Ito integral computation

Math 485
December 6, 2013

1 Goal:

To compute the explicit formula for some Ito integrals and verify the fact that Ito integral is a martingale with respect to the filtration generated by Brownian motion for “nice” integrands.

2 $\int_0^t B_s dB_s$:

Since intuitively, the “antiderivative” of $B_t$ is $B_t^2$, we apply Ito’s formula to $B_t^2$:

$$dB_t^2 = 2B_t dB_t + dt.$$ 

Hence

$$B_t^2 - B_0^2 = 2 \int_0^t B_s dB_s + \int_0^t ds,$$

which implies

$$\int_0^t B_s dB_s = \frac{1}{2}(B_t^2 - t).$$

This finishes the computation of $\int_0^t B_s dB_s$. To verify that it is a martingale we need to verify for $s \leq t$

$$E(\int_0^t B_s dB_s | B_s) = \int_0^s B_r dB_r,$$

which is equivalent to verifying

$$E(B_t^2 - t | B_s) = B_s^2 - s. \quad (1)$$
Thus we need to compute $E(B^2_t|B_s)$. We have

$$E(B^2_t|B_s) = E((B_s + (B_t - B_s))^2|B_s)$$

$$= E(B_s^2 + 2B_s(B_t - B_s) + (B_t - B_s)^2|B_s)$$

$$= B_s^2 + (t - s),$$

(By remembering that $B_t - B_s$ is independent of $B_s$).

It is easy to verify that (1) is true now.

3 $\int_0^t e^{B_s} dB_s$:

From the technique of section 2, it is easy to see that we can compute $\int_0^t B_s^k dB_s$ and verify that it is a martingale for any integer $k$. In this section we’ll compute a more challenging integral, namely $\int_0^t e^{B_s} dB_s$.

Again the idea is to apply Ito’s formula to the “antiderivative” of $e^{B_t}$, which is $e^{B_t}$.

$$de^{B_t} = e^{B_t}dB_t + \frac{1}{2}e^{B_t}dt.$$ 

Thus

$$e^{B_t} - e^{B_0} = \int_0^t e^{B_s} dB_s + \frac{1}{2} \int_0^t e^{B_s} ds.$$ 

So

$$\int_0^t e^{B_s} dB_s = e^{B_t} - 1 - \frac{1}{2} \int_0^t e^{B_s} ds.$$ 

This finishes the computation of the integral $\int_0^t e^{B_s} dB_s$. Notice then by computation, we mean rewrite the Ito integral into other expressions that do not involve the integral with respect to $dB_t$. To verify that $\int_0^t e^{B_s} dB_s$ is a martingale, we need to verify for $s \leq t$

$$E(e^{B_t} - 1 - \frac{1}{2} \int_0^t e^{B_s} dr|B_s) = e^{B_s} - 1 - \frac{1}{2} \int_0^s e^{B_r} dr. \quad (2)$$

So we need to compute two things: $E(e^{B_t}|B_s)$ and $E(\int_0^t e^{B_s} dr|B_s)$. The first computation is standard:

$$E(e^{B_t}|B_s) = E(e^{B_s} e^{B_t - B_s}|B_s)$$

$$= e^{B_s} e^{\frac{1}{2}(t-s)}.$$
The second computation requires more attention:

\[ E\left(\int_0^t e^{B_r} dr | B_s\right) = \int_0^t E(e^{B_r} | B_s) dr \]

Now for \( r \leq s \), \( E(e^{B_r} | B_s) = e^{B_r} \). But for \( r \leq s \), \( E(e^{B_r} | B_s) = e^{B_s} e^{\frac{1}{2}(r-s)} \) (by exactly the same computation just above). Thus

\[ E\left(\int_0^t e^{B_r} dr | B_s\right) = \int_0^s e^{B_r} dr + \int_s^t e^{B_s} e^{\frac{1}{2}(r-s)} dr. \]

\[ = \int_0^s e^{B_r} dr + e^{B_s} e^{-\frac{1}{2}} \left[2e^{\frac{1}{2}r}\right]_s^t \]

\[ = \int_0^s e^{B_r} dr + 2e^{B_s} \left[e^{\frac{1}{2}(t-s)} - 1\right]. \]

It is easy to verify the equality (2) now (See Homework 7 Problem 1 b).