3. Bayes’ rule in probability states that

\[ P(A|B) = \frac{P(B|A)P(A)}{P(B)} \]

a. Prove Bayes’ rule using the definition of conditional probability.

b. The chance of contracting a rare disease \( X \) is .97. However, the diagnosis for \( X \) is not very reliable. A person who contracted \( X \) has probability .9 of being diagnosed positive. A person who does not contract \( X \) has probability of .05 of being diagnosed positive. Using Bayes’ rule, find the probability that a person who was diagnosed positive did not contract \( X \).

Answer:

a.

\[ \frac{P(B|A)P(A)}{P(B)} = \frac{P(A \cap B)}{P(B)} = P(A|B). \]

b. Let \( A \) be the event that the person contracted \( X \) and \( B \) be the event that the person is diagnosed positive. Then \( P(B|A) = 0.9 \) and \( P(B|A^c) = 0.05 \). We want to find \( P(A^c|B) \). By Bayes’ rule:

\[ P(A^c|B) = \frac{P(B|A^c)P(A^c)}{P(B)}. \]

Note that

\[ P(B) = P(B|A)P(A) + P(B|A^c)P(A^c) = (0.9)(0.03) + (0.05)(0.97) = 0.0755. \]

Plug this in gives \( P(A^c|B) = 0.6423 \). Thus even for a rare disease with a non-precise diagnosis, the chance of a false positive is fairly high.
6. Frequent fliers of a certain airline fly a random number of miles each year, having a mean of 25,000 and standard deviation of 12,000 miles. If 30 such people are randomly chosen, approximate the probability that their mileages for this year will

a. exceed 25,000.
b. be between 23,000 and 27,000.

Answer:
Let $\bar{X}_n$ be the average mileage. Then

a. Set $\mu = 25,000, \sigma = 12,000, n = 30.$

$$P(\bar{X}_n \geq 25,000) = P\left(\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \geq \frac{\sqrt{n}(25000 - \mu)}{\sigma}\right) \approx P(Z \geq 0) = \frac{1}{2}.$$  

b.

$$P(23,000 \leq \bar{X}_n \leq 25,000) = P\left(\frac{\sqrt{n}(23000 - \mu)}{\sigma} \leq \frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \leq \frac{\sqrt{n}(27000 - \mu)}{\sigma}\right)$$

$$\approx P(-.91 \leq Z \leq .91) \approx .6372.$$  

7. An urn has 4 red balls and 5 blue balls. Three balls are drawn without replacement.

a. What is the probability that all of them are blue?

Answer:

$$P(3 \text{ blue}) = \frac{\binom{5}{3}}{\binom{9}{3}} = \frac{5}{42}.$$  

You can do this problem via conditioning by $P(3 \text{ blue}) = P(3 \text{ blue} | \text{ first 2 blue}) \times P(\text{first 2 blue}).$ $P(\text{first 2 blue})$ can be computed similarly to what we have done in class. $P(3 \text{ blue} | \text{ first 2 blue}) = \frac{4}{7}.$

8. Suppose we have a coin with probability $p$ of showing up Head. Let $X$ be the number of coin toss until we see the first Head. Let $Y$ be the result of the first toss, i.e. $Y = 1$ if the first toss shows head, 0 otherwise. Using the definitions given in class: $P(X = i | Y = j) = \frac{P(x=i,y=j)}{P(Y=j)}$ etc., find the following:

a. $P(X = k | Y = 1)$ (What are the possible values of $k$ here?)
b. $P(X = k | Y = 0)$ (What are the possible values of $k$ here?)
c. $E(X | Y = 1).$
d. $E(X|Y = 0)$ - Here you can use the fact that the series $p + 2(1 - p)p + 3(1 - p)^2p + 4(1 - p)^3p + ...$ is equal to $\frac{1}{p}$.

Answer:

a. 

$$P(X = k|Y = 1) = \frac{P(X = k, Y = 1)}{P(Y = 1)} = \frac{P(X = 1, Y = 1)}{P(Y = 1)}, k = 1$$

$$= 0 \text{ otherwise.}$$

(Note that the event $\{X = k, Y = 1\}$ is empty for $k \geq 2$). Also the event $\{X = 1, Y = 1\}$ is just $\{Y = 1\}$. Hence

$$P(X = k|Y = 1) = 1, k = 1$$

$$P(X = k|Y = 1) = 0 \text{ otherwise.}$$

b. 

$$P(X = k|Y = 0) = \frac{P(X = k, Y = 0)}{P(Y = 0)} = \frac{(1 - p)^{k-1}p}{1 - p} = (1 - p)^{k-2}p, k \geq 2.$$ 

Note: The event $\{X = 1, Y = 0\}$ is empty if the first trial is a failure: the number of trial until the first success cannot be 1. The event $\{X = k, Y = 0\}$ for $k \geq 2$ is simply $\{X = k\}$.

c. $E(X|Y = 1) = \sum_{k=1}^{\infty} kP(X = k|Y = 1) = 1.$

d. 

$$E(X|Y = 0) = \sum_{k=2}^{\infty} kP(X = k|Y = 0) = \sum_{k=2}^{\infty} k(1 - p)^{k-2}p$$

$$= 2p + 3(1 - p)p + 4(1 - p)^2p + ... = A.$$ 

Compare this with the series given in the question, we see that $A(1 - p) + p = \frac{1}{p}$. Solve for $A$ gives $A = \frac{1}{p} + 1$. This agrees with what we achieved in class.

10. A stock evolves according to the following model:

$$S_0 = 10;$$

$$S_{i+1} = S_iX_i.$$
where $X_i$ are iid random variables with distribution

$$X_i = \begin{cases} 1.5 & \text{with probability } 0.4 \\ 0.7 & \text{with probability } 0.6 \end{cases}$$

Compute $E(S_6|S_4)$. (This can be interpreted as the best guess for stock price on the 6th day given the stock price on the 4th day).

Answer:

$$E(S_6|S_4) = E(S_4X_5X_6|S_4) = S_4E(X_5X_6|S_4).$$

Note that $S_4 = S_0X_1X_2X_3$. Thus $X_5X_6$ is independent of $S_4$. Thus

$$E(S_6|S_4) = S_4E(X_5X_6) = S_4E(X_5)E(X_6) = S_4(1.02)^2.$$