1.95 c) 2.37 d) 12.30
Section 5.6 (page 97) 1,2,3.
1. See class note.
2.
\[ P(S_T > K) = P(S_0e^X > K) = P(X > \log\left(\frac{K}{S_0}\right)), \]

where \( X \) has distribution \( N((r - \frac{1}{2}\sigma^2)T, \sigma^2T) \).

Hence
\[ P(X > \log\left(\frac{K}{S_0}\right)) = P\left(Z > \frac{\log\left(\frac{K}{S_0}\right) - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) = N(d_2). \]

3.
\[
\begin{align*}
d_1 &= \frac{\log\left(\frac{S_0}{K}\right) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \\
    &= \frac{\log\left(\frac{F_0}{K}\right) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \\
    &= \frac{\log\left(\frac{F_0}{K}\right) - rT + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \\
    &= \frac{\log\left(\frac{F_0}{K}\right) + \frac{1}{2}\sigma^2T}{\sigma\sqrt{T}}.
\end{align*}
\]

Similarly for \( d_2 \).
Section 5.7 (page 103) 1,2.
1. I think the book’s answer is incorrect. We are looking at $X$ having distribution $\text{Bin}(60, 0.75)$ and we are asking $P(X \geq 40)$. Since $E(X) = 60(0.75) = 45$ and $\text{Var}(X) = 60(0.75)(0.25) = 11.25$ we have

$$P(X \geq 40) \approx P(Z \geq \frac{40 - 45}{\sqrt{11.25}})$$

$$\approx P(Z \geq -1.49) = .9319.$$

2. Let $n$ denote the number of up movements. Then

$$S_{100} = S_0 \left( e^{0.02} \right)^n \left( e^{-0.02} \right)^{100-n}$$

and we require $S_{100} \geq 1.8S_0$. In other words we require

$$(e^{0.02})^n e^{-2} \geq 1.8.$$

So $n \geq 64.6$. So we are looking for $P(\text{Bin}(100, 0.6) \geq 65)$. Using normal approximation as in problem 1 gives the answer as 0.17.

Section 6.1 (page 113) 1, 2, 3.

1. $\frac{\partial V}{\partial t} = rbe^{rt}; \frac{\partial V}{\partial S} = a; \frac{\partial^2 V}{\partial S^2} = 0.$

Plug in to the PDE gives

$$rbe^{rt} + arS + 0 - r(aS + be^{rt}) = 0.$$

2. $\frac{\partial V}{\partial t} = ae^{at}S^2; \frac{\partial V}{\partial S} = 2ae^{at}S; \frac{\partial^2 V}{\partial S^2} = 2ae^{at}.$

Plug in to the PDE gives

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S}rS + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} - rV = (a - r + \sigma^2 + 2r)S^2.$$

With the choice $a = -(\sigma^2 + r)$ it is clear that the RHS = 0.

3. $\frac{\partial V}{\partial t} = rV + e^{rt} \frac{\partial G}{\partial t}; \frac{\partial V}{\partial S} = e^{rt} \frac{\partial G}{\partial S}; \frac{\partial^2 V}{\partial S^2} = e^{rt} \frac{\partial^2 G}{\partial S^2}.$

Plug into the PDE gives

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S}rS + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} - rV = rV + e^{rt}(\frac{\partial G}{\partial t} + \frac{\partial G}{\partial S}rS + \frac{1}{2} \frac{\partial^2 G}{\partial S^2}) - rV = 0.$$
2 Additional problem

1. Verify the following given processes solve the corresponding stochastic differential equations:

a. $X_t = e^{B_t}$ solves

$$dX_t = \frac{1}{2} X_t dt + X_t dB_t.$$ 

$$dX_t = e^{B_t} dB_t + \frac{1}{2} e^{B_t} dt = X_t dB_t + \frac{1}{2} X_t dt.$$ 

b. $X_t = \frac{B_t}{1+t}$ solves

$$dX_t = - \frac{1}{1+t} X_t dt + \frac{1}{1+t} dB_t.$$ 

$$dX_t = \frac{-B_t}{(1+t)^2} dt + \frac{1}{1+t} dB_t = \frac{-X_t}{1+t} dt + \frac{1}{1+t} dB_t.$$ 

c. $X_t = \sin(B_t)$ solves

$$dX_t = - \frac{1}{2} X_t dt + \sqrt{1-X_t^2} dB_t.$$ 

$$dX_t = \cos(B_t) dB_t - \frac{1}{2} \sin(B_t) dt = \sqrt{1-\sin^2(B_t)} dB_t - \frac{1}{2} \sin(B_t) dt = \sqrt{1-X_t^2} dB_t - \frac{1}{2} X_t dt.$$