1. Compute $\phi_X(t) := E\left(\exp(itX)\right)$, where $\exp(x) := e^x$, $i$ is the imaginary number: $i^2 = -1$ and $X$ has $N(\mu, \sigma^2)$ distribution. Recall that the density of $N(\mu, \sigma^2)$ is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right).$$

$\phi(t)$ is called the characteristic function of $X$.

Ans:

We have showed in class that for $X$ having distribution $N(0, \sigma^2)$, $E(e^X) = e^{1/2\sigma^2}$.

Note that

$$e^{itX} = e^{it(X-\mu)+it\mu} = e^{it\mu} e^{it(X-\mu)},$$

where $it(X - \mu)$ has "distribution" $N(0, i^2t^2\sigma^2)$ (The $i^2$ is treated purely as a symbol here, nevertheless the calculation is still correct).

Apply the above result, we have

$$e^{itX} = e^{it\mu + \frac{1}{2}i^2t^2\sigma^2} = e^{it\mu - \frac{1}{2}t^2\sigma^2}.$$ 

2. Suppose $\mu = 0$. Compute $\phi^{(4)}_X(t)$: the 4th derivative of $\phi_X$ with respect to $t$.

Ans: Use some online differentiating solver (Wolfram Alpha comes to mind) we get

$$\phi^{(4)}_X(t) = \sigma^4 e^{-\frac{1}{2}t^2\sigma^2} (\sigma^4 t^4 - 6\sigma^2 t^2 + 3).$$

b) Use the following fact:

$$E[X^k] = (-i)^k \phi^{(k)}_X(0)$$

to compute $E[(B_t)^4]$ where $B$ is a Brownian motion.
Ans: Recall that $B_t$ has distribution $N(0, t)$, replacing $t = 0$ and $\sigma^2 = t$ into the expression of part a, we get $E[(B_t)^4] = 3t^2$.

3. Let $0 \leq s \leq t \leq T$ and $B$ a Brownian motion. Compute the followings:

a) $E(B_s^2 B_t)$

Ans:

$$E(B_s^2 B_t) = E(B_s^2(B_s + B_t - B_s)) = E(B_s^3) + E(B_s^2(B_t - B_s))$$

$$= E(B_s^3) + E(B_s^2)E(B_t - B_s) = 0 + s \times 0 = 0.$$

b) $E(B_s^2 B_s)$

Ans:

$$E(B_s B_s^2) = E(B_s(B_s + B_t - B_s)^2) = E(B_s(B_s^2 + 2B_s(B_t - B_s) + (B_t - B_s)^2))$$

$$= E(B_s^3) + E(2B_s^2(B_t - B_s)) + E(B_s(B_t - B_s)^2)$$

$$= E(B_s^3) + E(2B_s^2)E(B_t - B_s) + E(B_s)E((B_t - B_s)^2)$$

$$= 0 + s \times 0 + 0(t - s) = 0.$$

c) $E(\exp(\sigma B_t - \frac{1}{2}\sigma^2 t))$, where $\sigma$ is a constant.

Ans: Since $\sigma B_t$ has distribution $N(0, t^2)$, $E(\exp(\sigma B_t)) = e^{\frac{1}{2}\sigma^2 t}$. Thus $E(\exp(\sigma B_t - \frac{1}{2}\sigma^2 t)) = 1$.

d) $E(\exp(\int_0^t \sin(s)dB_s))$.

Ans: Recall that $\int_0^t \sin(s)dB_s$ has distribution $N(0, \int_0^t \sin^2(s)ds)$ since $\sin(s)$ is deterministic. Therefore

$$E(\exp(\int_0^t \sin(s)dB_s)) = \exp(\frac{1}{2} \int_0^t \sin^2(s)ds)$$

4. Use Ito formula to compute the following

a) $d\sin(B_t)$

$$d\sin(B_t) = \cos(B_t)dB_t - \frac{1}{2} \sin(B_t)dt.$$

b) $d\exp(B_t)$

$$d\exp(B_t) = \exp(B_t)dB_t + \frac{1}{2} \exp(B_t)dt.$$
5. Let $0 = t_0 < t_1 < t_2 < \ldots < t_n = T$. Show that

$$E\left[\left(\sum_{i=0}^{n-1} (B_{t_{i+1}} - B_{t_i})^2 - T\right)^2\right] = 2\sum_{i=0}^{n-1} (t_{i+1} - t_i)^2.$$ 

Ans: Note that

$$\left[\sum_{i=0}^{n-1} (B_{t_{i+1}} - B_{t_i})^2 - T\right]^2 = \left[\sum_{i=0}^{n-1} (B_{t_{i+1}} - B_{t_i})^2\right]^2 - 2\left(\sum_{i=0}^{n-1} (B_{t_{i+1}} - B_{t_i})^2\right)T + T^2.$$ 

And

$$E(2\sum_{i=0}^{n-1} (B_{t_{i+1}} - B_{t_i})^2)T = 2T\sum_{i=0}^{n-1} (t_{i+1} - t_i) = 2T^2.$$ 

Hence

$$E\left[\left(\sum_{i=0}^{n-1} (B_{t_{i+1}} - B_{t_i})^2 - T\right)^2\right] = -T^2.$$ 

Now

$$\left[\sum_{i=0}^{n-1} (B_{t_{i+1}} - B_{t_i})^2\right]^2 = \left[\sum_{i=0}^{n-1} (B_{t_{i+1}} - B_{t_i})^2\right]\left[\sum_{j=0}^{n-1} (B_{t_{j+1}} - B_{t_j})^2\right]$$

$$= \sum_{i=0}^{n-1} (B_{t_{i+1}} - B_{t_i})^4 + \sum_{i \neq j, i,j=0}^{n-1} (B_{t_{i+1}} - B_{t_i})(B_{t_{j+1}} - B_{t_j}).$$

Apply Problem 2, noting that $B_{t_{i+1}} - B_{t_i}$ has distribution $N(0, t_{i+1} - t_i)$ we have

$$E\left[\sum_{i=0}^{n-1} (B_{t_{i+1}} - B_{t_i})^4\right] = 3\sum_{i=0}^{n-1} (t_{i+1} - t_i)^2.$$ 

Also if $i \neq j$, $B_{t_{i+1}} - B_{t_i}$ and $B_{t_{j+1}} - B_{t_j}$ are independent. So

$$E\left[\sum_{i \neq j, i,j=0}^{n-1} (B_{t_{i+1}} - B_{t_i})(B_{t_{j+1}} - B_{t_j})\right] = \sum_{i \neq j, i,j=0}^{n-1} (t_{i+1} - t_i)(t_{j+1} - t_j).$$

The last thing to do is to simplify $\sum_{i \neq j, i,j=0}^{n-1} (t_{i+1} - t_i)(t_{j+1} - t_j)$.

We have

$$\sum_{i \neq j, i,j=0}^{n-1} (t_{i+1} - t_i)(t_{j+1} - t_j) = \sum_{i=0}^{n-1} \sum_{j=0, j \neq i}^{n-1} (t_{i+1} - t_i)(t_{j+1} - t_j)$$

$$= \sum_{i=0}^{n-1} \left[(t_{i+1} - t_i)(\sum_{j=0, j \neq i}^{n-1} (t_{j+1} - t_j))\right] = \sum_{i=0}^{n-1} \left[(t_{i+1} - t_i)(T - (t_{i+1} - t_i))\right]$$

$$= T \sum_{i=0}^{n-1} (t_{i+1} - t_i) - \sum_{i=0}^{n-1} (t_{i+1} - t_i)^2 = T^2 - \sum_{i=0}^{n-1} (t_{i+1} - t_i)^2.$$
Putting all these results together, we get the desired calculation.

6. Suppose $S_t$ satisfies

$$dS_t = \sin(S_t)dt + \exp(\sqrt{St} - t)dB_t.$$

Compute

a) $d\log(S_t)$

$$d\log(S_t) = \frac{1}{S_t}dS_t - \frac{1}{2S_t^2}[\exp(\sqrt{St} - t)]^2dt$$

$$= \frac{1}{S_t}[\sin(S_t)dt + \exp(\sqrt{St} - t)dB_t] - \frac{1}{2S_t^2}[\exp(\sqrt{St} - t)]^2dt$$

$$= \frac{1}{S_t} \exp(\sqrt{St} - t)dB_t$$

$$+ \left[\frac{1}{S_t}(\sin(S_t)dt) - \frac{1}{2S_t^2}[\exp(2\sqrt{St} - 2t)]\right]dt.$$

b) $d\exp(S_t^2)$

$$d\exp(S_t^2) = 2S_t \exp(S_t^2)dS_t + \frac{1}{2}(2 \exp(S_t^2) + 4S_t^2 \exp(S_t^2))[\exp(\sqrt{St} - t)]^2dt$$

$$= 2S_t \exp(S_t^2)[\sin(S_t)dt + \exp(\sqrt{St} - t)dB_t]$$

$$+ \frac{1}{2}(2 \exp(S_t^2) + 4S_t^2 \exp(S_t^2))[\exp(\sqrt{St} - t)]^2dt$$

$$= \ldots$$

c) $d\sqrt{St}$

$$d\sqrt{St} = \frac{1}{2\sqrt{St}}dS_t - \frac{1}{4S_t^3}[\exp(\sqrt{St} - t)]^2dt$$

$$= \frac{1}{2\sqrt{St}}[\sin(S_t)dt + \exp(\sqrt{St} - t)dB_t]$$

$$- \frac{1}{4S_t^3}[\exp(\sqrt{St} - t)]^2dt$$

$$= \ldots$$

Please note that this exercise is only for you to symbolically practice Ito’s formula. There may not exist a process $S_t$ that satisfies the above dynamics, or even if it does exist, to make sense in the expression a,b and c.
7. (Extra credit - 5 pts) a) Look up precisely in what "weak sense" the convergence in the definition of Ito integral is. Explain in a few words your understanding of that mode of convergence.

b) Find 3 properties of Brownian motion we have not discussed in the lecture that you find interesting.