1 Textbook (Stampfli and Goodman)

A. Section 3.1, Page 49: 1,2,3.
B. Section 3.2 Page 51: 1,2,3,4.

2 Additional problems

In all of these problems, $Q$ will denote the risk neutral probability measure. We also take the length of one period, $\tau$ to be 1 and the risk free interest rate to be a constant $r$.

We consider the following formulation for the binominal model:

$$S_k = S_0 X_1 X_2 \ldots X_k, 1 \leq k \leq n$$

where for $0 < d \leq e^{r\tau} \leq u$, $X_i$ are i.i.d. with distribution

$$X_i = u \text{ with probability } q$$
$$X_i = d \text{ with probability } 1 - q.$$

1. (Extra credit - 5 pts) Suppose $r = 0$. Compute

$$E^Q((S_5 - S_3)^+ | S_3).$$

Interpretation: This is the price for a European call option entered at time $k = 3$ with strike price $S_3$ and expiration time $n = 5$. Note that $S_3$ is known at time $k = 3$ so this call option makes sense.
2. (Extra credit - 3 pts) Show that the conditional expectation $E(X|\mathcal{F}_k^S)$ is the best guess of $X$ given $S_0, S_1, S_2, ..., S_k$ in the following sense

$$E[(X - E(X|\mathcal{F}_k^S))^2] \leq E[(X - g(S_0, S_1, S_2, ..., S_k))^2], \text{ for all } g.$$ 

3. Show that in the binomial model, we always have

$$E^Q(e^{-r(j-i)}S_j|S_i) = S_i.$$ 

4. Consider the binomial model with $n = 10$ and a forward contract on $S$ entered at some time $k, 0 \leq k \leq 9$, strike price $K$ and expiration time $n = 10$.

   a) In your own words, explain what $F(6, 10)$ means.
   b) Find $F(6, 10)$ using the replicating portfolio approach.
   c) Compute $E^Q(e^{-r(S_{10} - K)}|S_6)$.
   d) Find $K$ such that $E^Q(e^{-r(S_{10} - K)}|S_6) = 0$. Compare your answer with the answer in part b.
   e) Let $V$ be the value of a forward contract entered at time 0, with strike price $F(0, 10)$ (so that $V_0 = 0$). Compute $V_6$.
   f) Compute $E^Q(e^{-r(S_{10} - F(0, 10))|S_6})$ (Remember $F(0, 10)$ is a known constant). Compare your answer with the answer in part e.
   g) Compute $E^Q(e^{-rV_6})$. (You should get 0 for the answer here. This is an instance of the rule $E((E(X|Y)) = E(X)$.

5. The Put-Call parity principle says: Holding a long position on a European Call Option and a short position on a European Put Option is the same as holding a long position on a Forward Contract (on the same stock $S$, with the same expiration date $n$ and strike price $K$). Suppose $S$ follows the multi-period Binomial model.

   a) Express the Put-Call parity principle in terms of $V_{\text{put}}, V_{\text{call}}$ and $V_{\text{forward}}$.
   b) Prove the Put-Call parity principle.

Answer: