
Smooth distributions are globally finitely spanned

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Dedicated to Alberto Isidori on his 65th birthday

Summary. A smooth distribution on a smooth manifold M is, by definition, a map that assigns to each point x of M a linear subspace $\Delta(x)$ of the tangent space $T_x M$, in such a way that, locally, there exist smooth sections f_1, \dots, f_d of Δ such that the linear span of $f_1(x), \dots, f_d(x)$ is $\Delta(x)$ for all x . We prove that a much weaker definition of “smooth distribution,” in which it is only required that for each $x \in M$ and each $v \in \Delta(x)$ there exist a smooth section f of Δ defined near x such that $f(x) = v$, suffices to imply that there exists a finite family $\{f_1, \dots, f_d\}$ of smooth global sections of Δ such that $\Delta(x)$ is spanned, for every $x \in M$, by the values $f_1(x), \dots, f_d(x)$. The result is actually proved for general singular subbundles E of an arbitrary smooth vector bundle V , and we give a bound on the number d of global spanning sections, by showing that one can always take $d = \text{rank } E \cdot (1 + \dim M)$, where $\text{rank } E$ is the maximum dimension of the fibers $E(x)$.

1 Introduction

“Smooth singular distributions”—or, simply, “smooth distributions”—on a smooth manifold M are often defined as maps $Mx \mapsto \Delta(x)$ such that

- (i) for each $x \in M$, $\Delta(x)$ is a linear subspace of $T_x M$,
- (ii) Δ is locally finitely spanned.

Condition (ii) can be assigned a precise meaning in several different ways. For example, we could take (ii) to mean

(LFS) for every $x_* \in M$ there exist an open neighborhood U of x_* in M and a finite sequence (f_1, \dots, f_d) of smooth vector fields on U such that

$$\Delta(x) = \text{span}\{f_1(x), \dots, f_d(x)\} \quad \text{for all } x \in U.$$

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Naturally, it would also be possible to require a condition that in principle appears to be stronger, namely, that Δ is “globally finitely spanned,” in the sense that the sequence (f_1, \dots, f_d) exist globally, that is

(GFS) *There exists a finite sequence (f_1, \dots, f_d) of smooth vector fields on M such that*

$$\Delta(x) = \text{span}\{f_1(x), \dots, f_d(x)\} \quad \text{for all } x \in M.$$

(This is actually the requirement used by A. Isidori in [1], p. 14.)

Alternatively, one could require the seemingly much weaker condition that Δ is “determined by the values of its smooth sections,” in the sense that

(DVSS) *For every $x_* \in M$ and every $v \in \Delta(x_*)$ there exists a smooth vector field f on some open neighborhood U of x_* such that $f(x_*) = v$ and $f(x) \in \Delta(x)$ for all $x \in U$.*

The purpose of this note is to prove that the three conditions are equivalent, and to give a bound for the number d of smooth global sections of Δ that are required to span $\Delta(x)$ for each x .

We will actually prove the result for a general smooth singular subbundle E of an arbitrary smooth vector bundle V over a smooth manifold M . The bound on the number d of global sections of E needed to span the space $E(x)$ at each point x of M will turn out to be $\rho(E) \cdot (1 + \dim M)$, where $\rho(E)$ is the rank of E . (In particular, a singular distribution Δ on a smooth manifold of dimension m is always spanned at every point by $m^2 + m$ smooth global sections.)

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2 Statement and proof of the main result

Throughout this note, the word “smooth” means “of class C^∞ ,” and “smooth manifold” means “smooth Hausdorff paracompact finite-dimensional manifold without boundary.” If M is a smooth manifold, then TM is the tangent bundle of M and, for each $x \in M$, $T_x M$ is the tangent space of M at x . If S is a subset of a real linear space X , then $\text{span } S$ denotes the linear span of S .

Our precise definitions are as follows.

Let V be a smooth vector bundle over a smooth manifold M . A **singular subbundle** of V on an open subset U of M is a map $Mx \mapsto E(x)$ such that $E(x)$ is a linear subspace of $V(x)$ for each $x \in M$. A **smooth section** of a singular subbundle E of V defined on U is a map $Ux \mapsto \sigma(x) \in E(x)$ which is smooth as a section of V .

Definition 1. Let V be a smooth vector bundle over a smooth manifold M and let E be a singular subbundle of V over M . We say that E is **smooth** if for every $x \in M$ there exists a smooth section σ of E , defined on a neighborhood U of x , such that $v = \sigma(x)$.

The **rank** of a subbundle E of V is the maximum of the dimensions of the fibers of E . \square

We will use $\rho(E)$ to denote the rank of E .

In particular, a **smooth distribution** on M is a smooth singular subbundle of TM . (This means, that Δ is a map $Mx \mapsto \Delta(x)$ such that (i) and (ii) above hold, where (ii) is interpreted to mean ‘‘Condition DVSS holds.’’)

The following is then our main result.

Theorem 1. Let V be a smooth vector bundle over a smooth manifold M of dimension m , and let E be a smooth singular subbundle of V in the sense of Definition 1. Let $d = \rho(E)(m + 1)$. Then there exist d smooth global sections $\sigma_1, \dots, \sigma_d$ of E such that the values $\sigma_1(x), \dots, \sigma_d(x)$ linearly span $E(x)$ for every $x \in E$.

Proof. Let $r = \rho(E)$. For each x , use $\delta(x)$ to denote the dimension of the space $E(x)$. Define $\Omega_k = \{x : \delta(x) \geq k\}$, for each $k \in \{0, 1, \dots, r + 1\}$. Then the Ω_k are open subsets of M . (Indeed, if $x \in \Omega_k$ then we may pick k linearly independent members v_1, \dots, v_k of $E(x)$, and smooth sections $\sigma_1, \dots, \sigma_k$ of E , defined on a neighborhood U of x , such that $\sigma_j(x) = v_j$ for $j = 1, \dots, k$. Since $\sigma_1(x), \dots, \sigma_k(x)$ are linearly independent, it follows that $\sigma_1(x'), \dots, \sigma_k(x')$ are linearly independent for all x' in some neighborhood U' of x . Then $\delta(x') \geq k$ for all $x' \in U'$, so $U' \subseteq \Omega_k$.)

Furthermore, it is clear that

$$M = \Omega_0 \supseteq \Omega_1 \supseteq \Omega_2 \supseteq \dots \supseteq \Omega_{r-1} \supseteq \Omega_r \supseteq \Omega_{r+1} = \emptyset.$$

We will construct $m + 1$ -tuples $(\sigma_1^k, \dots, \sigma_{m+1}^k)$ of smooth global sections of E , for each $k \in \{1, \dots, r\}$, such that

(*) for every $k \in \{1, \dots, r\}$ the following is true

(# $_k$) for every $x \in \Omega_k$, the linear span of the set of $k(m + 1)$ values $\sigma_i^j(x)$, $i \in \{1, \dots, m + 1\}$, $j \in \{1, \dots, k\}$, is of dimension $\geq k$.

It is then easy to see that, once this construction is carried out, the $r(m + 1)$ sections σ_i^j , $i \in \{1, \dots, m + 1\}$, $j \in \{1, \dots, r\}$, will have the desired property. (Indeed, suppose $x \in M$, and let $k = \delta(x)$. Then $x \in \Omega_k$, so the linear span of the $\sigma_i^j(x)$ for $i \in \{1, \dots, m + 1\}$ and $j \in \{1, \dots, k\}$ is of dimension $\geq k$, and then *a fortiori* the linear span $L(x)$ of the $\sigma_i^j(x)$, for $i \in \{1, \dots, m + 1\}$ and $j \in \{1, \dots, r\}$, is of dimension $\geq k$. But $E(x)$ is of dimension k , and $L(x) \subseteq E(x)$. So $L(x) = E(x)$.)

We construct $(\sigma_1^k, \dots, \sigma_r^k)$ by induction on k , as follows. Suppose we have constructed the σ_i^j for $j < k$ and $i \in \{1, \dots, m + 1\}$, in such a way that

Conditions $(\#_j)$ are satisfied for all $j \in \{1, \dots, k-1\}$. (In particular, this assumption is vacuously true if $k=1$.)

For each $x \in \Omega_k$, let $A(x)$ be the linear span of the $\sigma_i^j(x)$ for $j < k$, $i \in \{1, \dots, m+1\}$.

Now fix a point $x \in \Omega_k$. Then our inductive hypothesis implies that $\dim A(x) \geq k-1$. Since $\dim E(x) \geq k$ (because $x \in \Omega_k$), we may pick a vector $v \in E(x)$ such that the linear span of $A(x) \cup \{v\}$ has dimension $\geq k$, and then we may pick a smooth section ν^x of E , defined on some open neighborhood W_x of x , such that $\nu^x(x) = v$. After multiplying ν^x by a smooth function $\varphi : M \rightarrow \mathbb{R}$ which is such that $\varphi(x) > 0$ and the support of φ is a compact subset of W_x , we may assume that ν^x is a global smooth section of E .

Since $\text{span}(A(x) \cup \{v\})$ has dimension $\geq k$, we may assume, after shrinking W_x , if necessary, that

(*) *the linear span of $A(x') \cup \{\nu^x(x')\}$ has dimension $\geq k$ for every $x' \in W_x$.*

This implies, in particular, that $W_x \subseteq \Omega_k$. Then $\mathcal{W} = \{W_x\}_{x \in \Omega_k}$ is an open covering of Ω_k .

We now come to the key step of our proof, namely, the construction of a refinement of \mathcal{Z} of \mathcal{W} with a very special property. (Recall that a **refinement** of an open covering $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ of a topological space T is an open covering $\mathcal{V} = \{V_\beta\}_{\beta \in B}$ of T such that for every $\beta \in B$ there exists an $\alpha \in A$ such that $V_\beta \subseteq U_\alpha$.)

Lemma 1. *There exists $\mathcal{Z} = \{Z_\lambda\}_{\lambda \in L}$ such that*

- *\mathcal{Z} is an open covering of Ω_k ;*
- *\mathcal{Z} is a refinement of \mathcal{W} ;*
- *the index set L is a union*

$$L = L_1 \cup L_2 \cup \dots \cup L_{m+1},$$

such that each L_i has the property that $Z_\lambda \cap Z_{\lambda'} = \emptyset$ whenever $\lambda \in L_i$, $\lambda' \in L_i$, and $\lambda \neq \lambda'$. (In other words, each $\mathcal{Z}_i = \{Z_\lambda\}_{\lambda \in L_i}$ is a family of pairwise disjoint open subsets of Ω_k .)

Proof. Let us prove the existence of a \mathcal{Z} with the above properties. Equip M with a Riemannian metric. (This is possible because we are assuming that M is paracompact.) Then let \mathcal{T} be a triangulation of Ω_k that refines \mathcal{W} , in the sense that every closed face F of \mathcal{T} is a subset of W_x for some x . (The existence of a triangulation of Ω_k follows from the well known fact that every smooth manifold is triangulable, cf. [2]. Then a triangulation that refines the covering \mathcal{W} can easily be constructed by successive barycentric subdivisions.)

For $j = 0, \dots, m$, let T_j be the set of all j -dimensional open faces of \mathcal{T} , and let $|T_j|$ be the union of the members of T_j . Then each $F \in T_j$ is both relatively open and relatively closed in $|T_j|$. Given a face $F \in T_j$, let $C(F)$ be the union of the closures in Ω_k of all the faces $G \in T_j$ other than F . Then $C(F)$ is a relatively closed subset of Ω_k , because it is the union of a locally

finite family of compact subsets of Ω_k . Furthermore, $F \cap C(F) = \emptyset$, because F is relatively open in T_j . If $x \in F$, then the Riemannian distance $\text{dist}(x, C(F))$ from x to $C(F)$ is strictly positive, because $x \notin C(F)$ and $C(F)$ is closed. Let B_x be the open ball in Ω_k with center x and radius $\frac{1}{2}\text{dist}(x, C(F))$. Let $B(F)$ be the union of the B_x for all $x \in F$.

We now prove that the $B(F)$, as F varies over T_j , are pairwise disjoint. To show this, let us assume that $F, F' \in T_j$ and $F \neq F'$, and let us show that $B(F) \cap B(F') = \emptyset$. Suppose $B(F) \cap B(F') \neq \emptyset$. Pick $y \in B(F) \cap B(F')$. Pick $x \in F$ such that $y \in B_x$ and $x' \in F'$ such that $y \in B_{x'}$. Let $\delta = \text{dist}(x, C(F))$, $\delta' = \text{dist}(x', C(F'))$. Assume, without loss of generality, that $\delta \geq \delta'$. Then

$$\text{dist}(x, x') \leq \text{dist}(x, y) + \text{dist}(y, x') < \frac{1}{2}\delta + \frac{1}{2}\delta' \leq \delta.$$

So $\text{dist}(x, x') < \delta$, which is a contradiction, since $\delta = d(x, C(F))$ and x' belongs to $C(F)$.

By construction, each $F \in T_j$ is a subset of W_x for some $x \in \Omega_k$. So we may pick, for each $F \in T_j$, a point $x_F \in \Omega_k$ such that $F \subseteq W_{x_F}$. We then define $\tilde{B}(F) = B(F) \cap W_{x_F}$. It is clear that

1. for every $j \in \{0, \dots, m\}$, and every $F \in T_j$, $\tilde{B}(F)$ is an open subset of Ω_k such that $F \subseteq \tilde{B}(F) \subseteq W_{x_F}$;
2. for every $j \in \{0, \dots, m\}$, the sets $\tilde{B}(F)$, for $F \in T_j$, are pairwise disjoint;
3. $\bigcup_{j=0}^m \bigcup_{F \in T_j} \tilde{B}(F) = \Omega_k$.

Now let $L_i = T_{i-1}$ for $i = 1, 2, \dots, m+1$, and then let $L = L_1 \cup \dots \cup L_{m+1}$. For $F \in L_i$, let $Z_F = \tilde{B}(F)$. Then $\mathcal{Z} = \{Z_F\}_{F \in L}$ is an open covering of Ω_k . Furthermore, \mathcal{Z} is a refinement of \mathcal{W} , because if $F \in L$ then $Z_F \subseteq W_{x_F}$. Finally, each $\mathcal{Z}_i = \{Z_F\}_{F \in L_i}$ is clearly a family of pairwise disjoint sets. So \mathcal{Z} has all the desired properties, and the proof of our lemma is complete. \square

We now return to the proof of the main theorem. Let Z_i be the union of all the members of \mathcal{Z}_i . For each i and each member Z of \mathcal{Z}_i pick a point $\bar{x}(i, Z) \in \Omega_k$ such that $Z \subseteq W_{\bar{x}(i, Z)}$. Then define a smooth section μ_i of E on Z_i by letting $\mu_i(x) = \nu^{\bar{x}(i, Z)}(x)$ if $x \in Z$, $Z \in \mathcal{Z}_i$. (This is possible because (a) if $x \in Z_i$ then there exists one and only one $Z \in \mathcal{Z}_i$ such that $x \in Z$, (b) $\nu^{\bar{x}(i, Z)}$ is a smooth section of E on $W_{\bar{x}(i, Z)}$, and (c) $Z \subseteq W_{\bar{x}(i, Z)}$.) It then follows from (*) that, whenever $i = 1, \dots, m+1$, $Z \in \mathcal{Z}_i$, and $x \in Z$, then $\text{span}(A(x) \cup \{\mu_i(x)\})$ has dimension $\geq k$. In particular,

(**) for every i , the linear span of $A(x) \cup \{\mu_i(x)\}$ has dimension $\geq k$ for every $x \in Z_i$.

We now construct a smooth function $\varphi_i : M \mapsto \mathbb{R}$ such that

- (1) $\varphi_i(x) > 0$ for all $x \in Z_i$,
- (2) if we let $\theta_i(x) = \varphi_i(x)\mu_i(x)$ for $x \in Z_i$, and $\theta_i(x) = 0$ for $x \in M$, $x \notin Z_i$, then θ_i is a smooth section of V on M .

(The existence of φ_i is easy to prove. Indeed, it is clearly sufficient to construct φ_i on each connected component of M , so we may as well assume that M is connected. Since Z_i is open in M , we can express Z_i as a countable union $\bigcup_{\ell=1}^{\infty} K_\ell$ of compact sets such that $K_\ell \subseteq \text{Int } K_{\ell+1}$ for each ℓ . For each ℓ , find a smooth nonnegative function $\psi_\ell : M \mapsto \mathbb{R}$ such that $\psi_\ell \equiv 1$ on K_ℓ and $\psi_\ell \equiv 0$ on $M \setminus K_{\ell+1}$. We construct φ_i by letting $\varphi_i = \sum_{\ell=1}^{\infty} \varepsilon_\ell \psi_\ell$, where the ε_ℓ are strictly positive numbers that converge to zero sufficiently fast. The maps $\theta_{i,\ell}$ given by $\theta_{i,\ell}(x) = \psi_\ell(x)\mu_i(x)$ are then smooth compactly supported global sections of V on M , and all we need is to find the ε_ℓ so that the series $\sum_{\ell=1}^{\infty} \varepsilon_\ell \psi_\ell$ and $\sum_{\ell=1}^{\infty} \varepsilon_\ell \theta_{i,\ell}$ converge to smooth limits φ_i, θ_i in the spaces $C^\infty(M, \mathbb{R}), C^\infty(M, V)$ —where $C^\infty(M, V)$ is the space of all smooth sections of V —endowed with the topology of uniform convergence on compact sets of all the derivatives of all orders. And it is a well known fact that such ε_ℓ always exist.)

We now let $\sigma_i^k = \theta_i$ for $i = 1, \dots, m+1$. We have to verify that $(\#_k)$ holds. For this purpose, we pick $x \in \Omega_k$ and verify that the linear span $S(x)$ of the vectors σ_i^j , for $i \in \{1, \dots, m+1\}, j \in \{1, \dots, k\}$, has dimension $\geq k$. Since \mathcal{Z} is a covering of Ω_k , we may pick an $F \in L$ such that $x \in Z_F$. Then F belongs to L_i for a unique i , and then $x \in Z_i$. It follows that $\text{span}(A(x) \cup \{\mu_i(x)\})$ has dimension $\geq k$. Since $x \in Z_i$, we have $\varphi_i(x) > 0$, and then the fact that $\theta_i(x) = \varphi_i(x)\mu_i(x)$ implies that $\text{span}(A(x) \cup \{\theta_i(x)\}) = \text{span}(A(x) \cup \{\mu_i(x)\})$, so $\text{span}(A(x) \cup \{\theta_i(x)\})$ also has dimension $\geq k$. In other words, we have shown that $\text{span}(A(x) \cup \{\sigma_i^k(x)\})$ has dimension $\geq k$. Since $A(x)$ is the linear span of the vectors σ_i^j , for $i \in \{1, \dots, m+1\}, j \in \{1, \dots, k-1\}$ it is clear that $A(x) \cup \{\sigma_i^k(x)\} \subseteq S(x)$. Hence $\dim S(x) \geq k$ as desired, concluding our proof. \square

References

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