1. Compute $\phi_X(t) := E(\exp(itX))$, where $\exp(x) := e^x$, $i$ is the imaginary number: $i^2 = -1$ and $X$ has $N(\mu, \sigma^2)$ distribution. Recall that the density of $N(\mu, \sigma^2)$ is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right).$$

$\phi(t)$ is called the characteristic function of $X$.

2. a) Suppose $\mu = 0$. Compute $\phi^{(4)}_X(t)$: the 4th derivative of $\phi_X$ with respect to $t$.
   b) Use the following fact:

$$E[X^k] = (-i)^k \phi_X^{(k)}(0)$$

   to compute $E[(B_t)^4]$ where $B$ is a Brownian motion.

3. Let $0 \leq s \leq t \leq T$ and $B$ a Brownian motion. Compute the followings:
   a) $E(B_s^2 B_t)$
   b) $E(B_s^2 B_s)$
   c) $E(\exp(\sigma B_t - \frac{1}{2}\sigma^2 t))$, where $\sigma$ is a constant.
   d) $E(\exp(\int_0^t \sin(s)dB_s))$.

4. Use Ito formula to compute the following
   a) $d\sin(B_t)$
   b) $d\exp(B_t)$

5. Let $0 = t_0 < t_1 < t_2 < ... < t_n = T$. Show that

$$E\left[\left(\sum_{i=0}^{n-1} (B_{t_{i+1}} - B_{t_i})^2 - T\right)^2\right] = 2 \sum_{i=0}^{n-1} (t_{i+1} - t_i)^2.$$
6. Suppose $S_t$ satisfies

$$dS_t = \sin(S_t)t^2 \, dt + \exp(\sqrt{S_t} - t) \, dB_t.$$  

Compute

a) $d\log(S_t)$  

b) $d\exp(S_t^2)$  

c) $d\sqrt{S_t}$  

Please note that this exercise is only for you to symbolically practice Ito’s formula. There may not exist a process $S_t$ that satisfies the above dynamics, or even if it does exist, to make sense in the expression a,b and c.

7. (Extra credit - 5 pts) a) Look up precisely in what ”weak sense” the convergence in the definition of Ito integral is. Explain in a few words your understanding of that mode of convergence.  

b) Find 3 properties of Brownian motion we have not discussed in the lecture that you find interesting.