Derivation of Black-Scholes PDE

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1 Goal:

To show that under the model

\[ dS_t = rS_t dt + \sigma S_t dB_t, \]

the price \( V(t, S_t) \) of a European-style derivative that pays \( \phi(S_t) \) at time \( T \) satisfies

\[
\begin{align*}
\frac{\partial}{\partial t} V(t, x) + \frac{\partial}{\partial x} V(t, x) &\left[ r x + \frac{1}{2} \frac{\partial^2}{\partial x^2} V(t, x) \sigma^2 x^2 \right] - r V = 0 \\
V(T, x) &\quad = \phi(x).
\end{align*}
\]

2 Ingredients

1. Ito’s formula
2. Game-theory portfolio: We hold \( \Delta_t \) shares of stock at time \( t \) and 1 share of \( V \). We choose \( \Delta_t \) such that the return of the portfolio is “deterministic”.
3. No arbitrage principle: If a portfolio \( \pi_t \) satisfies

\[ d\pi_t = \mu(t)\pi_t dt, \tag{2.1} \]

then we must have \( \mu(t) = r \), for all \( t \).

3 Derivation of Black-Scholes PDE

1. Apply Ito’s formula:

\[
dV_t = \frac{\partial}{\partial t} V(t, S_t) + \frac{\partial}{\partial x} V(t, S_t) dS_t + \frac{1}{2} \frac{\partial^2}{\partial x^2} V(t, S_t) \sigma^2 S_t^2 dt.
\]

Since

\[ dS_t = rS_t dt + \sigma S_t dB_t, \]

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grouping the \( d_t \) and \( dB_t \) terms together we have

\[
dV_t = \left[ \frac{\partial}{\partial t} V(t, S_t) + \frac{\partial}{\partial x} V(t, S_t) r S_t + \frac{1}{2} \frac{\partial^2}{\partial x^2} V(t, S_t) \sigma^2 S_t^2 \right] dt + \left[ \frac{\partial}{\partial x} V(t, S_t) \sigma S_t \right] dB_t. \tag{3.2}
\]

2.

a. The game-theory portfolio \( \pi_t \) satisfies:

\[
d\pi_t = \Delta_t dS_t + dV_t
\]

By self-financing requirement:

\[
d\pi_t = \Delta_t (r S_t dt + \sigma S_t dB_t) + dV_t.
\]

Replace \( dV_t \) by (3.2) and group \( d_t \), \( dB_t \) terms again we have

\[
d\pi_t = \left[ \Delta_t r S_t + \frac{\partial}{\partial t} V(t, S_t) + \frac{1}{2} \frac{\partial^2}{\partial x^2} V(t, S_t) \sigma^2 S_t^2 \right] dt
\]

\[+ \left[ \Delta_t \sigma S_t + \frac{\partial}{\partial x} V(t, S_t) \sigma S_t \right] dB_t. \tag{3.3}
\]

b. We choose \( \Delta_t = -\frac{\partial}{\partial x} V(t, S_t) \) to “kill” the \( dB_t \) term. Then

\[
d\pi_t = \left[ \frac{\partial}{\partial t} V(t, S_t) + \frac{1}{2} \frac{\partial^2}{\partial x^2} V(t, S_t) \sigma^2 S_t^2 \right] dt. \tag{3.3}
\]

3. To apply the no arbitrage principle, we need to rewrite the right hand side of (3.3) in the form of (2.1). We have

\[
d\pi_t = \left[ \frac{\partial}{\partial t} V(t, S_t) + \frac{1}{2} \frac{\partial^2}{\partial x^2} V(t, S_t) \sigma^2 S_t^2 \right] \frac{\pi_t}{\pi_t} dt. \tag{3.4}
\]

This is in the form of (2.1) with

\[
\mu(t) = \frac{\left[ \frac{\partial}{\partial t} V(t, S_t) + \frac{1}{2} \frac{\partial^2}{\partial x^2} V(t, S_t) \sigma^2 S_t^2 \right]}{\pi_t}
\]

Therefore, by the no arbitrage principle, we conclude that

\[
\left[ \frac{\partial}{\partial t} V(t, S_t) + \frac{1}{2} \frac{\partial^2}{\partial x^2} V(t, S_t) \sigma^2 S_t^2 \right] \frac{\pi_t}{\pi_t} = r.
\]

But \( \pi_t = \Delta_t S_t + V_t = -\frac{\partial}{\partial x} V(t, S_t) S_t + V_t \). So we have

\[
\frac{\partial}{\partial t} V(t, S_t) + \frac{1}{2} \frac{\partial^2}{\partial x^2} V(t, S_t) \sigma^2 S_t^2 = r(-\frac{\partial}{\partial x} V(t, S_t) S_t + V_t).
\]

In other words

\[
\frac{\partial}{\partial t} V(t, S_t) + \frac{1}{2} \frac{\partial^2}{\partial x^2} V(t, S_t) \sigma^2 S_t^2 + r \frac{\partial}{\partial x} V(t, S_t) S_t - rV_t = 0.
\]

This is the Black-Scholes PDE.