1 Textbook (Stampfli and Goodman)

A. Section 3.1, Page 49: 1, 2, 3.
Answer:
1. $2.66. The two Tuesday nodes are $2.18 and $3.63.
2. $pu + (1 − p)d = 1.04$. A formula on page 48 of the textbook states that

\[ S_0(pu + (1 − p)d)^2 = E(S_2) = 27.15 \]

So that $S_0 = 25.1$. The values for $S_1$ are $uS_0 = 30.12$ and $dS_0 = 20.08$.
3. Since $pu + (1 − p)d = 1$, $E(S_{k+1}|S_k = x) = x$.

B. Section 3.2 Page 51: 1, 2, 3, 4.
Answer:
1. $39.90$
2. $q = \frac{e^{0.06-0.8}}{1.7-0.8} = 0.2909$. The expiration values are 0, 137, 450. $V = \$29.92$.
3. $8.81$.

2 Additional problems

In all of these problems, $Q$ will denote the risk neutral probability measure. We also take the length of one period, $\tau$ to be 1 and the risk free interest rate to be a constant $r$.

We consider the following formulation for the binomial model:

\[ S_k = S_0X_1X_2...X_k, 1 \leq k \leq n \]
where for $0 < d \leq e^{rt} \leq u$, $X_i$ are i.i.d. with distribution

\[ X_i = u \text{ with probability } q \]
\[ X_i = d \text{ with probability } 1 - q. \]

1. (Extra credit - 5 pts) Suppose $r = 0$. Compute

\[ E^Q((S_5 - S_3)^+|S_3). \]

Interpretation: This is the price for a European call option entered at time $k = 3$ with strike price $S_3$ and expiration time $n = 5$. Note that $S_3$ is known at time $k = 3$ so this call option makes sense.

Answer:

\[ E^Q((S_5 - S_3)^+|S_3) = E^Q(S_3X_4X_5 - S_3)^+|S_3) \]
\[ = E^Q(S_3(X_4X_5 - 1)^+|S_3). \]

There are two cases:

Case 1: $ud > 1$. Then $(X_4X_5 - 1)^+ = 0$ on $(\omega_0, \omega_1, \omega_2, \omega_3, d, d)$ and $(X_4X_5 - 1)^+ = X_4X_5 - 1$ otherwise.

Hence

\[ E^Q((S_5 - S_3)^+|S_3) = S_3(u^2 - 1)q^2 + 2S_3(ud - 1)q(1 - q). \]

Case 2: $ud \leq 1$. Then $X_4X_5 - 1 > 0$ only on $(\omega_0, \omega_1, \omega_2, \omega_3, u, u)$. Hence

\[ E^Q((S_5 - S_3)^+|S_3) = S_3(u^2 - 1)q^2. \]

2. (Extra credit - 3 pts) Show that the conditional expectation $E(X|F^S_k)$ is the best guess of $X$ given $S_0, S_1, S_2, ..., S_k$ in the following sense

\[ E[(X - E(X|F^S_k))^2] \leq E[(X - g(S_0, S_1, S_2, ..., S_k))^2], \text{ for all } g. \]

Answer: First, similar to Hmwk 2 Problem 1, we have

\[ E[E(X|F^S_k)X] = E[E(X|F^S_k)^2]. \]
Reason:

\[ E \left[ E(X \mid \mathcal{F}_k^S) X \right] = E \left[ E \left( E(X \mid \mathcal{F}_k^S) X \mid \mathcal{F}_k^S \right) \right] \]

\[ = E \left[ E(X \mid \mathcal{F}_k^S) E \left( X \mid \mathcal{F}_k^S \right) \right] (\text{Since } E(X \mid \mathcal{F}_k^S) \text{ is a function of } S_0, S_1, \ldots, S_k) \]

\[ = E \left[ E(X \mid \mathcal{F}_k^S)^2 \right]. \]

Use this result and follow exactly the same line as part b of Hmwk 2 Problem 1, we have

The LHS of the inequality is equal to

\[ E \left[ (X - E(X \mid \mathcal{F}_k^S))^2 \right] = E \left[ X^2 - 2XE(X \mid \mathcal{F}_k^S) + E(X \mid \mathcal{F}_k^S)^2 \right] \]

\[ = E(X^2) - 2E \left[ E(X \mid \mathcal{F}_k^S)^2 \right] + E \left[ E(X \mid \mathcal{F}_k^S)^2 \right] \text{ (by part a)} \]

\[ = E(X^2) - E \left[ E(X \mid \mathcal{F}_k^S)^2 \right]. \]

The RHS of the inequality is equal to

\[ E \left[ (X - g(S_0, S_1, \ldots, S_k))^2 \right] = E \left[ X^2 - 2Xg(S_0, S_1, \ldots, S_k) + g(S_0, S_1, \ldots, S_k)^2 \right] \]

\[ + E \left[ g(S_0, S_1, \ldots, S_k)^2 \right] \]

Thus we only need to show

\[ E \left[ E(X \mid \mathcal{F}_k^S)^2 \right] - 2E \left[ E(X \mid \mathcal{F}_k^S)g(S_0, S_1, \ldots, S_k) \right] + E \left[ g(S_0, S_1, \ldots, S_k)^2 \right] \geq 0, \]

but this is obvious.

3. Show that in the binomial model, we always have

\[ E^Q(e^{-r(j-i)}S_j \mid S_i) = S_i. \]

Answer:

\[ E^Q(e^{-r(j-i)}S_j \mid S_i) = E^Q(e^{-r(j-i)}S_iX_{i+1}X_{i+2} \ldots X_j \mid S_i) \]

\[ = S_iE^Q(e^{-r}X_{i+1}e^{-r}X_{i+2} \ldots e^{-r}X_k) \]

\[ = S_i \left[ E^Q(e^{-r}X_1)^k \right]. \]

Thus we only need to show \( E^Q(e^{-r}X_1) = 1 \). But it is obvious since

\[ e^{-r}(uq + d(1 - q)) = e^{-r} \left( u \frac{e^r - d}{u - d} + d \frac{u - e^r}{u - d} \right) = 1. \]
4. Consider the binomial model with \( n = 10 \) and a forward contract on \( S \) entered at some time \( k, 0 \leq k \leq 9 \), strike price \( K \) and expiration time \( n = 10 \).
   a) In your own words, explain what \( F(6, 10) \) means.
   It is the forward price for a forward contract entered at time \( k = 6 \).
   b) Find \( F(6, 10) \) using the replicating portfolio approach.
   It is \( S_6 e^{4r} \).
   c) Compute \( E^Q(e^{-4r}(S_{10} - K)|S_6) \).
   It is \( S_6 - e^{-4r}K \).
   d) Find \( K \) such that \( E^Q(e^{-4r}(S_{10} - K)|S_6) = 0 \). Compare your answer with the answer in part b.
   \( K = S_6 e^{4r} \).
   e) Let \( V \) be the value of a forward contract entered at time 0, with strike price \( F(0, 10) \) (so that \( V_0 = 0 \)). Compute \( V_6 \).
   By the arguments we did in class before, \( V_6 = S_6 - e^{-4r}F(0, 10) \).
   f) Compute \( E^Q(e^{-4r}(S_{10} - F(0, 10))|S_6) \) (Remember \( F(0, 10) \) is a known constant). Compare your answer with the answer in part e.
   \[
   E^Q(e^{-4r}(S_{10} - F(0, 10))|S_6) = S_6 - e^{-4r}F(0, 10).
   \]
   g) Compute \( E^Q(e^{-6r}V_6) \). (You should get 0 for the answer here. This is an instance of the rule \( E((E(X|Y)) = E(X) \).
   \[
   E^Q(e^{-6r}V_6) = E^Q(e^{-6r}(S_6 - e^{-4r}F(0, 10)))
   = E^Q(e^{-6r}S_6 - e^{-10r}F(0, 10)).
   \]
   Note that \( E^Q(e^{-6r}S_6) = S_0 \), by property of risk neutral measure, while \( E^Q(e^{-10r}F(0, 10)) = S_0 \), as well by definition of \( F(0, 10) \).

5. The Put-Call parity principle says: Holding a long position on a European Call Option and a short position on a European Put Option is the same as holding a long position on a Forward Contract (on the same stock \( S \), with the same expiration date \( n \) and strike price \( K \)). Suppose \( S \) follows the multi-period Binomial model.
   a) Express the Put-Call parity principle in terms of \( V^\text{put}, V^\text{call} \) and \( V^\text{forward} \).
b) Prove the Put-Call parity principle.

Answer:

a) 
\[ V_k^{\text{call}} - V_k^{\text{put}} = V_k^{\text{forward}}, \text{ for all } 0 \leq k \leq n. \]

b) By the risk neutral pricing formula:

\[ V_k^{\text{call}} = \mathbb{E}^Q((S_n - K)^+|S_k); \]
\[ V_k^{\text{put}} = \mathbb{E}^Q((S_n - K)^-|S_k); \]
\[ V_k^{\text{forward}} = \mathbb{E}^Q(S_n - K|S_k). \]

But it's easy to check that \((S_n - K)^+ - (S_n - K)^- = S_n - K\). Indeed

\[ \max(x, 0) - \max(-x, 0) = x \]

since if \(x \geq 0\) then \(\max(x, 0) = x\) and if \(x < 0\) then \(\max(-x, 0) = -x\). Using linearity of conditional expectation, the result follows.