1 Textbook (Stampfli and Goodman)

A. Section 3.3, Page 54: 1,2,3,4.
Ans:
1. a) 14.79 b) 7.40 c) 12.26 d) 11.54 e) 9.04 f) 42.02 g) 27.84
2. a) 0.75 b) 16.09 c) 11.25 d) 1.45 e) 1.79 f) 20.91 g) 3.33
3. a) 1.16 b) 19.90 c) 15.59 d) 1.67 e) 2.96 f) 26.11
4. $V = \frac{2}{1.06}(K - 90)$ if the option was held to expiration. We solve for

$$ (K - 100)^+ \geq \frac{2}{1.06}(K - 90) $$

to obtain $K \geq 102.33$.

B. Section 3.4 Page 58: 1,2,3.
Ans:
1) 23.14 2) 28.59 3) 5.29

C. Section 3.5 Page 61: 1,2,3.
1) (Using $r = 0$) 109.33 2) (Using $r = 0$) 109.33 3) 58.64

D. Repeat Page 61: 1,2,3 for Asian option.
1) 100 2) 100 3) 50

2 Additional problems

1. (Extra credit - 5 pts) Show that in a Binomial model, the discounted value process $e^{-rk}V_k^\text{put}$ for $0 \leq k \leq n$ of an American put option is a super-martingale under the risk neutral probability.
Ans:
By the risk neutral pricing formula,

\[ V_k = \max(E^Q(e^{-rT}V_{k+1}|\mathcal{F}_k^S), (S_k - K)^-) \geq E^Q(e^{-rT}V_{k+1}|\mathcal{F}_k^S). \]

Therefore,

\[ e^{-rT}V_k \geq E^Q(e^{-rT(k+1)}V_{k+1}|\mathcal{F}_k^S). \]

Thus, the discounted value process is a super-martingale.

2. Consider a Binomial model with \( S_0 = 81, u = \frac{4}{3}, d = \frac{2}{3}, r = 0, n = 4. \) Compute the replicating portfolio for a European call option with strike price \( k = 30, \) expiration time \( n = 4. \) (You can either draw a tree and list the portfolio composition at each node, or you can give the answer in the form \( \Delta_2(ud) = x, b_2(ud) = y, \) for all possibilities of events at each time \( k \)).

Ans: It is enough to give the number of shares and the option value at each node, since the amount of money at the same node can be found by \( b = V - \Delta S. \)

\[
\begin{align*}
\Delta_3(uuu) &= 1, V_3(uuu) = 162 \\
\Delta_3(uud) &= 1, V_3(uud) = 66 \\
\Delta_3(udd) &= 1, V_3(udd) = 18 \\
\Delta_3(ddd) &= \frac{1}{8}, V_3(ddd) = 1 \\
\Delta_2(uu) &= 1, V_2(uu) = 114 \\
\Delta_2(ud) &= 1, V_2(ud) = 42 \\
\Delta_2(dd) &= .708, V_2(dd) = 9.5 \\
\Delta_1(u) &= 1, V_1(u) = 78 \\
\Delta_1(d) &= .90277, V_1(d) = 25.75.
\end{align*}
\]

3. Let \( X_i \) be i.i.d. with distribution

\[
\begin{align*}
X_i &= 1 \text{ with probability } \frac{1}{2} \\
&= -1 \text{ with probability } \frac{1}{2}
\end{align*}
\]

Classify whether the following is a martingale, sub-martingale, super-martingale with respect to the filtration \( \{\mathcal{F}_i^S\} \), or neither. Justify your answer.

a) \( S_k^1 \), where \( S_k^1 = \sum_{i=0}^{k} X_i. \)

\[
E(S_{k+1}^1|\mathcal{F}_k^S) = S_k + E(X_{k+1}|\mathcal{F}_k^S) = S_k + E(X_{k+1}) = S_k.
\]
It is a martingale.

b) \((S_k^1)^2\).

\[
E((S_{k+1}^1)^2 | \mathcal{F}_k^S) = (S_k^1)^2 + E(2S_k^1 X_{k+1} + (X_{k+1})^2 | \mathcal{F}_k^S) = (S_k^1)^2 + 2S_k E(X_{k+1}) + E((X_{k+1})^2) = (S_k^1)^2 + E((X_{k+1})^2) > S_k^1.
\]

It is a sub-martingale.

c) \(S_k^2\), where \(S_k^2 = \sum_{i=0}^{k} \sin(X_i)\).

\[
E(S_{k+1}^2 | \mathcal{F}_k^S) = S_k + E(\sin(X_{k+1}) | \mathcal{F}_k^S) = S_k^2 + E(\sin(X_{k+1})) = S_k^2.
\]

It is a martingale.

d) \(S_k^3\), where \(S_k^3 = \sum_{i=0}^{k} \cos(X_i)\).

\[
E(S_{k+1}^3 | \mathcal{F}_k^S) = S_k + E(\cos(X_{k+1}) | \mathcal{F}_k^S) = S_k^2 + E(\cos(X_{k+1})) > S_k^3.
\]

It is a sub-martingale.

e) \(S_k^4\), where \(S_k^4 = \sum_{i=0}^{k} (-1)^i X_i\).

\[
E(S_{k+1}^4 | \mathcal{F}_k^S) = S_k^4 + E((-1)^i X_{k+1} | \mathcal{F}_k^S) = S_k^4 + E((-1)^i X_{k+1}) = S_k^4.
\]

It is a martingale.