# Contents

**Generalized differentials, variational generators, and the maximum principle with state constraints**

# Generalized differentials, variational generators, and the maximum principle with state constraints

Héctor J. Sussmann[⋆]

Department of Mathematics, Rutgers University, Piscataway, NJ, U.S.A.
`sussmann@math.rutgers.edu`

**Summary.** We present the technical background material for a version of the Pontryagin Maximum Principle with state space constraints and very weak technical hypotheses, based on a primal approach that uses generalized differentials and packets of needle variations. In particular, we give a detailed account of two theories of generalized differentials, the "generalized differential quotients" (GDQs) and the "approximate generalized differential quotients" (AGDQs), and prove the corresponding open mapping and separation theorems. We state—but do not prove—the resulting version of the Maximum Principle. The result does not require the time-varying vector fields corresponding to the various control values to be continuously differentiable, Lipschitz, or even continuous with respect to the state, since all that is needed is that they be "co-integrably bounded integrally continuous." This includes the case of vector fields that are continuous with respect to the state, as well as large classes of discontinuous vector fields, containing, for example, rich sets of single-valued selections for almost semicontinuous differential inclusions. Uniqueness of trajectories is not required, since our methods deal directly with multivalued maps. The dynamical reference vector field and reference Lagrangian are only required to be "differentiable" along the reference trajectory in a very weak sense, namely, that of possessing suitable "variational generators." This includes—but is much more general than—the conditions of the classical cases when the reference vector field and Lagrangian are differentiable with respect to the state and the variational generator can be taken to be the singleton of the classical differential, as well as the case when they are Lipschitz and the variational generator can be chosen to be the Clarke generalized Jacobian. In addition, for the Lagrangian one can chose the variational generator to be the Michel-Penot subdifferential. For the functions defining the state space constraints, all that is needed is the existence of a variational generator in a slightly different technical sense, which includes as a special case the object often referred to as $\partial_x^{>} g$ in the literature, as well as many non-Lipschitz cases. The conclusion yields finitely additive measures, as in earlier work by other authors, and a Hamiltonian maximization inequality valid also at the jump times of the adjoint covector.

---

[⋆] Research supported in part by NSF Grants DMS01-03901 and DMS-05-09930.

## 1 Introduction

In a series of previous papers (cf. [20, 21, 22, 23]), we have developed a "primal" approach to the non-smooth Pontryagin Maximum Principle, based on generalized differentials, flows, and general variations. The method used is essentially the one of classical proofs of the Maximum Principle such as that of Pontryagin and his coauthors (cf. Pontryagin *et al.* [15], Berkovitz [1]), based on the construction of packets of needle variations, but with a refinement of the "topological argument," and with concepts of differential more general than the classical one, and usually set-valued.

In this article we apply this approach to optimal control problems with state space constraints, and at the same time we state the result in a more concrete form, dealing with a specific class of generalized derivatives (the "generalized differential quotients"), rather than in the abstract form used in some of the previous work.

The paper is organized as follows. In §2 we introduce some of our notations, and review some background material, especially the basic concepts about finitely additive vector-valued measures on an interval. In §3 we review the theory of "Cellina continuously approximable" (CCA) set-valued maps, and prove the CCA version—due to A. Cellina—of some classical fixed point theorems due to Leray-Schauder, Kakutani, Glicksberg and Fan. In §4 we define the notions of generalized differential quotient (GDQ), and approximate generalized differential quotient (AGDQ), and prove their basic properties, especially the chain rule, the directional open mapping theorem, and the transversal intersection property. In §5 we define the two types of variational generators that will occur in the maximum principle, and state and prove theorems asserting that various classical generalized derivatives—such as classical differentials, Clarke generalized Jacobians, subdifferentials in the sense of Michel-Penot, and (for functions defining state space constraints) the object often referred to as $\partial_x^> g$ in the literature—are special cases of our variational generators. In §6 we discuss the classes of discontinuous vector fields studied in detail in [24]. In §7 we state the main theorem. The rather lengthy proof will be given in a subsequent paper.

***Acknowledgement.*** The author is grateful to Palo Nistri and Gianna Stefani, the organizers of the Cetraro 2004 C.I.M.E. Summer School, for inviting him to present the ideas that have led to this paper, to Arrigo Cellina for his useful comments on the condition that is here called "Cellina continuous approximability," and to the audience of the Cetraro lectures for their numerous contributions. He also thanks G. Stefani for the invitation to give a series of lectures in Firenze in the Summer of 2005, and the participants of the Firenze lectures, especially Andrei Sarychev, who explained to the author why it was important to use finitely additive measures.

# 2 Preliminaries and background

## 2.1 Review of some notational conventions and definitions

***Integers and real numbers.*** We use $\mathbb{Z}$, $\mathbb{R}$ to denote, respectively, the set of all integers and the set of all real numbers, and write $\mathbb{N}\stackrel{\mathrm{def}}{=}\{n \in \mathbb{Z} : n > 0\}$, $\mathbb{Z}_+\stackrel{\mathrm{def}}{=}\mathbb{N} \cup \{0\}$. Also, $\bar{\mathbb{R}}$, $\mathbb{R}_+$, $\bar{\mathbb{R}}_+$, denote, respectively, the extended real line $\mathbb{R}\cup\{-\infty, +\infty\}$, the half-line $[0, +\infty\,[$ , and the extended half-line $[0, +\infty]$ (i.e., $[0, +\infty\,[\,\cup\{+\infty\}$).

***Intervals.*** An *interval* is an arbitrary connected subset of $\mathbb{R}$. If $a, b \in \mathbb{R}$ and $a \leq b$, then $\mathrm{INT}([a, b])$ is the set of all intervals $J$ such that $J \subseteq [a, b]$. Hence $\mathrm{INT}([a, b])$ consists of the intervals $[\alpha, \beta]$, $[\alpha, \beta\,[$ , $]\alpha, \beta]$ and $]\alpha, \beta\,[$ , with $a \leq \alpha < \beta \leq b$, as well as the singletons $\{\alpha\}$, for $a \leq \alpha \leq b$), and the empty set. A *nontrivial interval* is one whose length is strictly positive, that is, one that contains at least two distinct points.

***Euclidean spaces and matrices.*** The expressions $\mathbb{R}^n$, $\mathbb{R}_n$ will be used to denote, respectively, the set of all real column vectors $x = (x_1, \ldots, x_n)^\dagger$ (where "$\dagger$" stands for "transpose") and the set of all real row vectors $p = (p_1, \ldots, p_n)$. We refer to the members of $\mathbb{R}_n$ as *covectors*. Also, $\mathbb{R}^{m\times n}$ is the space of all real matrices with $m$ rows and $n$ columns.

If $n \in \mathbb{Z}_+$, $x \in \mathbb{R}^n$, $r \in \mathbb{R}$, and $r > 0$, we use $\bar{\mathbb{B}}^n(x, r)$, $\mathbb{B}^n(x, r)$ to denote, respectively, the closed and open balls in $\mathbb{R}^n$ with center $x$ and radius $r$. We write $\bar{\mathbb{B}}^n(r)$, $\mathbb{B}^n(r)$ for $\bar{\mathbb{B}}^n(0, r)$, $\mathbb{B}^n 0, (r)$, and $\bar{\mathbb{B}}^n$, $\mathbb{B}^n$ for $\bar{\mathbb{B}}^n(1)$, $\mathbb{B}^n(1)$. Also, we will use $\mathbb{S}^n$ to denote the $n$-dimensional unit sphere, so $\mathbb{S}^n = \{(x_1, \ldots, x_{n+1})^\dagger \in \mathbb{R}^{n+1} : \sum_{j=1}^{n+1} x_j^2 = 1\}$.

***Topological spaces, metric spaces, metric balls.*** We will use throughout the standard terminology of point-set topology: a *neighborhood* of a point $x$ in a topological space $X$ is any subset $S$ of $X$ that contains an open set $U$ such that $x \in U$. In the special case of a metric space $X$, we use $\mathbb{B}_X(x, r)$, $\bar{\mathbb{B}}_X(x, r)$, to denote, respectively, the open ball and the closed ball with center $x$ and radius $r$.

***Quasidistance and Hausdorff distance.*** If $X$ is a topological space, then $Comp^0(X)$ will denote the set of all compact subsets of $X$ (including the empty set), and $Comp(X)$ will be the set of all nonempty members of $Comp^0(X)$.

If $X$ is a metric space, with distance function $d_X$, then we can define the "quasidistance" $\varDelta_X^{qua}(A, B)$ from a set $A \in Comp^0(X)$ to another set $B \in Comp^0(X)$ by letting

$$\varDelta_X^{qua}(A, B) = \sup\Big\{\inf\{d_X(x, x') : x' \in B\} : x \in A\Big\}. \qquad (2.1.1)$$

(This function is not a distance because, for example, it is not symmetric, since $\varDelta_X^{qua}(A, B) = 0$ but $\varDelta_X^{qua}(B, A) \neq 0$ if $A \subseteq B$ and $A \neq B$. Furthermore, $\varDelta_X^{qua}$ can take the value $+\infty$, since $\varDelta_X^{qua}(A, B) = +\infty$ if $A \neq \emptyset$ but $B = \emptyset$.)

**Definition 2.1** *Suppose that $X$ is a metric space. The* ***Hausdorff distance*** *$\varDelta_X(K,L)$ between two nonempty subsets $K$, $L$ of $X$ is the number*

$$\varDelta_X(K,L) = \max\Big(\varDelta_X^{qua}(K,L), \varDelta_X^{qua}(L,K)\Big)\,. \qquad \square$$

It is then clear that the function $\varDelta_X$, restricted to $Comp(X) \times Comp(X)$, is a metric.

***Linear spaces and linear maps.*** The abbreviations "FDRLS" and "FDNRLS" will stand for the expressions "finite-dimensional real linear space," and "finite-dimensional normed real linear space," respectively. If $X$ and $Y$ are real linear spaces, then $Lin(X,Y)$ will denote the set of all linear maps from $X$ to $Y$. We use $X^\dagger$ to denote $Lin(X,\mathbb{R})$, i.e., the dual space of $X$. If $X$ is a FDNRLS, then $X^{\dagger\dagger}$ is identified with $X$ in the usual way.

If $X$ and $Y$ are FDNRLSs, then $Lin(X,Y)$ is a FDNRLS, endowed with the operator norm $\|\cdot\|_{op}$ given by

$$\|L\|_{op} = \sup\{\|L\cdot x\| : x \in X, \|x\| \le 1\}\,. \tag{2.1.2}$$

Also, we write $\boldsymbol{L}(X)$ for $Lin(X,X)$, the space of all linear maps $L : X \mapsto X$.

We identify $Lin(\mathbb{R}^n,\mathbb{R}^m)$ with $\mathbb{R}^{m\times n}$ in the usual way, by assigning to each matrix $M \in \mathbb{R}^{m\times n}$ the linear map $\mathbb{R}^n \ni x \mapsto M\cdot x \in \mathbb{R}^m$. In particular, $\boldsymbol{L}(X)$ is identified with $\mathbb{R}^{n\times n}$. Also, we identify $\mathbb{R}_n$ with the dual $(\mathbb{R}^n)^\dagger$ of $\mathbb{R}^n$, by assigning to a $y \in \mathbb{R}_n$ the linear functional $\mathbb{R}^n \ni x \mapsto y\cdot x \in \mathbb{R}$.

If $X, Y$ are FDRLSs, and $L \in Lin(X,Y)$, then the *adjoint* of $L$ is the map $L^\dagger : Y^\dagger \mapsto X^\dagger$ such that $L^\dagger(y) = y \circ L$ for $y \in Y^\dagger$. In the special case when $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$, so $L \in \mathbb{R}^{m\times n}$, the map $L^\dagger$ goes from $\mathbb{R}_m$ to $\mathbb{R}_n$, and is given by $L^\dagger(y) = y\cdot L$ for $y \in \mathbb{R}_m$.

***Manifolds, tangent spaces, differentials.*** If $M$ is a manifold of class $C^1$, and $x \in M$, then $T_xM$ will denote the tangent space of $M$ at $x$. It follows that if $M$, $N$ are manifolds of class $C^1$, $x \in M$, $F$ is an $N$-valued map defined on a neighborhood $U$ of $x$ in $M$, and $F$ is classically differentiable at $x$, then the differential $DF(x)$ belongs to $Lin(T_xM, T_{F(x)}N)$.

***Single- and set-valued maps.*** Throughout this paper, the word "map" always stands for "set-valued map." The expression "ppd map" refers to a "possibly partially defined (that is, not necessarily everywhere defined) ordinary (that is, single-valued) map." The precise definitions are as follows. A *set-valued map* is a triple $F = (A,B,G)$ such that $A$ and $B$ are sets and $G$ is a subset of $A \times B$. If $F = (A,B,G)$ is a set-valued map, we say that $F$ *is a set-valued map from $A$ to $B$*. In that case, we refer to the sets $A$, $B$, $G$ as the *source*, *target*, and *graph* of $F$, respectively, and we write $A = \mathrm{So}(F)$, $B = \mathrm{Ta}(F)$, $G = \mathrm{Gr}(F)$. If $x \in \mathrm{So}(F)$, we write $F(x) = \{y : (x,y) \in \mathrm{Gr}(F)\}$. The set $\mathrm{Do}(F) = \{x \in \mathrm{So}(F) : F(x) \neq \emptyset\}$ is the ***domain*** of $F$. If $A$, $B$ are sets, we use $SVM(A,B)$ to denote the set of all set-valued maps from $A$ to $B$, and write $F : A \mapsto\!\!\!\rightarrow B$ to indicate that $F \in SVM(A,B)$. A *ppd map from*

*$A$ to $B$* is an $F \in SVM(A,B)$ such that $F(x)$ has cardinality zero or one for every $x \in A$. We write $F : A \hookrightarrow B$ to indicate that $F$ is a ppd map from $A$ to $B$. If $F : A \mapsto\!\!\!\!\to B$, and $C \subseteq A$, then the *restriction* of $F$ to $C$ is the set-valued map $F \lceil C$ defined by $F \lceil C \stackrel{\text{def}}{=} (C, B, \mathrm{Gr}(F) \cap (C \times B))$.

If $F_1$ and $F_2$ are set-valued maps, then the *composite* $F_2 \circ F_1$ is defined if and only if $\mathrm{Ta}(F_1) = \mathrm{So}(F_2)$ and, in that case, $\mathrm{So}(F_2 \circ F_1) \stackrel{\text{def}}{=} \mathrm{So}(F_1)$ , $\mathrm{Ta}(F_2 \circ F_1) \stackrel{\text{def}}{=} \mathrm{Ta}(F_2)$, and

$$\mathrm{Gr}(F_2 \circ F_1) \stackrel{\text{def}}{=} \Big\{(x,z) : (\exists y)\Big((x,y) \in \mathrm{Gr}(F_1) \text{ and } (y,z) \in \mathrm{Gr}(F_2)\Big)\Big\}\,.$$

If $A$ is a set, then $\mathbb{I}_A$ denotes the *identity map* of $A$, that is, the triple $(A, A, \Delta_A)$, where $\Delta_A$ is the set of all pairs $(x,x)$, for all $x \in A$.

***Epimaps and constraint indicator maps.*** If $f : S \hookrightarrow \mathbb{R}$ is a ppd function, then

- The *epimap* of $f$ is the set-valued map $\check{f} : S \mapsto\!\!\!\!\to \mathbb{R}$ whose graph is the epigraph of $f$, so that $\check{f}(s) = \{f(s)+v : v \in \mathbb{R}, v \geq 0\}$ whenever $s \in \mathrm{Do}(f)$, and $\check{f}(s) = \emptyset$ if $s \in S \backslash \mathrm{Do}(f)$.
- The *constraint indicator map* of $f$ is the set-valued map $\chi_f^{co} : S \mapsto\!\!\!\!\to \mathbb{R}$ such that $\chi_f^{co}(s) = \emptyset$ if $f(s) \leq 0$ or $s \in S \backslash \mathrm{Do}(f)$, and $\chi_f^{co}(s) = [0, +\infty\,[$ if $f(s) > 0$.

***Cones and multicones.*** A *cone* in a FDRLS $X$ is a nonempty subset $C$ of $X$ such that $r \cdot c \in C$ whenever $c \in C, r \in \mathbb{R}$ and $r \geq 0$. If $X$ is a FDRLS, a *multicone* in $X$ is a nonempty set of convex cones in $X$. A multicone $\mathcal{C}$ is *convex* if every member $C$ of $\mathcal{C}$ is convex.

***Polars.*** Let $X$ be a FDNRLS. The *polar* of a cone $C \subseteq X$ is the closed convex cone $C^\dagger = \{\lambda \in X^\dagger : \lambda(c) \leq 0 \text{ for all } c \in C\}$. If $\mathcal{C}$ is a multicone in $X$, the *polar* of $\mathcal{C}$ is the set $\mathcal{C}^\dagger = \mathrm{Clos}\Big(\bigcup\{C^\dagger : C \in \mathcal{C}\}\Big)$, so $\mathcal{C}^\dagger$ is a (not necessarily convex) closed cone in $X^\dagger$.

***Boltyanskii approximating cones.*** If $X$ is a FDNRLS, $S \subseteq X$, and $x \in S$, a *Boltyanskii approximating cone to $S$ at $x$* is a convex cone $C$ in $X$ such that there exist an $n \in \mathbb{Z}_+$, a closed convex cone $D$ in $\mathbb{R}^n$, a neighborhood $U$ of $0$ in $\mathbb{R}^n$, a continuous map $F : U \cap D \mapsto S$, and a linear map $L : \mathbb{R}^n \mapsto X$, such that $F(h) = x + L \cdot h + o(\|h\|)$ as $h \to 0$ via values in $D$, and $C = L \cdot D$. A *limiting Boltyanskii approximating cone to $S$ at $x$* is a closed convex cone $C$ which is the closure of an increasing union $\bigcup_{j=1}^{\infty} C_j$ such that each $C_j$ is a Boltyanskii approximating cone to $S$ at $x$.

***Some function spaces.*** If $A$, $B$ are sets, we use $fn(A,B)$ to denote the set of all functions from $A$ to $B$. If $X$ is a real normed space and $A$ is a set, then $\mathcal{B}dfn(A,X)$ will denote the set of all bounded functions from $A$ to $X$. The space $\mathcal{B}dfn(A,X)$ is endowed with the norm $\|\cdot\|_{sup}$ given by

$\|f\|_{sup} = \sup\{\|f(t)\| : t \in A\}$. Then $\mathcal{B}dfn(A, X)$ is a Banach space if $X$ is a Banach space.

If, in addition, $A$ is a topological space, then $C^0(A, X)$ denotes the space of all bounded continuous functions from $A$ to $B$, endowed with the norm $\|\cdot\|_{sup}$. It is clear that $C^0(A, X)$ is a closed subspace of $\mathcal{B}dfn(A, X)$, so in particular $C^0(A, X)$ is a Banach space if $X$ is a Banach space.

***Tubes.*** If $X$ is a FDNRLS, $a, b \in \mathbb{R}$, $a \leq b$, $\xi \in C^0([a, b], X)$ and $\delta > 0$, we use $\mathcal{T}^X(\xi, \delta)$ to denote the *$\delta$-tube about $\xi$ in $X$*, defined by

$$\mathcal{T}^X(\xi, \delta) \stackrel{\text{def}}{=} \{(x, t) : x \in X, a \leq t \leq b, \|x - \xi(t)\| \leq \delta\}. \tag{2.1.3}$$

***Vector fields, trajectories, and flow maps.*** If $X$ is a FDNRLS, a *ppd time-varying vector field* on $X$ is a ppd map $X \times \mathbb{R} \ni (x, t) \hookrightarrow f(x, t) \in X$. A *trajectory*, or *integral curve*, of a ppd time-varying vector field $f$ on $X$ is a locally absolutely continuous map $\xi : I \mapsto X$, defined on a nonempty real interval $I$, such that for almost all $t \in I$ the following two conditions hold: (i) $(\xi(t), t) \in \mathrm{Do}(f)$, and (ii) $\dot{\xi}(t) = f(\xi(t), t)$. If $f$ is a ppd time-varying vector field on $X$, then $\mathrm{Traj}\,(f)$ will denote the set of all integral curves $\xi : I_\xi \mapsto X$ of $f$. If $S$ is a subset of $X \times \mathbb{R}$, then $\mathrm{Traj}\,(f, S)$ will denote the set of $\xi \in \mathrm{Traj}\,(f)$ such that $(\xi(t), t) \in S$ for all $t \in I_\xi$, and $\mathrm{Traj}_c(f, S)$ will denote the set of $\xi \in \mathrm{Traj}\,(f, S)$ whose domain $I_\xi$ is a compact interval.

The *flow map* of a ppd time-varying vector field $X \times \mathbb{R} \ni (x, t) \hookrightarrow f(x, t) \in X$ is the set-valued map $\Phi^f : \mathbb{R} \times \mathbb{R} \times X \mapsto\!\!\!\!\to X$ that assigns to each triple $(t, s, x) \in \mathbb{R} \times \mathbb{R} \times X$ the set $\Phi^f(t, s, x) = \{\xi(t) : \xi \in \mathrm{Traj}\,(f), \xi(s) = x\}$.

***Functions of bounded variation.*** Assume that $X$ is a real normed space, $a, b \in \mathbb{R}$, and $a < b$.

**Definition 2.2** *A function $\varphi \in fn([a, b], X)$ is **of bounded variation** if there exists a nonnegative real number $C$ such that $\sum_{j=1}^{m} \|\varphi(t_j) - \varphi(s_j)\| \leq C$ whenever $m \in \mathbb{N}$ and the finite sequences $\{s_j\}_{j=1}^m$, $\{t_j\}_{j=1}^m$ are such that $a \leq s_1 \leq t_1 \leq s_2 \leq t_2 \leq \cdots \leq s_m \leq t_m \leq b$.* □

We use $bvfn([a, b], X)$ to denote the set of all $\varphi \in fn([a, b], X)$ that are of bounded variation, and define the *total variation norm* $\|\varphi\|_{tv}$ of a function $\varphi \in fn([a, b], X)$ by letting $\|\varphi\|_{tv} = \|\varphi(b)\| + C(\varphi)$, where $C(\varphi)$ is the smallest $C$ having the property of Definition 2.2. Also, we let $bvfn^{0,b}([a, b], X)$ denote the set of all $\varphi \in bvfn([a, b], X)$ such that $\varphi(b) = 0$. Then $\|\varphi\|_{tv} = C(\varphi)$ if $\varphi \in bvfn^{0,b}([a, b], X)$. It is then easy to verify that

**Fact 2.3** *If $X$ is a Banach space, then the space $bvfn([a, b], X)$, endowed with the total variation norm $\|\cdot\|_{tv}$, is a Banach space, and $bvfn^{0,b}([a, b], X)$ is a closed linear subspace of $bvfn([a, b], X)$ of codimension one.* □

**Fact 2.4** *If $X$ is a Banach space and $f \in bvfn(([a, b], X)$, then $\lim_{s\uparrow t} f(s)$ exists for every $t \in\, ]a, b]$, and $\lim_{s\downarrow t} f(s)$ exists for every $t \in [a, b[$.* □

**Remark 2.5** The set $bvfn([a,b],X)$ is clearly a linear subspace of $\mathcal{B}dfn([a,b],X)$. The sup norm and the total variation norm are related by the inequality $\|\varphi\|_{sup} \le \|\varphi\|_{tv}$, which holds whenever $\varphi \in bvfn([a,b],X)$ . On the other hand, $bvfn([a,b],X)$ is clearly *not* closed in $\mathcal{B}dfn([a,b],X)$. ☐

***Measurable spaces and measure spaces.*** A *measurable space* is a pair $(S,\mathcal{A})$ such that $S$ is a set and $\mathcal{A}$ is a $\sigma$-algebra of subsets of $S$.

If $(S,\mathcal{A})$ is a measurable space, then a *nonnegative measure* on $(S,\mathcal{A})$ is a map $\mu : \mathcal{A} \mapsto [0,+\infty]$ that satisfies $\mu(\emptyset) = 0$ and is countably additive (i.e., such that $\mu(\bigcup_{j=1}^{\infty} A_j) = \sum_{j=1}^{\infty} \mu(A_j)$ whenever $\{A_j\}_{j\in\mathbb{N}}$ is a sequence of pairwise disjoint members of $\mathcal{A}$).

A *nonnegative measure space* is a triple $(S,\mathcal{A},\mu)$ is such that $(S,\mathcal{A})$ is a measurable space and $\mu$ is a nonnegative-measure on $(S,\mathcal{A})$. A nonnegative measure space $(S,\mathcal{A},\mu)$ is *finite* if $\mu(A) < \infty$ for all $A \in \mathcal{A}$.

***Measurability of set-valued maps; support functions.*** Assume that $(S,\mathcal{A})$ is a measurable space and $Y$ is a FDNRLS.

**Definition 2.6** *A set-valued map $\Lambda : S \mapsto\!\!\!\rightarrow Y$ is said to be* ***measurable*** *if the set $\{s \in S : \Lambda(s) \cap \Omega \neq \emptyset\}$ belongs to $\mathcal{A}$ for every open subset $\Omega$ of $Y$.* ☐

If $\Lambda$ has compact values, then we define the *support function* of $\Lambda$ to be the function $\sigma_\Lambda : S \times Y^\dagger \mapsto \mathbb{R}$ given by

$$\sigma_\Lambda(s,x) = \sup\{\langle x,y\rangle : y \in \Lambda(s)\} \ \text{ for } \ x \in Y^\dagger\,,\ \ s \in S. \tag{2.1.4}$$

(If $\Lambda(s) = \emptyset$ then we define $\sigma_\Lambda(s,y) = -\infty$.) The following fact is well known.

**Lemma 2.7** *Assume that $(S,\mathcal{A})$ is a measurable space, $Y$ is a FDNRLS, and $\Lambda : S \mapsto\!\!\!\rightarrow Y$ is a set-valued map with compact convex values. For each $y \in Y^\dagger$, let $\psi_y(s) = \sigma_\Lambda(s,y)$. Then $\Lambda$ is measurable if and only if the function $\psi_y : S \mapsto \mathbb{R} \cup \{-\infty\}$ is measurable for every $y \in Y^\dagger$.* ☐

***Integrable boundedness of set-valued maps.*** Assume that $(S,\mathcal{A},\nu)$ is a nonnegative measure space.

**Definition 2.8** *A* ***$\nu$-integrable bound*** *for a set-valued map $\Lambda : S \mapsto\!\!\!\rightarrow Y$ is a nonnegative $\nu$-integrable function $k : S \mapsto [0,+\infty]$ having the property that $\Lambda(s) \subseteq \{y \in Y : \|y\| \le k(s)\}$ for $\nu$-almost all $s \in S$. The map $\Lambda$ is said to be* ***$\nu$-integrably bounded*** *if there exists a $\nu$-integrable bound for $\Lambda$.* ☐

### 2.2 Generalized Jacobians, derivate containers, and Michel-Penot subdifferentials.

For future use, we will now review the definitions and basic properties of three classical "non-smooth" notions of set-valued derivative, namely, Clarke generalized Jacobians, Warga derivate containers, and Michel-Penot derivatives.

***Generalized Jacobians.*** Assume that $X$, $Y$ are FDNRLSs, $\Omega$ is an open subset of $X$, $F : \Omega \mapsto Y$ is a Lipschitz-continuous map, and $\bar{x}_* \in \Omega$.

**Definition 2.9** *The* ***Clarke generalized Jacobian*** *of* $F$ *at* $\bar{x}_*$ *is the subset* $\partial F(\bar{x}_*)$ *of* $Lin(X, Y)$ *defined as follows:*

- $\partial F(\bar{x}_*)$ *is the convex hull of the set of all limits* $L = \lim_{j\to\infty} DF(x_j)$*, for all sequences* $\{x_j\}_{j\in\mathbb{N}}$ *in* $\Omega$ *such that (1)* $\lim_{j\to\infty} x_j = \bar{x}_*$*, (2)* $F$ *is classically differentiable at* $x_j$ *for all* $j \in \mathbb{N}$*, and (3) the limit* $L$ *exists.* □

***Warga derivate containers.*** Assume that $X$, $Y$ are FDNRLSs, $\Omega$ is an open subset of $X$, $F : \Omega \mapsto Y$, and $\bar{x}_* \in \Omega$.

**Definition 2.10** *A* ***Warga derivate container*** *of* $F$ *at* $\bar{x}_*$ *is a compact subset* $\Lambda$ *of* $Lin(X, Y)$ *such that*

- *For every positive number* $\delta$ *there exist (1) an open neighborhood* $U_\delta$ *of* $\bar{x}_*$ *such that* $U_\delta \subseteq \Omega$*, and (2) a sequence* $\{F_j\}_{j\in\mathbb{N}}$ *of* $Y$*-valued functions of class* $C^1$ *on* $U_\delta$*, such that (i)* $\lim_{j\to\infty} F_j = F$ *uniformly on* $U_\delta$*, (ii)* $\operatorname{dist}(DF_j(x), \Lambda) \le \delta$ *for every* $(j, x) \in \mathbb{N}\times U_\delta$*.* □

***Michel-Penot subdifferentials.*** Assume that $X$ is a FDNRLS, $\Omega$ is an open subset of $X$, $f : \Omega \mapsto \mathbb{R}$ is a Lipschitz-continuous function, and $\bar{x}_* \in \Omega$. For $h \in X$, define

$$d^o f(\bar{x}_*, h) = \sup_{k\in X} \limsup_{t\downarrow 0} t^{-1}\Big(f(\bar{x}_* + t(k+h)) - f(\bar{x}_* + tk)\Big)\,, \qquad (2.2.1)$$

so that $X \ni h \mapsto d^o f(\bar{x}_*, h) \in \bar{\mathbb{R}}$ is a convex positively homogeneous function.

**Definition 2.11** *The* ***Michel-Penot subdifferential*** *of* $f$ *at* $\bar{x}_*$ *is the set* $\partial^o f(\bar{x}_*)$ *of all linear functionals* $\omega \in X^\dagger$ *having the property that the inequality* $d^o f(\bar{x}_*, h) \ge \langle \omega, h\rangle$ *holds whenever* $h \in X$*.* □

### 2.3 Finitely additive measures.

If $a, b \in \mathbb{R}$, $a < b$, and $X$ is a FDNRLS, we use $\mathcal{P}c([a,b], X)$ to denote the set of all piecewise constant $X$-valued functions on $[a,b]$, so that $f \in \mathcal{P}c([a,b], X)$ iff $f : [a,b] \mapsto X$ and there exists a finite partition $\mathcal{P}$ of $[a,b]$ into intervals such that $f$ is constant on each $I \in \mathcal{P}$. We let $\overline{\mathcal{P}c}([a,b], X)$ denote the set of all uniform limits of members of $\mathcal{P}c([a,b], X)$, so $\overline{\mathcal{P}c}([a,b], X)$ is a Banach space, endowed with the sup norm. Furthermore, $\overline{\mathcal{P}c}([a,b], X)$ is exactly the space of all $f : [a,b] \mapsto X$ such that the left limit $f(t-) = \lim_{s\to t, s<t} f(s)$ exists for all $t \in\, ]a,b]$, and the right limit $f(t+) = \lim_{s\to t, s>t} f(s)$ exists for all $t \in [a, b\,[$.

We define $\overline{\mathcal{P}c}_0([a,b], X)$ to be the set of all $f \in \overline{\mathcal{P}c}([a,b], X)$ that vanish on the complement of a countable (i.e., finite or countably infinite) set. Then $\overline{\mathcal{P}c}_0([a,b], X)$ is the closure in $\overline{\mathcal{P}c}([a,b], X)$ of the space $\mathcal{P}c_0([a,b], X)$ of all $f \in \mathcal{P}c([a,b], X)$ such that $f$ vanishes on the complement of a finite set.

We let $pc([a,b],X)$ be the quotient space $\overline{\mathcal{P}c}([a,b],X)/\overline{\mathcal{P}c}_0([a,b],X)$. Then every equivalence class $F \in pc([a,b],X)$ has a unique left-continuous member $F_-$, and a unique right-continuous member $F_+$, and of course $F_- \equiv F_+$ on the complement of a countable set. So $pc([a,b],X)$ can be identified with the set of all pairs $(f_-,f_+)$ of $X$-valued functions on $[a,b]$ such that $f_-$ is left-continuous, $f_+$ is right-continuous, and $f_- \equiv f_+$ on the complement of a countable set.

If $X$ is a FDNRLS, then we use $bvadd([a,b],X)$ to denote the dual space $pc([a,b],X^\dagger)^\dagger$ of $pc([a,b],X^\dagger)$. A *reduced additive $X$-valued interval function of bounded variation* on $[a,b]$ is a member of $bvadd([a,b],X)$. A measure $\mu \in bvadd([a,b],X)$ gives rise to a set function $\hat{\mu} : INT([a,b]) \mapsto X$, defined by $\langle \hat{\mu}(I), y\rangle = \mu(\chi_I^y)$ for $y \in X^\dagger$, where $\chi_I^y(t) = 0$ if $t \notin I$ and $\chi_I^y(t) = y$ if $t \in I$. We then associate to $\mu$ its *cumulative distribution* $cd_\mu$, defined by $cd_\mu(t) = -\hat{\mu}([t,b])$ for $t \in [a,b]$. Then $cd_\mu$ belongs to the space $bvfn^{0,b}([a,b],X)$ of all functions $\varphi : [a,b] \mapsto X$ that are of bounded variation (cf. Definition 2.2) and such that $\varphi(b) = 0$. The map

$$bvadd([a,b],X) \ni \mu \mapsto cd_\mu \in bvfn^{0,b}([a,b],X)$$

is a bijection. The dual Banach space norm $\|\mu\|$ of a $\mu \in bvadd([a,b],X)$ coincides with $\|cd_\mu\|_{bv}$.

**Remark 2.12** The *non-reduced* additive $X$-valued interval functions of bounded variation on $[a,b]$ are the members of the dual space $\overline{\mathcal{P}c}([a,b],X^\dagger)^\dagger$. Then $bvadd([a,b],X)$ consists of those members of $\overline{\mathcal{P}c}([a,b],X^\dagger)^\dagger$ that vanish on every test function $F \in \overline{\mathcal{P}c}([a,b],X^\dagger)$ such that $F(t) = 0$ for all but finitely many values of $t$. For a reduced interval function $\mu \in bvadd([a,b],X)$, the measure $\hat{\mu}(\{t\})$ of every singleton is equal to zero, because the function $\chi_{\{t\}}^y$ belongs to $\overline{\mathcal{P}c}_0([a,b],X^\dagger)$ for every $y \in X^\dagger$. □

A $\mu \in bvadd([a,b],X)$ is a *left* (resp.*right*) *delta function* if there exist an $x \in X$ and a $t \in ]\,a,b]$ (resp. a $t \in [\,a,b\,[$ ) such that $\mu(F) = \langle F(t-), x\rangle$ (resp. $\mu(F) = \langle F(t+), x\rangle$) for all $F \in pc([a,b],X)$. We call $\mu$ *left-atomic* (resp. *right-atomic*) if it is the sum of a convergent series of left (resp. right) delta functions.

A $\mu \in bvadd([a,b],X)$ is *continuous* if the function $cd_\mu$ is continuous. Every $\mu \in bvadd([a,b],X)$ has a unique decomposition into the sum of a continuous part $\mu_{co}$, a left-atomic part $\mu_{at,-}$ and a right-atomic part $\mu_{at,+}$. (This resembles the usual decomposition of a countably additive measure into the sum of a continuous part and an atomic part. The only difference is that in the finitely additive setting there are left and right atoms rather than just atoms.)

If $Y$ is a FDNRLS, a *bounded $Y$-valued measurable pair on* $[a,b]$ is a pair $(\gamma_-,\gamma_+)$ of bounded Borel measurable functions from $[a,b]$ to $Y$ such that $\gamma_- \equiv \gamma_+$ on the complement of a finite or countable set. If $X,Y,Z$ are FDNRLSs, $Y \times X \ni (y,x) \mapsto \langle y,x\rangle \in Z$ is a bilinear map, $\mu \in bvadd([a,b],X)$, and $\gamma = (\gamma_-, \gamma_+)$ is a bounded $Y$-valued measurable pair on $[a,b]$, then the product measure $\gamma \cdot \mu$ is a member of $bvadd([a,b],Z)$ defined by multiplying the continuous part $\mu_{co}$ by $\gamma_-$ or $\gamma_+$, the left-atomic part by $\gamma_-$, and the right-atomic part by $\gamma_+$. In particular, the product $\gamma \cdot \mu$ is a well defined member of $bvadd([a,b],X)$ whenever $\mu \in bvadd([a,b],\mathbb{R})$ and $\gamma$ is a bounded $X$-valued measurable pair on $[a,b]$.

Finally, we briefly discuss the solutions of an "adjoint" Cauchy problem represented formally as

$$dy(t) = -y(t) \cdot L(t) \cdot dt + d\mu(t)\,, \quad y(b) = \bar{y}\,, \tag{2.3.1}$$

where $\mu$ belongs to $bvadd([a,b],X^\dagger)$, $L \in L^1([a,b],\boldsymbol{L}(X))$, and we are looking for solutions $y(\cdot) \in bvadd([a,b],X^\dagger)$.

This is done by rewriting our Cauchy problem as the integral equation

$$y(t) - V(t) = \int_t^b y(s) \cdot L(s) \cdot ds\,, \qquad \text{where} \quad V = cd_\mu\,. \tag{2.3.2}$$

Equation (2.3.2) is easily seen to have a unique solution $\pi$, given by

$$\pi(t) = \bar{y} \cdot M_L(b,t) - \int_{[t,b]} d\mu(s) \cdot M_L(s,t)\,, \tag{2.3.3}$$

where $M_L : [a,b] \times [a,b] \mapsto \boldsymbol{L}(X)$ is the fundamental solution of $\dot{M} = M \cdot L$, characterized by the identity $M_L(\tau,t) = \mathbb{I}_X + \int_t^\tau L(r) \cdot M_L(r,t)\,dr$.

## 3 Cellina continuously approximable (CCA) maps

The "Cellina continuously approximable" maps constitute a class of set-valued maps whose properties are similar to those of single-valued continuous maps. The most important such property is the fixed point theorem that, for single-valued continuous maps, is known as Brouwer's theorem in the finite-dimensional case, and as Schauder's theorem in the infinite-dimensional case. A class of set-valued maps with some of the desired properties was singled out in the celebrated Kakutani fixed point theorem (for the finite-dimensional case), and its infinite-dimensional generalization due to Fan and Glicksberg. This class, whose members are the upper semicontinuous maps with nonempty compact convex values, turns out to be insufficient for our purposes, because is lacks the crucial property that a composite of two maps belonging to the class also belongs to the class. (For example, if $f : \bar{\mathbb{B}}^n(0,1) \mapsto\!\!\!\!\rightarrow \bar{\mathbb{B}}^n(0,1)$ has nonempty convex values and a compact graph, and $g : \bar{\mathbb{B}}^n(0,1) \mapsto \bar{\mathbb{B}}^n(0,1)$ is single-valued and continuous, then $g$ also has a compact graph and nonempty convex values, so $g$ belongs to the class as well, but $g \circ f$ need not belong to the class, because the image of a convex set under a continuous map need not be convex. And yet it is obvious that $g \circ f$ has to have a fixed point, because the same standard argument used to prove the Kakutani theorem applies here as well: we can find a sequence of single-valued continuous maps $f_j$ that converge to $f$ in an appropriate sense, apply Brouwer's theorem to obtain fixed points $x_j$ of the maps $g \circ f_j$, and then pass to the limit.)

The previous example strongly suggests that there ought to exist a class of maps, larger than that of the Kakutani and Fan-Glicksberg theorems, which is closed under composition and such that the usual fixed point theorems hold. This class was introduced by A. Cellina in a series of papers around 1970 (cf. [3, 4, 5, 6]). We now study it in detail.

### 3.1 Definition and elementary properties

CCA maps are set-valued maps that are limits of single-valued continuous maps in the sense of an appropriate (non-Hausdorff) notion of convergence. We begin by defining this concept of convergence precisely.

***Inward graph convergence.*** If $K$, $Y$ are metric spaces and $K$ is compact, then $SVM_{comp}(K,Y)$ will denote the subset of $SVM(K,Y)$ whose members are the set-valued maps from $K$ to $Y$ that have a compact graph. We say that a sequence $\{F_j\}_{j\in\mathbb{N}}$ of members of $SVM_{comp}(K,Y)$ *inward graph-converges* to an $F \in SVM_{comp}(K,Y)$—and write $F_j \xrightarrow{\text{igr}} F$—if for every open subset $\Omega$ of $K \times Y$ such that $\text{Gr}(F) \subseteq \Omega$ there exists a $j_\Omega \in \mathbb{N}$ such that $\text{Gr}(F_j) \subseteq \Omega$ whenever $j \geq j_\Omega$.

The above notion of convergence is a special case of the following more general idea. Recall that $Comp^0(X)$ is the set of all compact subsets of $X$. Then we can define a topology $\mathcal{T}_{Comp^0(X)}$ on $Comp^0(X)$ by declaring a subset $\mathcal{U}$ of $Comp^0(X)$ to be *open* if for every $K \in \mathcal{U}$ there exists an open subset $U$ of $X$ such that $K \subseteq U$ and $\{J \in Comp^0(X) : J \subseteq U\} \subseteq \mathcal{U}$. (This topology is non-Hausdorff even if $X$ is Hausdorff, because if $J, K \in Comp^0(X)$, $J \subseteq K$, and $J \neq K$, then every neighborhood of $K$ contains $J$.) Inward graph convergence of a sequence $\{F_j\}_{j\in\mathbb{N}}$ of members of $SVM_{comp}(K,Y)$ to an $F \in SVM_{comp}(K,Y)$ is then equivalent to convergence to $\text{Gr}(F)$ of the sets $\text{Gr}(F_j)$ in the topology $\mathcal{T}_{Comp^0(X)}$.

The convergence of sequences and, more generally, of nets, in the space $\mathcal{T}_{Comp^0(X)}$ can be characterized as follows, in terms of the quasidistance $\Delta^{qua}$ defined in (2.1.1).

**Fact 3.1** *Let $(Z, d_Z)$ be a metric space, let $\mathbf{K} = \{K_\alpha\}_{\alpha\in A}$ be a net of members of $Comp^0(Z)$, indexed by a directed set $(A, \preceq_A)$, and let $K \in Comp^0(Z)$. Then the net $\mathbf{K}$ converges to $K$ with respect to $\mathcal{T}_{Comp^0(Z)}$ if and only if* $\lim_\alpha \Delta_Z^{qua}(K_\alpha, K) = 0$. □

Fact 3.1 can be applied in the special case when the metric space $Z$ is a product $X \times Y$, equipped with the distance $d_Z : Z \times Z \mapsto \mathbb{R}_+$ given by

$$d_Z\Big((x,y),(x',y')\Big) = d_X(x,x') + d_Y(y,y')\,. \tag{3.1.1}$$

We then obtain the following equivalent characterization of inward graph convergence.

**Fact 3.2** *Let $X$, $Y$ be metric spaces, with distance functions $d_X$, $d_Y$, let $\mathbf{F} = \{F_\alpha\}_{\alpha\in A}$ be a net of members of $SVM_{comp}(X,Y)$, indexed by a directed set $(A, \preceq_A)$, and let $F \in SVM_{comp}(X,Y)$. Then the net $\mathbf{F}$ converges to $F$ in the inward graph convergence sense (that is, the graphs $\mathrm{Gr}(F_\alpha)$ converge to $\mathrm{Gr}(F)$ in $\mathcal{T}_{Comp^0(X\times Y)}$) if and only if $\lim_\alpha \Delta^{qua}_Z\Big(\mathrm{Gr}(F_\alpha), \mathrm{Gr}(F)\Big) = 0$, where $Z = X \times Y$, equipped with the distance $d_Z$ given by (3.1.1).* □

***Compactly graphed set-valued maps.*** Suppose that $X$ and $Y$ are metric spaces, and $F : X \mapsto\!\!\!\!\twoheadrightarrow Y$. Then $F$ is *compactly graphed* if, for every compact subset $K$ of $X$, the restriction $F\lceil K$ of $F$ to $K$ has a compact graph, i.e., has the property that the set $\mathrm{Gr}(F\lceil K) \stackrel{\mathrm{def}}{=} \{(x,y) : x \in K \wedge y \in F(x)\}$ is compact.

We recall that, if $X$, $Y$ are topological spaces, then a set-valued map $F : X \mapsto\!\!\!\!\twoheadrightarrow Y$ is said to be *upper semicontinuous* if the inverse image of every closed subset $U$ of $Y$ is a closed subset of $X$. It is then easy to see that

**Fact 3.3** *If $X$ and $Y$ are metric spaces and $F : X \mapsto\!\!\!\!\twoheadrightarrow Y$ has compact values, then $F$ is upper semicontinuous if and only if it is compactly graphed.* □

***CCA maps.*** We are now, finally, in a position to define the notion of a "Cellina continuously approximable map."

**Definition 3.4** *Assume that $X$ and $Y$ are metric spaces, and $F : X \mapsto\!\!\!\!\twoheadrightarrow Y$. We say that $F$ is* ***Cellina continuously approximable*** *(abbr. "CCA") if $F$ is compactly graphed and*

- *For every compact subset $K$ of $X$, the restriction $F\lceil K$ is a limit—in the sense of inward graph-convergence—of a sequence of continuous single-valued maps from $K$ to $Y$.* □

We will use the expression $\mathrm{CCA}(X,Y)$ to denote the set of all CCA set-valued maps from $X$ to $Y$. It is easy to see that

**Fact 3.5** *If $f : X \hookrightarrow Y$ is a ppd map, then the following are equivalent:*

*(1) $f \in \mathrm{CCA}(X,Y)$,*
*(2) $f$ is everywhere defined and continuous,*
*(3) $f$ is everywhere defined and compactly graphed.* □

***Composites of CCA maps.*** The following simple observation will play a crucial role in the theory of GDQs and AGDQs.

**Theorem 3.6** *Assume that $X$, $Y$, $Z$ are metric spaces. Let $F \in \mathrm{CCA}(X,Y)$, $G \in \mathrm{CCA}(Y,Z)$. Then the composite map $G \circ F$ belongs to $\mathrm{CCA}(X,Z)$.*

*Proof.* Let $H = G \circ F$. We prove first that $H$ is compactly graphed. Let $K$ be a compact subset of $X$, and let $J = \mathrm{Gr}(H\lceil K)$. A pair $(x,z)$ belongs to $J$ if and only if there exists $y \in Y$ such that $(x,y) \in \mathrm{Gr}(F\lceil K)$

and $(y,z) \in \mathrm{Gr}(G)$. Let $Q = \pi(\mathrm{Gr}(F\lceil K))$, where $\pi$ is the projection $X \times Y \ni (x,y) \mapsto y \in Y$. Then $(x,z) \in J$ iff there exists $y \in Q$ such that $(x,y) \in \mathrm{Gr}(F\lceil K)$ and $(y,z) \in \mathrm{Gr}(G\lceil Q)$. Equivalently, $(x,z) \in J$ iff there exists a point $p = (x,y,\tilde{y},z) \in S$ such that $\Pi(p) = (x,z)$ and $p \in A$, where $A = \{(x,y,\tilde{y},z) \in X \times Y \times Y \times Z : y = \tilde{y}\}$, $S = \mathrm{Gr}(F\lceil K) \times \mathrm{Gr}(G\lceil Q)$, and $\Pi$ is the projection $X \times Y \times Y \times Z \in (x,y,\tilde{y},z) \mapsto (x,z) \in X \times Z$.

So $J = \Pi(S \cap A)$. Since $S$ is compact and $A$ is closed in $X \times Y \times Y \times Z$, the set $S \cap A$ is compact, so $J$ is compact, since $\Pi$ is continuous. Hence $H$ is compactly graphed.

We now fix a compact subset $K$ of $X$, let $h = H\lceil K$, and show that there exists a sequence $\{h_j\}_{j\in\mathbb{N}}$ of continuous maps from $K$ to $Z$ such that $h_j \xrightarrow{\mathrm{igr}} h$. For this purpose, we let $f = F\lceil K$, and use the fact that $F$ is a CCA map to construct a sequence $\{f_j\}_{j\in\mathbb{N}}$ of continuous maps from $K$ to $Y$ such that $f_j \xrightarrow{\mathrm{igr}} f$ as $j \to \infty$. Then the set $B = \mathrm{Gr}(f) \cup \left(\bigcup_{j=1}^{\infty} \mathrm{Gr}(f_j)\right)$ is clearly compact. (*Proof:* Let $\mathcal{O}$ be a set of open subsets of $X \times Y$ such that $B \subseteq \bigcup\{\Omega : \Omega \in \mathcal{O}\}$. Then $\mathrm{Gr}(f) \subseteq \bigcup\{\Omega : \Omega \in \mathcal{O}\}$. Since $\mathrm{Gr}(f)$ is compact, we may pick a finite subset $\mathcal{O}_0$ of $\mathcal{O}$ such that $\mathrm{Gr}(f) \subseteq \bigcup\{\Omega : \Omega \in \mathcal{O}_0\}$. Since the set $\bigcup\{\Omega : \Omega \in \mathcal{O}_0\}$ is open and contains $\mathrm{Gr}(f)$, there exists a $j^* \in \mathbb{N}$ such that $\mathrm{Gr}(f_j) \subseteq \bigcup\{\Omega : \Omega \in \mathcal{O}_0\}$ whenever $j > j^*$. For $j = 1, \ldots, j^*$, use the compactness of $\mathrm{Gr}(f_j)$ to pick a finite subset $\mathcal{O}_j$ of $\mathcal{O}$ such that $\mathrm{Gr}(f_j) \subseteq \bigcup\{\Omega : \Omega \in \mathcal{O}_j\}$. Let $\hat{\mathcal{O}} = \bigcup\{\mathcal{O}_j : j = 0, \ldots j^*\}$. Then $\hat{\mathcal{O}}$ is a finite subset of $\mathcal{O}$, and $B \subseteq \bigcup\{\Omega : \Omega \in \hat{\mathcal{O}}\}$.)

Let $C = \pi(B)$, where $\pi$ is the projection defined above. Then $C$ is a compact subset of $Y$, and the fact that $G$ is a CCA map implies that there exists a sequence $\{g_j\}_{j\in\mathbb{N}}$ of continuous maps $g_j : C \mapsto Z$ such that $g_j \xrightarrow{\mathrm{igr}} g$, where $g = G\lceil C$.

We now define $h_j = g_j \circ f_j$, and begin by observing that the $h_j$ are well defined continuous maps from $K$ to $Z$. (The reason that $h_j$ is well defined is that if $x \in K$, then $(x, f_j(x)) \in \mathrm{Gr}(f_j) \subseteq B$, so $(x, f_j(x)) \in B$, and then $f_j(x) \in C$, so $g_j(f_j(x))$ is defined. The continuity of $h_j$ then follows because it is a composite of continuous maps.)

To conclude the proof, we have to establish that $h_j \xrightarrow{\mathrm{igr}} h$. Let us first define $\alpha_j = \sup\{\Xi(x,z) : (x,z) \in \mathrm{Gr}(h_j)\}$, where $\Xi$ is the map given by $\Xi(x,z) = \inf\{d(x,\tilde{x}) + d(z,\tilde{z}) : (\tilde{x},\tilde{z}) \in \mathrm{Gr}(h)\}$. We want to show that $\alpha_j \to 0$ as $j \to \infty$. Suppose not. Then by passing to a subsequence we may assume that $\alpha_j \geq 2\bar{\alpha}$ for all $j$, for some strictly positive $\bar{\alpha}$. For each $j$, pick $(x_j, z_j) \in \mathrm{Gr}(h_j)$ such that $\Xi(x_j, z_j) \geq \bar{\alpha}$. Let $y_j = f_j(x_j)$, so $z_j = g_j(y_j)$. The point $(x_j, y_j)$ then belongs to $\mathrm{Gr}(f_j)$, so $\Theta(x_j, y_j) \to 0$, where $\Theta$ was defined above. Hence we can find $(\tilde{x}_j, \tilde{y}_j) \in \mathrm{Gr}(f)$ such that $d(x_j, \tilde{x}_j) + d(y_j, \tilde{y}_j) \to 0$. Similarly, we can define $\hat{\Theta}(y,z) = \inf\{d(y,\tilde{y}) + d(z,\tilde{z}) : (\tilde{y},\tilde{z}) \in \mathrm{Gr}(g)\}$, and conclude that $\hat{\Theta}(y_j, z_j) \to 0$, since $g_j \xrightarrow{\mathrm{igr}} g$, so we can find points $(\tilde{y}_j^{\#}, \tilde{z}_j)$, belonging to $\mathrm{Gr}(g)$, such that $d(y_j, \tilde{y}_j^{\#}) + d(z_j, \tilde{z}_j) \to 0$. So all four quantities $d(x_j, \tilde{x}_j)$,

$d(y_j,\tilde{y}_j)$, $d(y_j,\tilde{y}_j^{\#})$, and $d(z_j,\tilde{z}_j)$, go to 0. Since $\mathrm{Gr}(f)$ and $\mathrm{Gr}(g)$ are compact we may assume, after passing to a subsequence, that the $(\tilde{x}_j,\tilde{y}_j)$ converge to a limit $(\tilde{x},\tilde{y})\in\mathrm{Gr}(f)$, and the $(\tilde{y}_j^{\#},\tilde{z}_j)$ converge to a limit $d(\tilde{y}^{\#},\tilde{z})\in\mathrm{Gr}(g)$. Since $d(y_j,\tilde{y}_j)\to 0$ and $d(y_j,\tilde{y}_j^{\#})\to 0$, we have $d\tilde{y}_j,\tilde{y}_j^{\#})\to 0$, so $\tilde{y}=\tilde{y}^{\#}$. So $(\tilde{x},\tilde{y})\in\mathrm{Gr}(F)$ and $(\tilde{y},\tilde{z})\in\mathrm{Gr}(G)$, from which it follows that $(\tilde{x},\tilde{z})\in\mathrm{Gr}(H)$. But $d(x_j,\tilde{x}_j)\to 0$ and $\tilde{x}_j\to\tilde{x}$, so $d(x_j,\tilde{x})\to 0$. Similarly, $d(z_j,\tilde{z})\to 0$. Hence $\Xi(x_j,z_j)\to 0$ contradicting the inequalities $\Xi(x_j,z_j)\geq\bar{\alpha}>0$. So $\alpha_j\to 0$, and our proof is complete. □

### 3.2 Fixed point theorems for CCA maps

***The space of compact connected subsets of a compact metric space.*** Recall that, if $X$ is a metric space, then $\mathit{Comp}(X)$ denotes the set of all nonempty compact subsets of $X$. The Hausdorff distance $\Delta_X$ was introduced in Definition 2.1. We write $\mathit{Comp}_c(X)$ to denote the set of all connected members of $\mathit{Comp}(X)$. We will need the following fact about $\mathit{Comp}(X)$.

**Proposition 3.7** *Let $X$ be a compact metric space. Then (I) $(\mathit{Comp}(X),\Delta_X)$ is compact, and (II) $\mathit{Comp}_c(X)$ is a closed subset of $\mathit{Comp}(X)$.*

*Proof.* We first prove (I). Let $X$ be compact, and let $D$ be the diameter of $X$, that is, $D=\max\{d_X(x,x'):x,x'\in X\}$. Let $\{K_j\}_{j\in\mathbb{N}}$ be a sequence in $\mathit{Comp}(X)$. For each $j\in\mathbb{N}$, let $\varphi_j:X\mapsto\mathbb{R}$ be the function given by $\varphi_j(x)=d_X(x,K_j)$. Then each $\varphi_j$ is a Lipschitz function on $X$, with Lipschitz constant 1. Furthermore, the bounds $0\leq\varphi_j(x)\leq D$ clearly hold. Hence $\{\varphi_j\}_{j\in\mathbb{N}}$ is a uniformly bounded equicontinuous sequence of continuous real-valued functions on the compact space $X$. Therefore the Ascoli-Arzelà theorem implies that there exist an infinite subset $J$ of $\mathbb{N}$ and a continuous function $\varphi:X\mapsto\mathbb{R}$ such that the $\varphi_j$ converge uniformly to $\varphi$ as $j\to\infty$ via values in $J$. Define $K=\{x:\varphi(x)=0\}$. Then $K$ is a compact subset of $X$.

Let us show that $K\neq\emptyset$. For this purpose, use the fact that each $K_j$ is nonempty to find a member $x_j$ of $K_j$. Since $X$ is compact, there exists an infinite subset $J'$ of $J$ such that the limit $x=\lim_{j\to\infty,j\in J'}x_j$ exists. Since $\varphi_j(x_j)=0$, and $\varphi_j\to\varphi$ uniformly, it follows that $\varphi(x)=0$, so $x\in K$, proving that $K\neq\emptyset$, so that $K\in\mathit{Comp}(X)$.

We now show that $K_j\to_J K$ in the Hausdorff metric, where "$\to_J$" means "converges as $j$ goes to $\infty$ via values in $J$." First, we prove that $\Delta_X^{qua}(K,K_j)\to_J 0$. By definition, $\Delta_X^{qua}(K,K_j)=\sup\{\varphi_j(x):x\in K\}$. Since $\varphi_j\to_J\varphi$ uniformly on $X$, it follows that $\varphi_j\to_J\varphi$ uniformly on $K$. But $\varphi\equiv 0$ on $K$, so $\varphi_j\to_J 0$ uniformly on $K$, and then $\sup\{\varphi_j(x):x\in K\}\to_J 0$, that is, $\Delta_X^{qua}(K,K_j)\to_J 0$.

Next, we prove that $\Delta_X^{qua}(K_j,K)\to_J 0$. If this was not so, there would exist an infinite subset $J'$ of $J$ and an $\alpha$ such that $\alpha>0$ and

$$\Delta_X^{qua}(K_j,K)\geq\alpha\quad\text{whenever}\quad j\in J'\,. \tag{3.2.1}$$

For each $j \in J'$, pick $x_j \in K_j$ such that $\mathrm{dist}_X(x_j, K) = \Delta_X^{qua}(K_j, K)$. Then, using the compactness of $X$, pick an infinite subset $J''$ of $J'$ such that the limit $x = \lim_{j\to\infty, j\in J''} x_j$ exists. Clearly, $\varphi_j(x_j) = 0$, because $x_j \in K_j$. Hence $\varphi(x) = 0$, so $x \in K$. But $d_X(x_j, x) \to 0$ as $j \to_{J''} \infty$. Hence $\mathrm{dist}_X(x_j, K) \to 0$ as $j \to_{J''} \infty$, contradicting (3.2.1). This proves (I).

We now prove (II). Let $\{K_j\}_{j\in\mathbb{N}}$ be a sequence in $\mathrm{Comp}(X)$ that converges to a $K \in Comp(X)$ and is such that all the $K_j$ are connected. We have to prove that $K$ is connected. Suppose $K$ was not connected. Then there would exist open subsets $U_1, U_2$ of $X$ such that $K \subseteq U_1 \cup U_2$, $U_1 \cap U_2 = \emptyset$, $K \cap U_1 \neq \emptyset$, and $K \cap U_2 \neq \emptyset$. The fact that $K_j \to K$ clearly implies that there exists a $j_*$ such that, if $j \geq j_*$, then (a) $K_j \subseteq U_1 \cup U_2$, (b) $K_j \cap U_1 \neq \emptyset$, and (c) $K_j \cap U_2 \neq \emptyset$. But then, if we pick any $j$ such that $j \geq j_*$, the set $K_j$ is not connected, and we have reached a contradiction. This completes the proof of (II). □

***Connected sets of zeros.*** The following result is a very minor modification of a theorem of Leray and Schauder—stated in [14] and proved by F. Browder in [2]—according to which: *if $K \subseteq \mathbb{R}^n$ is compact convex, $0 \in \mathrm{Int}\, K$, $R > 0$, and $H : K \times [0, R] \mapsto \mathbb{R}^n$ is a continuous map such that $H(x, 0) = x$ whenever $x \in K$ and $H$ never vanishes on $\partial K \times [0, R]$, then there exists a compact connected subset $Z$ of $K \times [0, R]$ such that $H(x, t) = 0$ whenever $(x, t) \in Z$, and the intersections $Z \cap (K \times \{0\})$, $Z \cap (K \times \{R\})$ are nonempty.*

Our version allows $H$ to be a set-valued CCA map, and in addition allows 0 to belong to the boundary of $K$, but requires that 0 be a limit of interior points $v_j$ such that $H$ never takes the value $v_j$ on $\partial K \times [0, R]$.

**Theorem 3.8** *Let $n \in \mathbb{Z}_+$, and let $K$ be a compact convex subset of $\mathbb{R}^n$. Assume that $R > 0$ and $H : K \times [0, R] \mapsto\!\!\!\!\rightarrow \mathbb{R}^n$ is a CCA map. Assume, moreover, that*

*(1)* *$H(x, 0) = \{x\}$ whenever $x \in K$,*
*(2)* *there exists a sequence $\{v_j\}_{j\in\mathbb{N}}$ of interior points of $K$ such that*
  *(2.1)* $\lim_{j\to\infty} v_j = 0$,
  *(2.2)* *$H(x, t) \neq v_j$ whenever $x \in \partial K$, $t \in [0, R]$, $j \in \mathbb{N}$.*

*Then there exists a compact connected subset $Z$ of $K \times [0, R]$ such that*

*(a)* *$0 \in H(x, t)$ whenever $(x, t) \in Z$,*
*(b)* $Z \cap (K \times \{0\}) \neq \emptyset$,
*(c)* $Z \cap (K \times \{R\}) \neq \emptyset$.

**Remark 3.9** If 0 is an interior point of $K$, and $H$ never takes the value 0 on $\partial K \times [0, R]$, then Hypothesis (2) is automatically satisfied, since in that case we can take $v_j = 0$. If in addition $H$ is single-valued, then Theorem 3.8 specializes to the result of [14] and [2]. □

**Remark 3.10** Any point $(\xi, \tau)$ of intersection of $Z \cap (K \times \{0\})$ must satisfy $\tau = 0$ and $0 \in H(\xi, 0)$. Since $H(\xi, 0) = \{\xi\}$, $\xi$ must be 0. So Conclusion (b) is equivalent to the assertion that $(0, 0) \in Z$. □

*Proof of Theorem 3.8.* Pick a sequence $\{H_k^1\}_{k\in\mathbb{N}}$ of ordinary continuous maps $H_k^1 : K\times[0,R]\mapsto\mathbb{R}^n$ such that $H_k^1 \xrightarrow{\text{igr}} H$ as $k\to\infty$. Then, for each $k$, pick a sequence $\{H_{k,\ell}^2\}_{k\in\mathbb{N}}$ of polynomial maps $H_{k,\ell}^2 : \mathbb{R}^n\times\mathbb{R}\mapsto\mathbb{R}^n$ such that

$$\sup\{\|H_{k,\ell}^2(x,t)-H_k^1(x,t)\| : (x,t)\in K\times[0,R]\}\le 2^{-\ell}\,.$$

Let $H_k^3 = H_{k,k}^2$, and define $H_k^4(x,t) = H_k^3(x,t)+x-H_k^3(x,0)$. Then the $H_k^4$ are polynomial maps from $\mathbb{R}^n\times\mathbb{R}$ to $\mathbb{R}^n$ such that $H_k^4(x,0)=x$ for all $x\in\mathbb{R}^n$. We claim that

$$H_k^4\lceil(K\times[0,R]) \xrightarrow{\text{igr}} H \qquad\text{as}\quad k\to\infty\,. \tag{3.2.2}$$

To prove (3.2.2), we let $\alpha_k = \sup\{\theta_k(\xi,\tau) : (\xi,\tau)\in K\times[0,R]\}$, where

$$\begin{aligned}&\theta_k(\xi,\tau) =\\ &\min\Big\{\|\xi-x\|+|\tau-t|+\|H_k^4(\xi,\tau)-y\| : (x,t)\in K\times[0,R],\ y\in H(x,t)\Big\},\end{aligned} \tag{3.2.3}$$

and show that $\alpha_k\to 0$. Assume that $\alpha_k$ does not go to 0. Then assume, after passing to a subsequence if necessary, that $\alpha_k\ge 3\beta$ for a strictly positive $\beta$. Then we may pick $(\xi_k,\tau_k)\in K\times[0,R]$ such that $\theta_k(\xi_k,\tau_k)\ge 2\beta$ for all $k$. After passing once again to a subsequence, we may assume that the limit $(\bar\xi,\bar\tau)=\lim_{k\to\infty}(\xi_k,\tau_k)$ exists and belongs to $K\times[0,R]$. Then (3.2.3) implies, since $\theta_k(\xi_k,\tau_k)\ge 2\beta$, that $\|\xi_k-\bar\xi\|+|\tau_k-\bar\tau|+\|H_k^4(\xi_k,\tau_k)-y\|\ge 2\beta$ whenever $y\in H(\bar\xi,\bar\tau)$. If $k$ is large enough then $\|\xi_k-\bar\xi\|+|\tau_k-\bar\tau|\le\beta$. So we may assume, after passing to a subsequence, that $\|H_k^4(\xi_k,\tau_k)-y\|\ge\beta$ whenever $y\in H(\bar\xi,\bar\tau)$.

On the other hand, if $y\in H(\bar\xi,\bar\tau)$. then

$$\begin{aligned}\beta &\le \|H_k^4(\xi_k,\tau_k)-y\|\\ &\le \|H_k^4(\xi_k,\tau_k)-H_k^1(\xi_k,\tau_k)\|+\|H_k^1(\xi_k,\tau_k)-y\|\\ &= \|H_k^3(\xi_k,\tau_k)+\xi_k-H_k^3(\xi_k,0)-H_k^1(\xi_k,\tau_k)\|+\|H_k^1(\xi_k,\tau_k)-y\|\\ &= \|H_k^3(\xi_k,\tau_k)-H_k^1(\xi_k,\tau_k)\|+\|\xi_k-H_k^3(\xi_k,0)\|+\|H_k^1(\xi_k,\tau_k)-y\|\\ &= \|H_{k,k}^2(\xi_k,\tau_k)-H_k^1(\xi_k,\tau_k)\|+\|\xi_k-H_{k,k}^2(\xi_k,0)\|+\|H_k^1(\xi_k,\tau_k)-y\|\\ &\le 2^{-k}+\|\xi_k-H_k^1(\xi_k,0)\|+\|H_k^1(\xi_k,0)-H_{k,k}^2(\xi_k,0)\|+\|H_k^1(\xi_k,\tau_k)-y\|\\ &\le 2^{1-k}+\|\xi_k-H_k^1(\xi_k,0)\|+\|H_k^1(\xi_k,\tau_k)-y\|\\ &= 2^{1-k}+\|\xi_k-u_k\|+\|v_k-y\|\,,\end{aligned}$$

where $u_k = H_k^1(\xi_k,0)$, $v_k = H_k^1(\xi_k,\tau_k)$. Since $(\xi_k,0,u_k)\in\mathrm{Gr}(H_k^1)$ and $H_k^1\xrightarrow{\text{igr}} H$, we may pick points $(\tilde\xi_k,\tilde\tau_k,\tilde u_k)\in\mathrm{Gr}(H)$ such that

$$\|\xi_k-\tilde\xi_k\|+\tilde\tau_k+\|u_k-\tilde u_k\|\to 0 \qquad\text{as}\quad t\to\infty\,. \tag{3.2.4}$$

We may then pass to a subsequence and assume that the limit $(\tilde\xi_\infty,\tilde\tau_\infty,\tilde u_\infty)$ of the sequence $\{(\tilde\xi_k,\tilde\tau_k,\tilde u_k)\}_{k\in\mathbb{N}}$ exists and belongs to $\mathrm{Gr}(H)$. Then (3.2.4) implies that $\xi_k \to \tilde{\xi}_\infty$ (from which it follows that $\tilde{\xi}_\infty = \bar{\xi}$), $\tilde{\tau}_\infty = 0$, and, finally $\tilde{u}_\infty = \lim_{k\to\infty} u_k = \lim_{k\to\infty} H^1_k(\xi_k, 0)$.

Since $(\bar{\xi}, 0, \tilde{u}_\infty) = (\tilde{\xi}_\infty, \tilde{\tau}_\infty, \tilde{u}_\infty) \in \mathrm{Gr}(H)$, we conclude that $\tilde{u}_\infty \in H(\bar{\xi}, 0)$, so $\tilde{u}_\infty = \bar{\xi}$. Since $\xi_k \to \bar{\xi}$ and $u_k \to \bar{\xi}$, we see that $\lim_{k\to\infty} \|\xi_k - u_k\| = 0$.

Next, since $(\xi_k, \tau_k, v_k) \in \mathrm{Gr}(H^1_k)$ and $H^1_k \xrightarrow{\text{igr}} H$, we may pick points $(\hat{\xi}_k, \hat{\tau}_k, \hat{v}_k) \in \mathrm{Gr}(H)$ such that

$$\|\xi_k - \hat{\xi}_k\| + |\tau_k - \hat{\tau}_k| + \|v_k - \hat{v}_k\| \to 0 \qquad \text{as} \quad t \to \infty\,. \tag{3.2.5}$$

It is then possible to pass to a subsequence and assume that the limit $(\hat{\xi}_\infty, \hat{\tau}_\infty, \hat{v}_\infty) = \lim_{k\to\infty}(\hat{\xi}_k, \hat{\tau}_k, \hat{v}_k)$ exists and belongs to $\mathrm{Gr}(H)$. Then (3.2.5) implies that $\xi_k \to \hat{\xi}_\infty$ (so that $\hat{\xi}_\infty = \bar{\xi}$), $\tau_k \to \hat{\tau}_\infty$ (so that $\hat{\tau}_\infty = \bar{\tau}$), and $\hat{v}_\infty = \lim_{k\to\infty} v_k = \lim_{k\to\infty} H^1_k(\xi_k, \tau_k)$ (so that $\|v_k - \hat{v}_\infty\| \to 0$ as $k \to \infty$). Since $(\bar{\xi}, \bar{\tau}, \hat{v}_\infty) = (\hat{\xi}_\infty, \hat{\tau}_\infty, \hat{v}_\infty) \in \mathrm{Gr}(H)$, we conclude that $\hat{v}_\infty \in H(\bar{\xi}, \bar{\tau})$. Hence we can apply the inequality $\beta \le 2^{1-k} + \|\xi_k - u_k\| + \|v_k - y\|$ with $y = \hat{v}_\infty$, and conclude that

$$\beta \le 2^{1-k} + \|\xi_k - u_k\| + \|v_k - \hat{v}_\infty\|\,. \tag{3.2.6}$$

However, we know that $\lim_{k\to\infty} \|\xi_k - u_k\| = 0$, and $\lim_{k\to\infty} \|v_k - \hat{v}_\infty\| = 0$. So the right-hand side of (3.2.6) goes to zero as $k \to \infty$, contradicting the fact that $\beta > 0$. This contradiction completes the proof of (3.2.2).

The set

$$Q = H(\partial K \times [0, R]) = \{y \in \mathbb{R}^n : (\exists x \in \partial K)(\exists t \in [0, R])(y \in H(x, t))\}$$

is compact, and our hypotheses imply that the points $v_j$ do not belong to $Q$. Let $Q_k = H^4_k(\partial K \times [0, R])$, so $Q_k$ is also compact. We claim that

(\$) *for every* $j \in \mathbb{N}$ *there exists a* $\kappa(j) \in \mathbb{N}$ *such that* $v_j \notin Q_k$ *whenever* $k \ge \kappa(j)$.

To see this, suppose that $j$ is such that $v_j \in Q_k$ for infinitely many values of $k$. Then we may assume, after passing to a subsequence, that $v_j \in Q_k$ for all $k$. Let $v_j = H^4_k(x_k, t_k)$, $x_k \in \partial K$, $t_k \in [0, R]$. Since $H^4_k \lceil (K \times [0, R]) \xrightarrow{\text{igr}} H$, we may pick $(\tilde{x}_k, \tilde{t}_k, \tilde{v}_k) \in \mathrm{Gr}(H)$ such that $\|x_k - \tilde{x}_k\| + \|t_k - \tilde{t}_k\| + \|v_j - \tilde{v}_k\| \to 0$. Since $\mathrm{Gr}(H)$ is compact, we may pass to a subsequence and assume that the limit $(\tilde{x}_\infty, \tilde{t}_\infty, \tilde{v}_\infty) = \lim_{k\to\infty}(\tilde{x}_k, \tilde{t}_k, \tilde{v}_k)$ exists and belongs to $\mathrm{Gr}(H)$. But then $\tilde{x}_\infty = \lim_{k\to\infty} x_k$, so in particular $\tilde{x}_\infty \in \partial K$, because $x_k \in \partial K$, and $\tilde{y}_\infty = \lim_{k\to\infty} t_k$. In addition, $\tilde{v}_k = v_j$. So $v_j \in H(\tilde{x}_\infty, \tilde{t}_\infty)$ and $(\tilde{x}_\infty, \tilde{t}_\infty) \in \partial K \times [0, R]$. Hence $v_j \in Q$, and we have reached a contradiction, proving (\$).

We now pick, for each $j$, an index $k(j)$ such that $k(j) \ge \kappa(j)$ and $k(j) \ge j$, and let $H^5_j = H^4_{k(j)}$. Then each $H^5_j$ is a polynomial map such that $H^5_j(x, 0) = x$ whenever $x \in \mathbb{R}^n$, and $H^5_j(x, t) \ne v_j$ whenever $(x, t)$ belongs to $\partial K \times [0, R]$. Furthermore, $H^5_j \lceil (K \times [0, R]) \xrightarrow{\text{igr}} H$ as $j \to \infty$. Since the set

$P_j = H_j^5(\partial K \times [0,R])$ is compact, $v_j \notin P_j$, and $v_j \in \mathrm{Int}(K)$, we may pick for each $j$ an $\varepsilon_j$ such that $0 < \varepsilon_j < 2^{-j}$ with the property that the ball $B_j = \{v \in \mathbb{R}^n : \|v - v_j\| < \varepsilon_j\}$ is a subset of $\mathrm{Int}(K)$ and does not intersect $P_j$. It follows from Sard's theorem that, for any given $j$, almost every $v \in \mathbb{R}^n$ is a regular value of both maps $\mathbb{R}^n \times \mathbb{R} \ni (x,t) \mapsto H_j^5(x,t) \in \mathbb{R}^n$ and $\mathbb{R}^n \ni x \mapsto H_j^5(x,R) \in \mathbb{R}^n$. So we may pick $w_j \in B_j$ which is a regular value of both maps. Since $v_j \to 0$ as $j \to \infty$ and $\|w_j - v_j\| < \varepsilon_j < 2^{-j}$, we can conclude that $\lim_{j\to\infty} w_j = 0$.

We now fix a $j$. Let $S = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : H_j^5(x,t) = w_j\}$. Then $S$ is the set of zeros of the polynomial map

$$\mathbb{R}^n \times \mathbb{R} \ni (x,t) \mapsto H_j^5(x,t) - w_j \in \mathbb{R}^n \,,$$

which does not have 0 as a regular value. It follows that $S$ is a closed embedded one-dimensional submanifold of $\mathbb{R}^n \times \mathbb{R}$, so each connected component of $S$ is a closed embedded one-dimensional submanifold of $\mathbb{R}^n \times \mathbb{R}$ which is diffeomorphic to $\mathbb{R}$ or to the circle $\mathbb{S}^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. Since $H_j^5(w_j,0) = w_j$, the point $(w_j,0)$ belongs to a connected component $C$ of $S$. Since $C$ is diffeomorphic to $\mathbb{R}$ or $\mathbb{S}^1$, the set $\mathcal{X}$ of all smooth vector fields $X$ on $C$ such that $\|X(x)\| = 1$ for every $x \in C$ has exactly two members. Fix an $X \in \mathcal{X}$, so the other member of $\mathcal{X}$ is $-X$. The vector $X(w_j,0)$ is then tangent to $C$ at $(w_j,0)$, and therefore belongs to the kernel of $DH_j^5(w_j,0)$. On the other hand, the differential at $w_j$ of the map $\mathbb{R}^n \ni x \mapsto H_j^5(x,0) \in \mathbb{R}^n$ is the identity map, which is injective. It follows that the vector $X(w_j,0)$ is not tangent to $\mathbb{R}^n \times \{0\}$. Hence $X(w_j,0) = (\omega,r)$, with $\omega \in \mathbb{R}^n$, $r \in \mathbb{R}$, and $r \neq 0$. We may then assume, after relabeling $-X$ as $X$, if necessary, that $r > 0$.

Next, still keeping $j$ fixed, we let $\gamma_q$ be, for each $q \in C$, the maximal integral curve of $X$ such that $\gamma_q(0) = q$. Then each $\gamma_q$ is defined, in principle, on an interval $I_q = ]\alpha_q, \beta_q[$, where $-\infty \le \alpha_q < 0 < \beta_q \le +\infty$. It turns out, however, that the numbers $\alpha_q$, $\beta_q$ cannot be finite. (For example, suppose $\beta_q$ was finite. Then the limit $p = \lim_{t\uparrow\beta_q} \gamma_q(t)$ would exist, as a limit in the ambient space $\mathbb{R}^n \times \mathbb{R}$, because $\gamma_q$ is Lipschitz. Then $p$ would have to belong to $C$, since $C$ is closed, and $p$ would also be the limit in $C$ of $\gamma_q(t)$ as $t \uparrow \beta_q$, because $C$ is embedded. Hence we would be able to extend $\gamma_q$ to a continuous map from the interval $I_q = ]\alpha_q, \beta_q]$ to $C$ such that $\gamma_q(\beta_q) = p$, and concatenate this with an integral curve $\tilde{\gamma} : [\beta_q, \beta_q + \varepsilon[ \mapsto C$ such that $\tilde{\gamma}(\beta_q) = p$, thereby obtaining an extension of $\gamma_q$ to a larger interval, and contradicting the maximality. of $\gamma_q$. A similar argument works for $\alpha_q$. So $\alpha_q = -\infty$ and $\beta_q = +\infty$.) Therefore $I_q = \mathbb{R}$ for every $q \in C$. Clearly, the set $A_q = \gamma_q(\mathbb{R})$ is an open submanifold of $C$. Furthermore, if $q, q' \in C$ then the sets $A_q$, $A_{q'}$ are either equal or disjoint. Since $C$ is connected, all the sets $A_q$ coincide and are equal to $C$. In particular, if we let $\bar{q} = (w_j,0)$, and write $\gamma = \gamma_{\bar{q}}$, then $\gamma(\mathbb{R}) = C$. Write $\gamma(t) = (\xi(t), \tau(t))$, $\xi(t) \in \mathbb{R}^n$, $\tau(t) \in \mathbb{R}$. Then there exists a positive number $\delta$ such that $\xi(t) \in \mathrm{Int}(K)$ for $-\delta < t < \delta$ and $t\tau(t) > 0$ for $0 < |t| < \delta$. It follows, after making $\delta$ smaller, if necessary, that $\gamma(t)$ is an interior point of $K \times [0,R]$ for $0 < t < \delta$. If $C$ is diffeomorphic to $\mathbb{S}^1$, then $\gamma$ is periodic, so there exists a smallest time $T > 0$ such that $\gamma_{\bar{q}}(T) = \gamma(0)$. Then $\gamma(T-h) = \gamma(-h)$ for small positive $h$, so $\tau(T-h) = \tau(-h) < 0$ for such $h$, implying that $\gamma(t) \notin K \times [0,R]$ when $t < T$ and $T - t$ is small enough. It follows that it is not true that $\gamma(t) \in K \times [0,R]$ for all $t \in [0,T]$. If we let $M = \{t \in [0,T] : \gamma(t) \notin K \times [0,R]\}$, then $M$ is a nonempty relatively open subset of $[0,T]$. Let $T_0 = \inf M$. Then $T_0 > 0$, because $\gamma(t) \in K \times [0,R]$ when $0 \le t < \delta$. Therefore $T_0 \notin M$, because if $T_0 \in M$ then the facts that $M$ is relatively open in $[0,T]$ and $T_0 > 0$ would imply that $T_0 - h \in M$ for small positive $h$, contradicting the fact that $T_0 = \inf M$. It follows that

(&) *$T_0 > 0$, $\gamma(t) \in K \times [0,R]$ for $0 \le t \le T_0$, $\gamma(T_0 + h_\ell) \notin K \times [0,R]$ for a sequence $\{h_\ell\}_{\ell \in \mathbb{N}}$ of positive numbers converging to $0$, and $\gamma$ is an injective map on $[0,T_0]$.*

So we have proved the existence of a $T_0$ for which (&) is true, under the hypothesis that $C$ is diffeomorphic to $\mathbb{S}^1$.

We now show, still keeping $j$ fixed, that a $T_0$ for which (&) holds also exists if $C$ is diffeomorphic to $\mathbb{R}$. To prove this, we define a set $M$ by letting $M = \{t \in [0,+\infty[\, : \gamma(t) \notin K \times [0,R]\}$. Then $M$ is a relatively open subset of $[0,+\infty[$. Furthermore, $M \neq \emptyset$. (*Proof.* If $M$ was empty, then $\gamma(t)$ would belong to $K \times [0,R]$ for all positive $t$. So we could pick a sequence $\{t_\ell\}_{\ell \in \mathbb{N}}$ of positive numbers converging to $+\infty$ and such that $\gamma(t_\ell)$ converges to a limit $q$. But then $q \in C$, because $C$ is closed, and the equality $\lim_{\ell \to \infty} \gamma(t_\ell) = q$ also holds in $C$, because $C$ is embedded. Since $C$ is embedded, there exists a neighborhood $U$ of $q$ in $\mathbb{R}^n \times \mathbb{R}$ which is diffeomorphic to a product $-]\rho,\rho[^{n+1}$ under a map $\Phi : U \mapsto -]\rho,\rho[^{n+1}$ that sends $q$ to $0$ and is such that $\Phi(U \cap C)$ is the arc $A = \{(s,0,\ldots,0) : -\rho < s < \rho\}$. Then $\gamma(t_\ell) \in A$ if $\ell$ is large enough. But $A$ itself, suitably parametrized, is an integral curve $]\alpha,\beta[\, \ni t \mapsto \zeta(t)$ of $X$ such that $\alpha < 0 < \beta$ and $\zeta(0) = q$. It follows that for large enough $\ell$ there exist $h_\ell \in ]\alpha,\beta[$ such that $h_\ell \to 0$ as $\ell \to \infty$ and $\zeta(h_\ell) = \gamma(t_\ell)$. Let $T \in \mathbb{R}$ be such that $\gamma(T) = q$. Then $\gamma(T + h_\ell) = \zeta(h_\ell) = \gamma(t_\ell)$. Since the $t_\ell$ go to $+\infty$, but the $T + h_\ell$ are bounded, there must exist at least one $\ell$ such that $T + h_\ell \neq t_\ell$. Since $\gamma(T + h_\ell) = \zeta(h_\ell) = \gamma(t_\ell)$, it follows that $\gamma$ is periodic and then $C = \gamma(\mathbb{R})$ is compact, contradicting the assumption that $C$ is diffeomorphic to $\mathbb{R}$.) Let $T_0 = \inf M$. Then $T_0 > 0$, because $\gamma(t) \in K \times [0,R]$ when $0 \le t < \delta$. Therefore $T_0 \notin M$, because if $T_0 \in M$ then the facts that $M$ is relatively open in $[0,+\infty[$ and $T_0 > 0$ would imply that $T_0 - h \in M$ for small positive $h$, contradicting the fact that $T_0 = \inf M$. Hence (&) holds.

So we have shown that

(&&) *For every $j$ there exist a positive number $T_0^j$ and a smooth curve $[0,+\infty[\, \ni s \mapsto \gamma^j(s) = (\xi^j(s), \tau^j(s)) \in \mathbb{R}^n \times \mathbb{R}$ such that*

(&&.1) $\gamma^j(0) = (w_j, 0)$;

(&&.2) $\gamma_j(s) \in K \times [0,R]$ *for* $0 \le s \le T_0^j$;

(&&.3) *there exists a sequence $\{h_\ell\}_{\ell \in \mathbb{N}}$ of positive numbers, converging to $0$, such that $\gamma^j(T_0^j + h_\ell) \notin K \times [0,R]$ for every $\ell$;*

(&&.4) $\gamma^j$ *is an injective map on* $[0, T_0^j]$;
(&&.5) $H_j^5(\gamma^j(s)) = w_j$ *for every* $s \in [0, T_0^j]$.

We now let $Z_j = \gamma^j([0, T_0^j])$ for every $j \in \mathbb{N}$. Then each $Z_j$ is a compact connected subset of $K \times [0, R]$, such that $(w_j, 0) \in Z_j$ and the function $H_j^5 - w_j$ vanishes on $Z_j$. Furthermore, we claim that $Z_j \cap (K \times \{R\}) \neq \emptyset$. (*Proof.* We show that $\gamma^j(T_0^j) \in K \times \{R\}$. To see this, observe that (&&.2) implies that $\gamma^j(T_0^j) \in K \times [0, R]$, and (&&.3) implies that $\gamma^j(T_0^j)$ is not an interior point of $K \times [0, R]$, so $\gamma^j(T_0^j) \in \partial(K \times [0, R])$. On the other hand, it is clear that $\partial(K \times [0, R]) = (\partial K \times [0, R]) \cup (K \times \{0, R\})$. But $\gamma^j(T_0^j)$ cannot belong to $\partial K \times [0, R]$, because $H_j^5(\gamma^j(T_0^j)) = w_j$ and $H_j^5$ never takes the value $w_j$ on $\partial K \times [0, R]$ (because $w_j \in B_j$ and $B_j \cap P_j = \emptyset$). So $\gamma^j(T_0^j)$ belongs to $(K \times \{0\}) \cup (K \times \{R\})$. But $\gamma^j(T_0^j)$ cannot belong to $K \times \{0\}$, because $\gamma^j(T_0^j) \neq \gamma^j(0)$ (thanks to (&&.4)), $\gamma^j(0) = (w_j, 0)$, and $(w_j, 0)$ is the only point of $K \times \{0\}$ where $H_j^5 - w_j$ vanishes (since $H_j^5(x, 0) = x$ for all $x$). So $\gamma^j(T_0^j) \in K \times \{R\}$, as desired.)

Since $\mathbf{Z} = \{Z_j\}_{j \in \mathbb{N}}$ is a sequence of nonempty compact connected subsets of $K \times [0, R]$, Proposition 3.7 implies that we may assume, after passing to a subsequence, that $\mathbf{Z}$ converges in the Hausdorff metric to a nonempty compact connected subset $Z$ of $K \times [0, R]$. We now show that $Z$ satisfies the three properties of the conclusion of our theorem. First. we prove that $0 \in H(x, t)$ whenever $(x, t) \in Z$. Pick a point $(x, t)$ of $Z$. Then $\mathrm{dist}((x, t), Z_j)$ goes to 0 as $j \to \infty$. So we may pick $(x_j, t_j) \in Z_j$ such that $x_j \to x$ and $t_j \to t$. Since $(x_j, t_j) \in Z_j$, the point $((x_j, t_j), w_j)$ belongs to $\mathrm{Gr}(H_j^5 \lceil (K \times [0, R]))$. Since $H_j^5 \lceil (K \times [0, R]) \xrightarrow{\mathrm{igr}} H$, we may pick points $((\tilde{x}_j, \tilde{t}_j), \tilde{w}_j)$ in $\mathrm{Gr}(H)$ such that

$$\lim_{j \to \infty} \Big( \|x_j - \tilde{x}_j\| + |t_j - \tilde{t}_j| + \|w_j - \tilde{w}_j\| \Big) = 0 \,. \tag{3.2.7}$$

Since $(x_j, t_j, w_j) \to (x, t, 0)$, (3.2.7) implies that $((\tilde{x}_j, \tilde{t}_j), \tilde{w}_j) \to ((x, t), 0)$. Since $\mathrm{Gr}(H)$ is compact, $((x, t), 0)$ belongs to $\mathrm{Gr}(H)$, so $0 \in H(x, t)$, as desired. Next we show that $Z \cap (K \times \{0\}) \neq \emptyset$. To see this, it suffices to observe that $(w_j, 0) \in Z_j$ and $w_j \to 0$, so $(0, 0) \in Z$. Finally, we prove that $Z_j \cap (K \times \{R\}) \neq \emptyset$. For this purpose, we use the fact that $Z_j \cap (K \times \{R\}) \neq \emptyset$ to pick points $z_j \in K$ such that $(z_j, R) \in Z_j$. Using the compactness of $K$, pick an infinite subset $J$ of $\mathbb{N}$ such that $z = \lim_{j \to \infty, j \in J} z_j$ exists and belongs to $K$. Then, since $(z_j, R) \in Z_j$, $(z_j, R) \to (z, R)$, and $Z_j \to Z$ in the Hausdorff metric, it follows that $(z, R) \in Z$, concluding our proof. □

***Kakutani-Fan-Glicksberg (KFG) maps.*** An important class of examples of CCA maps consists of those that we will call *Kakutani-Fan-Glicksberg* (abbreviated "KFG") *maps*, because they occur in the celebrated finite-dimensional Kakutani fixed point theorem as well as in its infinite-dimensional version due to Fan and Glicksberg.

**Definition 3.11** *If $X$ is a metric space and $C$ is a convex subset of a normed space, a **KFG map** from $X$ to $C$ is a compactly-graphed set-valued map $F : X \mapsto\!\!\!\!\rightarrow C$ such that $F(x)$ is convex and nonempty whenever $x \in X$.* $\square$

**Remark 3.12** It follows from Fact 3.3 that *a set-valued map $F : X \mapsto\!\!\!\!\rightarrow C$ from a metric space $X$ to a convex subset $C$ of a normed space is a KFG map if and only if it is an upper semicontinuous map with nonempty compact convex values.* $\square$

The following result is due to A. Cellina, cf. [3, 4, 6].

**Theorem 3.13** *If $X$ is a metric space, $C$ is a convex subset of a normed space $Y$, $F : X \mapsto\!\!\!\!\rightarrow C$, and $F$ is a KFG map, then $F$ is a CCA map.*

*Proof.* The definition of a KFG map implies that $\mathrm{Gr}(F \lceil K)$ is compact and nonempty whenever $K$ is a nonempty compact subset of $X$, which is one of the two conditions needed for $F$ to be a CCA map. To prove the other condition, we fix a nonempty compact subset $K$ of $X$ and prove that there exists a sequence $\{F_j\}_{j=1}^{\infty}$ of continuous maps $F_j : K \mapsto C$ such that $F_j \xrightarrow{\text{igr}} F \lceil K$ as $j \to \infty$.

For each positive number $\varepsilon$, select a finite subset $S_\varepsilon$ of $K$ such that $K \subseteq \bigcup_{s \in S_\varepsilon} \mathbb{B}_X(s, \varepsilon)$. For $x \in K$, $s \in S_\varepsilon$, let $\psi_{s,\varepsilon}(x) = \max\Big(0, \varepsilon - d_X(x,s)\Big)$, so $\psi_{s,\varepsilon} : K \mapsto \mathbb{R}$ is continuous and nonnegative and $\psi_{s,\varepsilon}(x) > 0$ if and only if $x \in \mathbb{B}_X(s, \varepsilon)$. Define $\varphi_{s,\varepsilon}(x) = \Big(\sum_{s' \in S_\varepsilon} \psi_{s',\varepsilon}(x)\Big)^{-1} \psi_{s,\varepsilon}(x)$, so the $\varphi_{s,\varepsilon}$ are continuous nonnegative real-valued functions on $K$ having the property that $\sum_{s \in S_\varepsilon} \varphi_{s,\varepsilon}(x) = 1$ for all $x \in K$. Using the fact that the sets $F(x)$ are nonempty, pick a $y_{s,\varepsilon} \in F(s)$ for each $s \in S_\varepsilon$. Define $H_\varepsilon : K \mapsto C$ by letting $H_\varepsilon(x) = \sum_{s \in S_\varepsilon} \varphi_{s,\varepsilon}(x) y_{s,\varepsilon}$. Then each $H_\varepsilon$ is continuous.

Now let $\{\varepsilon_j\}_{j \in \mathbb{N}}$ be a sequence of positive numbers that converges to zero. We claim that the $H_{\varepsilon_j} \xrightarrow{\text{igr}} F \lceil K$. To see this, we let

$$\alpha_j = \sup\{d_{X \times Y}(q, \mathrm{Gr}(F \lceil K)) : q \in \mathrm{Gr}(H_{\varepsilon_j})\},$$

and prove that $\alpha_j \to 0$. The proof will be by contradiction.

Assume that $\{\alpha_j\}$ does not go to zero. Then we may pass to a subsequence and assume that the $\alpha_j$ are bounded below by a fixed strictly positive number $\alpha$. Pick a $\beta$ such that $0 < \beta < \alpha$. Pick $q_j \in \mathrm{Gr}(H_{\varepsilon_j})$ such that

$$d_{X \times Y}(q_j, \mathrm{Gr}(F \lceil K)) \geq \beta. \tag{3.2.8}$$

Write $q_j = (x_j, y_j)$. Then the $x_j$ belong to $K$, so we may assume, after passing to a subsequence, that the limit $\bar{x} = \lim_{j \to \infty} x_j$ exists.

Fix a $\gamma$ such that $0 < \gamma$ and $2\gamma < \beta$. Pick a positive $\delta$ such that $d_Y(z, F(\bar{x})) < \gamma$ whenever $w \in K$, $z \in F(w)$, and $d_X(\bar{x}, w) \leq \delta$. (The existence of such a $\delta$ is easily proved: suppose, by contradiction, that there exist sequences $\{w_k\}$, $\{z_k\}$ in $K$ such that $z_k \in F(w_k)$, $w_k \to \bar{x}$ as $k \to \infty$, and

$d_Y(z_k, F(\bar{x})) \geq \gamma$; since $\mathrm{Gr}(F\lceil K)$ is compact we may assume, after passing to a subsequence, that the sequence $\{z_k\}$ converges to a limit $z$; since $z_k \in F(w_k)$, and $w_k \to \bar{x}$, the compactness of $\mathrm{Gr}(F\lceil K)$ also implies that $z \in F(\bar{x})$; since $z_k \to z$, we see that $d_Y(z_k, F(\bar{x})) \to 0$, and we have derived a contradiction.)

Now let $j^* \in \mathbb{N}$ be such that

$$2\,\varepsilon_j \leq \delta \quad \text{and} \quad d_X(x_j, \bar{x}) \leq \min\Big(\gamma, \frac{\delta}{2}\Big) \tag{3.2.9}$$

whenever $j \geq j^*$. If $j \geq j^*$, $x = x_j$, and $\varepsilon = \varepsilon_j$, then all the terms in the summation defining $H_\varepsilon$ for which $d_X(s, \bar{x}) \geq \delta$ vanish, because $d_X(s, \bar{x}) \geq \delta$ implies $d_X(x_j, s) \geq \frac{\delta}{2} \geq \varepsilon_j$ in view of (3.2.9), so $\varphi_{s,\varepsilon_j}(x_j) = 0$. Therefore, if we let $y_j = H_{\varepsilon_j}(x_j)$, we have

$$y_j \;=\; H_{\varepsilon_j}(x_j) \;=\; \sum_{s \in \hat{S}_{\varepsilon_j,\bar{x}}} \varphi_{s,\varepsilon_j}(x_j) y_{s,\varepsilon_j}\,, \tag{3.2.10}$$

where $\hat{S}_{\varepsilon_j,\bar{x}} \;=\; \Big\{s \in S_{\varepsilon_j} : d_X(s, \bar{x}) < \delta\Big\}$. For every $s \in \hat{S}_{\varepsilon_j,\bar{x}}$, the point $y_{s,\varepsilon_j}$ is in $F(s)$, so $\mathrm{dist}(y_{s,\varepsilon_j}, F(\bar{x})) < \gamma$. Therefore we may pick $\tilde{y}_{s,\varepsilon_j} \in F(\bar{x})$ such that $\|y_{s,\varepsilon_j} - \tilde{y}_{s,\varepsilon_j}\| \leq \gamma$. If we let $\tilde{y}_j = \sum_{s \in \hat{S}_{\varepsilon_j,\bar{x}}} \varphi_{s,\varepsilon_j}(x_j)\tilde{y}_{s,\varepsilon_j}$, and compare this with (3.2.10), we find $\|\tilde{y}_j - y_j\| \leq \sum_{s \in \hat{S}_{\varepsilon_j,\bar{x}}} \varphi_{s,\varepsilon_j}(x_j)\|\tilde{y}_{s,\varepsilon_j} - y_{s,\varepsilon_j}\| \leq \gamma$. On the other hand, $\tilde{y}_j$ clearly is a convex combination of points of $F(\bar{x})$, so $\tilde{y}_j \in F(\bar{x})$, because $F(\bar{x})$ is convex. Since $\|y_j - \tilde{y}_j\| \leq \gamma$ and $d_X(x_j, \bar{x}) \leq \gamma$ for $j \geq j^*$, and the point $\tilde{q}_j \overset{\mathrm{def}}{=} (\bar{x}, \tilde{y}_j)$ belongs to $\mathrm{Gr}(F\lceil K)$, we can conclude that $d_{X\times Y}(q_j, \mathrm{Gr}(F\lceil K)) \;\leq\; 2\,\gamma \;<\; \beta \quad \text{if} \quad j \;\geq\; j^*$ This, together with Formula (3.2.8), shows that the assumption that $\alpha_j$ does not go zero leads to a contradiction. So $\alpha_j \to 0$, and the proof is complete. □

***The Cellina, Kakutani, and Fan-Glicksberg fixed point theorems.*** Many fixed point properties of continuous maps are also valid for CCA maps, as we now show. Let us recall that, if $A$ is a set, and $F : A \mapsto\!\!\!\rightarrow A$, then a *fixed point* of $F$ is a point $a \in A$ such that $a \in F(a)$.

**Theorem 3.14** *(Cellina, cf. [5]) Let $K$ be a nonempty compact convex subset of a normed space $X$, and let $F : K \mapsto\!\!\!\rightarrow K$ be a CCA map. Then $F$ has a fixed point.*

*Proof.* Let $\{F_j\}_{j\in\mathbb{N}}$ be a sequence of continuous maps from $K$ to $K$ such that $F_j \xrightarrow{\mathrm{igr}} F$ as $j \to \infty$. By the Schauder fixed point theorem, there exist $x_j$ such that $F_j(x_j) = x_j$. Since $K$ is compact we may pass to a subsequence, if necessary, and assume that the sequence $\{x_j\}_{j\in\mathbb{N}}$ has a limit $x \in K$. Then $F_j(x_j) \to x$ as well, so $x \in F(x)$. □

**Corollary 3.15** *(The Kakutani-Fan-Glicksberg fixed point theorem, cf. Kakutani [13], Fan [10], Glicksberg [11].) Let $K$ be a nonempty compact convex subset of a normed space $X$. Let $F : K \mapsto\!\!\!\rightarrow K$ be a set-valued map with a compact graph and nonempty convex values. Then $F$ has a fixed point.*

*Proof.* Theorem 3.13 tells us that $F$ is a CCA map, and then Theorem 3.14 implies that $F$ has a fixed point. $\square$

# 4 GDQs and AGDQs

*We use* $\mathbf{\Theta}$ *to denote the class of all functions* $\theta : [0, +\infty[ \mapsto [0, +\infty]$ *such that*

- $\theta$ *is monotonically nondecreasing (that is,* $\theta(s) \leq \theta(t)$ *whenever* $s, t$ *are such that* $0 \leq s \leq t < +\infty$*);*
- $\theta(0) = 0$ *and* $\lim_{s \downarrow 0} \theta(s) = 0$.

*If* $X, Y$ *are FDNRLSs, we endow* $Lin(X, Y)$ *with the operator norm* $\| \cdot \|_{op}$ *defined in (2.1.2). If* $\Lambda \subseteq Lin(X, Y)$ *and* $\delta > 0$*, we define*

$$\Lambda^\delta = \{ L \in Lin(X, Y) : \mathrm{dist}(L, \Lambda) \leq \delta \} ,$$

*where* $\mathrm{dist}(L, \Lambda) = \inf\{ \|L - L'\|_{op} : L' \in \Lambda \}$*. Notice that if* $L \in Lin(X, Y)$*, then* $\mathrm{dist}(L, \emptyset) = +\infty$*. In particular, if* $\Lambda = \emptyset$ *then* $\Lambda^\delta = \emptyset$*. Notice also that* $\Lambda^\delta$ *is compact if* $\Lambda$ *is compact and* $\Lambda^\delta$ *is convex if* $\Lambda$ *is convex.*

## 4.1 The basic definitions

***Generalized differential quotients (GDQs).*** We assume that (1) $X$ and $Y$ are FDNRLSs, (2) $F : X \mapsto\!\!\!\to Y$ is a set-valued map; (3) $\bar{x}_* \in X$, (4) $\bar{y}_* \in Y$, and (5) $S \subseteq X$.

**Definition 4.1** *A* ***generalized differential quotient*** *(abbreviated "GDQ")* ***of*** $F$ ***at*** $(\bar{x}_*, \bar{y}_*)$ ***in the direction of*** $S$ *is a compact subset* $\Lambda$ *of* $Lin(X, Y)$ *having the property that for every neighborhood* $\hat{\Lambda}$ *of* $\Lambda$ *in* $Lin(X, Y)$ *there exist* $U$*,* $G$ *such that*

(I) $U$ *is a neighborhood of* $\bar{x}_*$ *in* $X$*;*
(II) $\bar{y}_* + G(x) \cdot (x - \bar{x}_*) \subseteq F(x)$ *for every* $x \in U \cap S$*;*
(III) $G$ *is a CCA set-valued map from* $U \cap S$ *to* $\hat{\Lambda}$*.* $\square$

We will use $GDQ(F, \bar{x}_*, \bar{y}_*, S)$ to denote the set of all GDQs of $F$ at $(\bar{x}_*, \bar{y}_*)$ in the direction of $S$.

**Remark 4.2** The set $\Lambda$ can, in principle, be empty. Actually, it is very easy to show that the following three conditions are equivalent:

(1) $\emptyset \in GDQ(F, \bar{x}_*, \bar{y}_*, S)$;
(2) *every compact subset of* $Lin(X, Y)$ *belongs to* $GDQ(F, \bar{x}, \bar{y}, S)$;
(3) $\bar{x}_*$ *does not belong to the closure of* $S$*.* $\square$

It is not hard to prove the following alternative characterization of GDQs.

**Proposition 4.3** *Let* $X, Y$ *be FDNRLSs, let* $F : X \mapsto\!\!\!\!\to Y$ *be a set-valued map, and let* $\Lambda$ *be a compact subset of* $Lin(X,Y)$*. Let* $\bar{x}_* \in X$*,* $\bar{y}_* \in Y$*,* $S \subseteq X$*. Then* $\Lambda \in GDQ(F, \bar{x}_*, \bar{y}, S)$ *if and only if there exists a function* $\theta \in \boldsymbol{\Theta}$*—called a* ***GDQ modulus for*** $(\Lambda, F, \bar{x}_*, \bar{y}_*, S)$*—having the property that*

*(*) for every* $\varepsilon \in ]\,0, +\infty\,[$ *such that* $\theta(\varepsilon) < \infty$ *there exists a set-valued map* $G^\varepsilon \in CCA(\bar{\mathbb{B}}_X(\bar{x}_*, \varepsilon) \cap S, Lin(X,Y))$ *such that for every* $x \in \bar{\mathbb{B}}_X(\bar{x}_*, \varepsilon) \cap S$ *the inclusions* $G^\varepsilon(x) \subseteq \Lambda^{\theta(\varepsilon)}$ *and* $\bar{y}_* + G^\varepsilon(x) \cdot (x - \bar{x}_*) \subseteq F(x)$ *hold.*

*Proof.* Assume that $\Lambda$ belongs to $GDQ(F, \bar{x}_*, \bar{y}_*, S)$. For each nonnegative real number $\varepsilon$, let $H(\varepsilon)$ be the set of all $\delta$ such that (i) $\delta > 0$, and (ii) there exists a $G \in CCA(\bar{\mathbb{B}}_X(\bar{x}_*, \varepsilon) \cap S, Lin(X,Y))$ with the property that $G(x) \subseteq \Lambda^\delta$ and $\bar{y}_* + G(x) \cdot (x - \bar{x}_*) \subseteq F(x)$ whenever $x \in \bar{\mathbb{B}}_X(\bar{x}_*, \varepsilon) \cap S$. Let $\theta_0(\varepsilon) = \inf H(\varepsilon)$, and then define $\theta(\varepsilon) = \theta_0(\varepsilon) + \varepsilon$. (Notice that the set $H(\varepsilon)$ could be empty, in which case $\theta_0(\varepsilon) = \theta(\varepsilon) = +\infty$.) It is clear that $\theta$ is monotonically non-decreasing, since $H(\varepsilon') \subseteq H(\varepsilon)$ whenever $0 \leq \varepsilon < \varepsilon'$. The fact that $\Lambda \in GDQ(F, \bar{x}_*, \bar{y}_*, S)$ implies that, given any positive $\delta$, there exist a neighborhood $U$ of $\bar{x}_*$ and a map $\tilde{G} \in CCA(U \cap S, \Lambda^\delta)$ such that $\bar{y}_* + \tilde{G}(x) \cdot (x - \bar{x}_*) \subseteq F(x)$ whenever $x \in U \cap S$. Find $\varepsilon$ such that $\bar{\mathbb{B}}_X(\bar{x}_*, \varepsilon) \subseteq U$, and let $G = \iota_2 \circ \tilde{G} \circ \iota_1$, where $\iota_1 : \bar{\mathbb{B}}_X(\bar{x}_*, \varepsilon) \cap S \mapsto U \cap S$ and $\iota_2 : \Lambda^\delta \mapsto Lin(X,Y)$ are the set inclusions. Then it is clear that $G$ belongs to $CCA(\bar{\mathbb{B}}_X(x_*, \varepsilon) \cap S, Lin(X,Y))$, and also that $G(x) \subseteq \Lambda^\delta$ and $\bar{y}_* + G(x) \cdot (x - \bar{x}_*) \subseteq F(x)$ whenever $x \in \bar{\mathbb{B}}_X(\bar{x}_*, \varepsilon) \cap S$. Therefore $\delta \in H(\varepsilon)$, so $\theta_0(\varepsilon) \leq \delta$. This proves that $\lim_{\varepsilon \downarrow 0} \theta_0(\varepsilon) = 0$, thus establishing that $\theta_0 \in \boldsymbol{\Theta}$, and then $\theta \in \boldsymbol{\Theta}$ as well. Finally, if $\theta(\varepsilon) < +\infty$, then we can pick a $\delta \in H(\varepsilon)$ such that $\theta_0(\varepsilon) \leq \delta \leq \theta(\varepsilon)$, and then find a $G$ belonging to $CCA(\bar{\mathbb{B}}_X(\bar{x}_*, \varepsilon) \cap S, Lin(X,Y))$ for which the conditions $G(x) \subseteq \Lambda^\delta$ and $\bar{y}_* + G(x) \cdot (x - \bar{x}_*) \subseteq F(x)$ hold whenever $x \in \bar{\mathbb{B}}_X(\bar{x}_*, \varepsilon) \cap S$. Since $\delta \leq \theta(\varepsilon)$, the map $G$ takes values in $\Lambda^{\theta(\varepsilon)}$. Hence we can choose $G^\varepsilon$ to be $G$, and the condition of (*) is satisfied.

To prove the converse, let $\theta$ be a GDQ modulus for $\Lambda, F, \bar{x}_*, \bar{y}_*, S$. Fix a positive number $\delta$. Pick an $\varepsilon$ such that $\theta(\varepsilon) < \delta$. Then pick $G^\varepsilon$ such that the conditions of (*) hold. Then the map $G^\varepsilon$ satisfies the requirement that $\bar{y}_* + G^\varepsilon \cdot (x - \bar{x}_*) \subseteq F(x)$ whenever $x \in \bar{\mathbb{B}}_X(\bar{x}_*, \varepsilon) \cap S$. Furthermore, $G^\varepsilon \in CCA(\bar{\mathbb{B}}_X(\bar{x}_*, \varepsilon) \cap S, Lin(X,Y))$, and $G^\varepsilon$ takes values in $\Lambda^{\theta(\varepsilon)}$. Since $\theta(\varepsilon) < \delta$, if $K$ is a compact subset of $\bar{\mathbb{B}}_X(\bar{x}_*, \varepsilon) \cap S$ and $\{G_j\}_{j \in \mathbb{N}}$ is a sequence of continuous maps from $K$ to $Lin(X,Y)$ such that $G_j \xrightarrow{\text{igr}} G^\varepsilon \lceil K$, it follows that $G_j$ takes values in $\Lambda^\delta$ if $j$ is large enough. Therefore $G^\varepsilon$ belongs to $CCA(\bar{\mathbb{B}}^n(\bar{x}_*, \varepsilon) \cap S, \Lambda^\delta)$. This shows that $\Lambda \in GDQ(F, \bar{x}_*, \bar{y}_*, S)$, concluding our proof. □

***Approximate generalized differential quotients (AGDQs)*** Motivated by the characterization of GDQs given in Proposition 4.3, we now define a slightly larger class of generalized differentials. First, if $X$, $Y$ are FDNRLSs, we let $Aff(X,Y)$ be the set of all affine maps from $X$ to $Y$, so the members of $Aff(X,Y)$ are the maps $X \ni x \mapsto A(x) = L \cdot x + h\,,\ L \in Lin(X,Y),\ h \in Y$.

(For a map $A$ of this form, the linear map $L \in Lin(X,Y)$ and the vector $h \in Y$ are the *linear part* and the *constant part* of $A$.) We identify $Aff(X,Y)$ with $Lin(X,Y) \times Y$ by identifying each $A \in Aff(X,Y)$ with the pair $(L,h) \in Lin(X,Y) \times Y$, where $L$, $h$ are, respectively, the linear part and the constant part of $A$.

**Definition 4.4** *Assume that $X, Y$ are FDNRLSs, $F : X \mapsto\!\!\!\!\rightarrow Y$ is a set-valued map, $\Lambda$ is a compact subset of $Lin(X,Y)$, $\bar{x}_* \in X$, $\bar{y}_* \in Y$, and $S \subseteq X$. We say that $\Lambda$ is an* ***approximate generalized differential quotient of*** *$F$* ***at*** *$(\bar{x}_*, \bar{y}_*)$* ***in the direction of*** *$S$—and write $\Lambda \in AGDQ(F, \bar{x}_*, \bar{y}_*, S)$—if there exists a function $\theta \in \mathbf{\Theta}$—called an* ***AGDQ modulus for*** *$(\Lambda, F, \bar{x}_*, \bar{y}_*, S)$—having the property that*

*(**) for every $\varepsilon \in ]\,0, +\infty\,[$ such that $\theta(\varepsilon) < \infty$ there exists a set-valued map $A^\varepsilon \in CCA(\bar{\mathbb{B}}_X(\bar{x}_*, \boldsymbol{\varepsilon}) \cap S, Aff(X,Y))$ such that*

$$L \in \Lambda^{\theta(\varepsilon)}\,, \quad \|h\| \le \theta(\varepsilon)\varepsilon\,, \quad \text{and} \quad \bar{y}_* + L \cdot (x - \bar{x}_*) + h \in F(x)$$

*whenever $x \in \bar{\mathbb{B}}_X(\bar{x}_*, \varepsilon) \cap S$ and $(L,h)$ belongs to $A^\varepsilon(x)$.* □

## 4.2 Properties of GDQs and AGDQs

***Retracts, quasiretracts and local quasiretracts*** In order to formulate and prove the chain rule, we first need some basic facts about retracts.

**Definition 4.5** *Let $T$ be a topological space and let $S$ be a subset of $T$. A* ***retraction from*** *$T$* ***to*** *$S$ is a continuous map $\rho : T \mapsto S$ such that $\rho(s) = s$ for every $s \in S$. We say that $S$ is a* ***retract of*** *$T$ if there exists a retraction from $T$ to $S$.* □

Often, the redundant phrase "continuous retraction" will be used for emphasis, instead of just saying "retraction."

It follows easily from the definition that

**Fact 4.6** *If $T$ is a Hausdorff topological space and $S$ is a retract of $T$, then $S$ is closed.* □

Also, it is easy to show that every retract is a "local retract" at any point, in the following precise sense:

**Fact 4.7** *If $T$ is a Hausdorff topological space, $S$ is a retract of $T$, and $s \in S$, then every neighborhood $U$ of $s$ contains a neighborhood $V$ of $s$ such that $S \cap V$ is a retract of $V$.* □

It will be convenient to introduce a weaker concept, namely, that of a "quasiretract," as well as its local version.

**Definition 4.8** *Let $T$ be a topological space and let $S$ be a subset of $T$. We say that $S$ is a* ***quasiretract of*** *$T$ if for every compact subset $K$ of $S$ there exist a neighborhood $U$ of $K$ and a continuous map $\rho : U \mapsto S$ such that $\rho(s) = s$ for every $s \in K$.* □

**Definition 4.9** *Assume that* $T$ *is a topological space,* $S \subseteq T$*, and* $\bar{s}_* \in T$*. We say that* $S$ *is a* ***local quasiretract of*** $T$ ***at*** $\bar{s}_*$ *if there exists a neighborhood* $U$ *of* $\bar{s}_*$ *such that* $S \cap U$ *is a quasiretract of* $U$*.* $\square$

It is then easy to verify the following facts.

**Fact 4.10** *If* $T$ *is a topological space and* $S \subseteq T$*, then*

*(1) if* $S$ *is a retract of* $T$ *then* $S$ *is a quasiretract of* $T$*;*
*(2) if* $S$ *is a quasiretract of* $T$ *and* $\Omega$ *is an open subset of* $T$ *then* $S \cap \Omega$ *is a quasiretract of* $\Omega$*.* $\square$

**Fact 4.11** *Assume that* $T$ *is a topological space,* $S \subseteq T$*, and* $\bar{s}_* \in T$*. Then the following are equivalent:*

*(a)* $S$ *is a local quasiretract of* $T$ *at* $\bar{s}_*$*;*
*(b) every neighborhood* $V$ *of* $\bar{s}_*$ *contains an open neighborhood* $U$ *of* $\bar{s}_*$ *in* $T$ *such that* $S \cap U$ *is a quasiretract of* $U$*.* $\square$

Fact 4.11 implies, in particular, that *being a local quasiretract is a local-homeomorphism invariant property of the germ of* $S$ *at* $\bar{s}_*$. Precisely,

**Corollary 4.12** *Assume that* $T, T'$ *are topological spaces,* $S \subseteq T$*,* $S' \subseteq T'$*,* $\bar{s}_* \in T$*, and* $\bar{s}'_* \in T'$*. Assume that there exist neighborhoods* $V$*,* $V'$ *of* $\bar{s}_*$*,* $\bar{s}'_*$ *in* $T, T'$*, and a homeomorphism* $h$ *from* $V$ *onto* $V'$ *such that* $h(S \cap V) = S' \cap V'$ *and* $h(\bar{s}_*) = \bar{s}'_*$*. Then* $S$ *is a local quasiretract of* $T$ *at* $\bar{s}_*$ *if and only if* $S'$ *is a local quasiretract of* $T'$ *at* $\bar{s}'_*$*.*

*Proof.* It clearly suffices to prove one of the two implications. Assume that $S$ is a local quasiretract of $T$ at $\bar{s}_*$. Then Fact 4.11 implies that there exists an open subset $U$ of $T$ such that $\bar{s}_* \in U$, $U \subseteq V$, and $S \cap U$ is a quasiretract of $U$. Let $U' = h(U)$. Since $h$ is a homeomorphism, $U'$ is a relatively open subset of $V'$ such that $\bar{s}'_* \in U'$, and $S' \cap U'$ is a quasiretract of $U'$. Since $V'$ is a neighborhood of $\bar{s}'_*$ in $T'$, it follows that $U'$ is a neighborhood of $\bar{s}'_*$ in $T'$, so Definition 4.9 tells us that $S'$ is a local quasiretract of $T'$ at $\bar{s}'_*$. $\square$

**Remark 4.13** The set $S = \{(x, y) \in \mathbb{R}^2 : y > 0\} \cup \{(0, 0)\}$ is a quasiretract of $\mathbb{R}^2$. (Indeed, if $K$ is a compact subset of $S$, then the convex hull $\hat{K}$ of $K$ is also compact, and $\hat{K} \subseteq S$ because $S$ is convex. Therefore $\hat{K}$ is a retract of $\mathbb{R}^2$. If $\rho : \mathbb{R}^2 \mapsto \hat{K}$ is a retraction, then $\rho$ maps $\mathbb{R}^2$ into $S$, and $\rho(s) = s$ for every $s \in K$.)

On the other hand, $S$ is not a retract of $\mathbb{R}^2$, because $S$ is not a closed subset of $\mathbb{R}$. This shows that the notion of quasiretract is strictly more general than that of a retract.

The same is true for the notions of "local quasiretract" and "local retract." For example, the set $S$ of our previous example is a local quasiretract at the origin, but it is not a local retract at $(0, 0)$, because there does not exist a neighborhood $V$ of $(0, 0)$ such that $S \cap V$ is a relatively closed subset of $V$. $\square$

***The chain rule.*** We now prove the *chain rule* for GDQs and AGDQs.

**Theorem 4.14** *For $i = 1, 2, 3$, let $X_i$ be a FDNRLS, and let $\bar{x}_{*,i}$ be a point of $X_i$. Assume that, for $i = 1, 2$, (i) $F_i : X_i \mapsto\!\!\!\!\rightarrow X_{i+1}$ is a set-valued map, (ii) $S_i$ is a subset of $X_i$, and (iii) $\Lambda_i \in AGDQ(F_i, \bar{x}_{*,i}, \bar{x}_{*,i+1}, S_i)$. Assume, in addition, that (iv) $F_1(S_1) \subseteq S_2$, and*

*(v) either (v.1) $S_2$ is a local quasiretract of $X_2$ at $\bar{x}_{*,2}$ or (v.2) there exists a neighborhood $U$ of $\bar{x}_{*,1}$ in $X_1$ such that the restriction $F_1 \lceil (U \cap S_1)$ of $F_1$ to $U \cap S_1$ is single-valued.*

*Then $\Lambda_2 \circ \Lambda_1 \in AGDQ(F_2 \circ F_1, \bar{x}_{*,1}, \bar{x}_{*,3}, S_1)$. Furthermore, if the sets $\Lambda_1$, $\Lambda_2$ belong to $GDQ(F_1, \bar{x}_{*,1}, \bar{x}_{*,2}, S_1)$ and $GDQ(F_2, \bar{x}_{*,2}, \bar{x}_{*,3}, S_2)$, respectively, then $\Lambda \in GDQ(F, \bar{x}_{*,1}, \bar{x}_{*,3}, S_1)$.*

*Proof.* We assume, as is clearly possible without loss of generality, that $\bar{x}_{*,i} = 0$ for $i = 1, 2, 3$. We let $F \overset{\text{def}}{=} F_2 \circ F_1$, $\Lambda \overset{\text{def}}{=} \Lambda_2 \circ \Lambda_1$. We will first prove the conclusion for AGDQs, and then indicate how to make a trivial modification to obtain the GDQ result.

To begin with, let us fix AGDQ moduli $\theta_1$, $\theta_2$ for $(\Lambda_1, F_1, 0, 0, S_1)$ and $(\Lambda_2, F_2, 0, 0, S_2)$, respectively. Also, let $\kappa_i = 1 + \sup\Big\{\|L\| : L \in \Lambda_i\Big\}$, for $i = 1, 2$. (We add 1 to make sure that $\kappa_i > 0$ even if $\Lambda_i = \{0\}$.) It is then easy to see that $\Lambda_2^{\delta_2} \circ \Lambda_1^{\delta_1} \subseteq \Lambda^{\kappa_2\delta_1 + \kappa_1\delta_2 + \delta_1\delta_2}$ if $\delta_1 \geq 0$, $\delta_2 \geq 0$. (Indeed, if $L_1 \in \Lambda_1^{\delta_1}$, $L_2 \in \Lambda_2^{\delta_2}$, we may pick $\tilde{L}_1 \in \Lambda_1$, $\tilde{L}_2 \in \Lambda_2$ such that $\|\tilde{L}_1 - L_1\| \leq \delta_1$ and $\|\tilde{L}_2 - L_2\| \leq \delta_2$. Then $\|\tilde{L}_2\tilde{L}_1 - L_2L_1\| \leq \|\tilde{L}_2\tilde{L}_1 - \tilde{L}_2L_1\| + \|\tilde{L}_2L_1 - L_2L_1\|$, so $\|\tilde{L}_2\tilde{L}_1 - L_2L_1\| \leq \|\tilde{L}_2\|\,\|\tilde{L}_1 - L_1\| + \|\tilde{L}_2 - L_2\|\,\|L_1\| \leq (\kappa_2 + \delta_2)\delta_1 + \kappa_1\delta_2$, showing that $L_2L_1 \in \Lambda^{\kappa_2\delta_1 + \kappa_1\delta_2 + \delta_1\delta_2}$.)

We now use Hypothesis $(v)$. If $S_2$ is a local quasiretract of $X_2$ at 0, then we choose a neighborhood $U$ of 0 in $X_2$ such that $S_2 \cap U$ is a quasiretract of $U$, and then we choose a positive number $\bar{\sigma}$ such that the open ball $\mathbb{B}_{X_2}(0, \bar{\sigma})$ is contained in $U$. Then Fact 4.10 implies that $S_2 \cap \mathbb{B}_{X_2}(0, \bar{\sigma})$ is a quasiretract of $\mathbb{B}_{X_2}(0, \bar{\sigma})$. If $S_2$ is not a local quasiretract of $X_2$, then Hypothesis $(v)$ guarantees that $F_1 \lceil (U \cap S_1)$ is single-valued for some neighborhood $U$ of 0 in $X_1$. In this case, we choose a positive $\bar{\varepsilon}$ such that $F_1$ is single-valued on $\bar{\mathbb{B}}_{X_1}(0, \bar{\varepsilon}) \cap S_1$, and then take $\bar{\sigma}$ to be equal to $\bar{\varepsilon}$.

Then, for $\varepsilon \in \,]\,0, +\infty\,[$, we define $\sigma_\varepsilon^0 = (\kappa_1 + 2\theta_1(\varepsilon))\varepsilon$, $\sigma_\varepsilon = \sigma_\varepsilon^0 + \varepsilon$,

$$\theta^0(\varepsilon) = \kappa_2\theta_1(\varepsilon) + \kappa_1\theta_2(\sigma_\varepsilon) + 3\theta_1(\varepsilon)\theta_2(\sigma_\varepsilon)\,, \qquad \theta(\varepsilon) = \begin{cases} \theta^0(\varepsilon) \text{ if } \sigma_\varepsilon < \bar{\sigma} \\ +\infty \;\text{ if } \sigma_\varepsilon \geq \bar{\sigma}\,. \end{cases}$$

Let us show that $\theta$ is an AGDQ modulus for $(\Lambda, F, 0, 0, S_1)$. For this purpose, we first observe that $\theta \in \boldsymbol{\Theta}$. We next fix a positive $\varepsilon$ such that $\theta(\varepsilon)$ is finite, and set out to construct a CCA map $A : \bar{\mathbb{B}}_{X_1}(0, \varepsilon) \cap S_1 \mapsto\!\!\!\!\rightarrow Lin(X_1, X_3) \times X_3$ such that

$$\Big(x \in \bar{\mathbb{B}}_{X_1}(0, \varepsilon) \cap S_1 \;\wedge\; (L, h) \in A(x)\Big) \Rightarrow \qquad\qquad \\ \Big(L \in \Lambda^{\theta(\varepsilon)} \;\wedge\; \|h\| \leq \theta(\varepsilon)\varepsilon \;\wedge\; L \cdot x + h \in F(x)\Big). \tag{4.2.1}$$

The fact that $\theta(\varepsilon) < +\infty$ clearly implies that $\sigma_\varepsilon < \bar{\sigma}$, $\theta(\varepsilon) = \theta^0(\varepsilon)$, $\theta_1(\varepsilon) < +\infty$, and $\theta_2(\sigma_\varepsilon) < +\infty$. We may therefore choose set-valued maps

$$\begin{aligned} &A_1 \in CCA(\bar{\mathbb{B}}_{X_1}(0,\varepsilon) \cap S_1, Lin(X_1,X_2) \times X_2)\,,\\ &A_2 \in CCA(\bar{\mathbb{B}}_{X_2}(0,\sigma_\varepsilon) \cap S_2, Lin(X_2,X_3) \times X_3)\,,\end{aligned}$$

such that the conditions

$$L_1 \in \Lambda_1^{\theta_1(\varepsilon)}\,, \quad \|h_1\| \le \theta_1(\varepsilon)\varepsilon\,, \quad L_1 \cdot x + h_1 \in F_1(x)\,, \tag{4.2.2}$$

$$L_2 \in \Lambda_2^{\theta_2(\sigma_\varepsilon)}, \quad \|h_2\| \le \theta_2(\sigma_\varepsilon)\sigma_\varepsilon\,, \quad L_2 \cdot y + h_2 \in F_2(y) \tag{4.2.3}$$

hold whenever $x \in \bar{\mathbb{B}}_{X_1}(0,\varepsilon) \cap S_1$, $(L_1,h_1) \in A_1(x)$, $y \in \bar{\mathbb{B}}_{X_2}(0,\sigma_\varepsilon) \cap S_2$, and $(L_2,h_2) \in A_2(y)$.

We then define our desired set-valued map $A$ from $\bar{\mathbb{B}}_{X_1}(0,\varepsilon) \cap S$ to $Lin(X_1,X_3) \times X_3$ as follows. For each $x \in \bar{\mathbb{B}}_{X_1}(0,\varepsilon) \cap S_1$, we let

$$A(x) = \Big\{(L_2 \cdot L_1, L_2h_1 + h_2) : (L_1h_1) \in A_1(x)\,,\ (L_2,h_2) \in A_2(L_1 \cdot x + h_1)\Big\}\,.$$

Assume that $x \in \bar{\mathbb{B}}_{X_1}(0,\varepsilon) \cap S_1$ and $(L,h) \in A(x)$, and let $z = L \cdot x + h$. Then there exist $(L_1,h_1) \in A_1(x)$ and $(L_2,h_2) \in A_2(L_1 \cdot x + h_1)$ such that $L = L_2 \cdot L_1$ and $h = L_2h_1 + h_2$. The fact that $(L_1,h_1) \in A_1(x)$ implies that $L_1 \in \Lambda_1^{\theta(\varepsilon)}$, $\|h_1\| \le \theta_1(\varepsilon)\varepsilon$, and $y \overset{\text{def}}{=} L_1 \cdot x + h_1 \in F_1(x)$. Then $y \in S_2$ (because $F_1(S_1) \subseteq S_2$), and $\|y\| \le (\kappa_1 + \theta_1(\varepsilon))\varepsilon + \theta_1(\varepsilon)\varepsilon = \sigma_\varepsilon^0 < \sigma_\varepsilon$, so

$$y \in \bar{\mathbb{B}}_{X_2}(0,\sigma_\varepsilon^0) \cap S_2 \subseteq \mathbb{B}_{X_2}(0,\sigma_\varepsilon) \cap S_2 \tag{4.2.4}$$

and then $L_2 \in \Lambda_2^{\theta(\sigma_\varepsilon)}$, $\|h_2\| \le \theta_2(\sigma_\varepsilon)\sigma_\varepsilon$, and $L_2 \cdot y + h_2 \in F_2(y)$. It follows that $L = L_2L_1 \in \Lambda^{\kappa_1\theta_2(\sigma_\varepsilon)+\kappa_2\theta_1(\varepsilon)+\theta_2(\sigma_\varepsilon)\theta_1(\varepsilon)} \subseteq \Lambda^{\theta(\varepsilon)}$. Also,

$$\begin{aligned}\|h\| &\le \|L_2\|\,\|h_1\| + \|h_2\|\\ &\le (\kappa_2 + \theta_2(\sigma_\varepsilon))\theta_1(\varepsilon)\varepsilon + \theta_2(\sigma_\varepsilon)\sigma_\varepsilon\\ &= (\kappa_2 + \theta_2(\sigma_\varepsilon))\theta_1(\varepsilon)\varepsilon + \theta_2(\sigma_\varepsilon)(\kappa_1 + 2\theta_1(\varepsilon))\varepsilon\\ &= \Big(\kappa_2\theta_1(\varepsilon) + \theta_2(\sigma_\varepsilon)\theta_1(\varepsilon) + \theta_2(\sigma_\varepsilon)\kappa_1 + 2\theta_2(\sigma_\varepsilon)\theta_1(\varepsilon)\Big)\varepsilon\\ &= \Big(\kappa_2\theta_1(\varepsilon) + \theta_2(\sigma_\varepsilon)\kappa_1 + 3\theta_2(\sigma_\varepsilon)\theta_1(\varepsilon)\Big)\varepsilon\\ &= \theta(\varepsilon)\varepsilon\,.\end{aligned}$$

Finally,

$$z = L\cdot x+h = L_2L_1\cdot x+L_2\cdot h_1+h_2 = L_2(L_1\cdot x+h_1)+h_2 = L_2\cdot y+h_2 \in F_2(y)\,.$$

Since $y \in F_1(x)$, we conclude that $z \in F(x)$. Hence $A$ satisfies (4.2.1).

To conclude our proof, we have to show that

$$A \in CCA(\bar{\mathbb{B}^n}(0,\varepsilon) \cap S_1, Lin(X_1,X_3) \times X_3)\,. \tag{4.2.5}$$

We let

$$\begin{aligned} \mathcal{Q}_{1,\varepsilon} &= \bar{\mathbb{B}}_{X_1}(0,\varepsilon)\cap S_1\,, & \mathcal{T}_{1,\varepsilon} &= \mathcal{Q}_{1,\varepsilon}\times Lin(X_1,X_2)\times X_2\times X_2\,,\\ \mathcal{R}_{1,\varepsilon} &= \bar{\mathbb{B}}_{X_2}(0,\sigma_\varepsilon)\cap S_2\,, & \mathcal{T}_{2,\varepsilon} &= \mathcal{Q}_{1,\varepsilon}\times Lin(X_1,X_2)\times X_2\times \mathcal{R}_{1,\varepsilon}\,, \end{aligned}$$

and let $\Psi_{1,\varepsilon}$ be the set-valued map with source $\mathcal{Q}_{1,\varepsilon}$ and target $\mathcal{T}_{1,\varepsilon}$ that sends each $x\in\mathcal{Q}_{1,\varepsilon}$ to the set $\Psi_{1,\varepsilon}(x)$ of all 4-tuples $(\xi,L_1,h_1,y)\in\mathcal{T}_{1,\varepsilon}$ such that $\xi=x$, $(L_1,h_1)\in A_1(x)$, and $y=L_1\cdot x+h_1$. We then observe that $\Psi_{1,\varepsilon}$ takes values in $\mathcal{T}_{2,\varepsilon}$. (This is trivial, because we have already established—cf. (4.2.4)—that if $x\in\mathcal{Q}_{1,\varepsilon}$, $(L_1,h_1)\in A_1(x)$, and $y=L_1\cdot x+h_1$, then $y\in\mathcal{R}_{1,\varepsilon}$.)

Let $\tilde{\Psi}_{1,\varepsilon}$ be "$\Psi_{1,\varepsilon}$ regarded as a set-valued map with target $\mathcal{T}_{2,\varepsilon}$." (Precisely, $\tilde{\Psi}_{1,\varepsilon}$ is the set-valued map with source $\mathcal{Q}_{1,\varepsilon}$, target $\mathcal{T}_{2,\varepsilon}$, and graph $\mathrm{Gr}(\Psi_{1,\varepsilon})$.)

We now show that $\tilde{\Psi}_{1,\varepsilon}\in CCA(\mathcal{Q}_{1,\varepsilon},\mathcal{T}_{2,\varepsilon})$. To prove this, we pick a compact subset $K$ of $\mathcal{Q}_{1,\varepsilon}$, and show that (a) $\mathrm{Gr}(\tilde{\Psi}_{1,\varepsilon}\lceil K)$ is compact, and (b) there exists a sequence $\mathbf{H}=\{H_j\}_{j\in\mathbb{N}}$ of continuous maps $H_j:K\mapsto\mathcal{T}_{2,\varepsilon}$ such that $H_j\xrightarrow{\mathrm{igr}}\tilde{\Psi}_{1,\varepsilon}\lceil K$ as $j\to\infty$.

The compactness of $\mathrm{Gr}(\tilde{\Psi}_{1,\varepsilon}\lceil K)$ follows from the fact that $\mathrm{Gr}(\tilde{\Psi}_{1,\varepsilon}\lceil K)$ is the image of $\mathrm{Gr}(A_1\lceil K)$ under the continuous map

$$\mathcal{Q}_{1,\varepsilon}\times Lin(X_1,X_2)\times X_2\ni(x,L_1,h_1)\mapsto(x,(x,L_1,h_1,L_1\cdot x+h_1))\in\mathcal{Q}_{1,\varepsilon}\times\mathcal{T}_{1,\varepsilon}\,.$$

To prove the existence of the sequence $\mathbf{H}$, we use the fact that $A_1$ belongs to $CCA(\mathcal{Q}_{1,\varepsilon},Lin(X_1,X_2)\times X_2)$ to produce a sequence $\{A_1^j\}_{j\in\mathbb{N}}$ of ordinary continuous maps from $K$ to $\mathbb{R}^{n_2\times n_1}\times\mathbb{R}^{n_2}$ such that $A_1^j\xrightarrow{\mathrm{igr}}A_1\lceil K$ as $j\to\infty$, and we write $A_1^j(x)=(L_1^j(x),h_1^j(x))$ for $x\in K$.

We will construct $\mathbf{H}$ in two different ways, depending on whether *(v.1)* or *(v.2)* holds.

First suppose that *(v.1)* holds. The set

$$\mathcal{K}=\{L_1\cdot x+h_1:(x,L_1,h_1)\in\mathrm{Gr}(A_1\lceil K)\}\tag{4.2.6}$$

is compact, and we know from (4.2.4) that every $y\in\mathcal{K}$ is a member of $\mathbb{B}_{X_2}(0,\sigma_\varepsilon)\cap S_2$. Since $\mathbb{B}_{X_2}(0,\bar{\sigma})\cap S_2$ is a quasiretract of $\mathbb{B}_{X_2}(0,\bar{\sigma})$, and $\sigma_\varepsilon<\bar{\sigma}$, Fact 4.10 implies that $\mathbb{B}_{X_2}(0,\sigma_\varepsilon)\cap S_2$ is a quasiretract of $\mathbb{B}_{X_2}(0,\sigma_\varepsilon)$. Hence there exist an open subset $\Omega$ of the ball $\mathbb{B}_{X_2}(0,\sigma_\varepsilon)$ and a continuous map $\rho:\Omega\mapsto\mathbb{B}_{X_2}(0,\sigma_\varepsilon)\cap S_2$ such that $\rho(y)=y$ whenever $y\in\mathcal{K}$. Since $A_1^j\xrightarrow{\mathrm{igr}}A_1\lceil K$, the functions $A_1^j$ must satisfy

$$\{L_1^j(x)\cdot x+h_1^j(x):x\in K\}\subseteq\Omega\tag{4.2.7}$$

for all sufficiently large $j$. (Otherwise, there would exist an infinite subset $J$ of $\mathbb{N}$ and $x_j\in K$ such that $y_j=L_1^j(x_j)\cdot x_j+h_1^j(x_j)\notin\Omega$. By making $J$ smaller—but still infinite—if necessary, we may assume that the sequence $\{(x_j,L_1^j,h_1^j)\}_{j\in J}$ converges to a limit $(x,L_1,h_1)\in\mathrm{Gr}(A_1\lceil K)$. Then if we let $y=L_1\cdot x+h_1$, we see that $y\in\mathcal{K}$. On the other hand, the $y_j$ are not in $\Omega$, so $y$ is not in $\Omega$ either, because $\Omega$ is open. Since $\mathcal{K} \subseteq \Omega$, we have reached a contradiction.)

So we may assume, after passing to a subsequence, that (4.2.7) holds for all $j \in \mathbb{N}$. We then define $H_j(x) = (x, L_1^j(x), h_1^j(x), \rho(L_1^j(x) \cdot x + h_1^j(x)))$ for $x \in K$, $j \in \mathbb{N}$. Then the $H_j$ are continuous maps from $K$ to $\mathcal{T}_{2,\varepsilon}$, because $\rho$ takes values in $\mathcal{R}_{1,\varepsilon}$.

We now show that $H_j \xrightarrow{\text{igr}} \tilde{\Psi}_{1,\varepsilon} \lceil K$ as $j$ goes to $\infty$. To prove this, we let $\nu_j = \sup\{\mathrm{dist}(q, \mathrm{Gr}(\tilde{\Psi}_{1,\varepsilon} \lceil K)) : q \in \mathrm{Gr}(H_j)\}$, and assume that $\nu_j$ does not go to zero. We may then assume, after passing to a subsequence, that there exists a $\bar{\nu}$ such that $0 < 2\bar{\nu} \leq \nu_j$ for all $j$. We can then pick $x_j \in K$ such that

$$\|x_j - x\| + \|L_1^j(x_j) - L_1\| + \|h_1^j(x_j) - h_1\| + \|\rho(L_1^j(x_j) \cdot x + h_1^j(x_j)) - y\| \geq \bar{\nu} \quad (4.2.8)$$

whenever $(x, L_1, h_1, y) \in \mathrm{Gr}(\tilde{\Psi}_{1,\varepsilon} \lceil K)$, $j \in \mathbb{N}$. Since $A_1^j \xrightarrow{\text{igr}} A_1 \lceil K$, we may clearly assume, after passing to a subsequence if necessary, that the sequence $\{(x_j, L_1^j(x_j), h_1^j(x_j))\}_{j \in \mathbb{N}}$ has a limit $(\bar{x}, \bar{L}_1, \bar{h}_1) \in \mathrm{Gr}(A_1 \lceil K)$.

Let $\bar{y}_* = \bar{L}_1 \cdot \bar{x} + \bar{h}_1$. Then $\bar{y}_* \in \mathcal{K}$, because of (4.2.6) and the fact that $(\bar{x}, \bar{L}_1, \bar{h}_1) \in \mathrm{Gr}(A_1 \lceil K)$. Therefore $\rho(\bar{y}_*) = \bar{y}_*$. Furthermore, $x_j \to \bar{x}$, $L_1^j(x_j) \to \bar{L}_1$, and $h_1^j(x_j) \to \bar{h}_1$. Hence $L_1^j(x_j) \cdot x_j + h_1^j(x_j)$ converges to $\bar{L}_1(\bar{x}) \cdot \bar{x} + \bar{h}_1 = \bar{y}_*$. But then $\lim_{j\to\infty} \Big(\rho(L_1^j(x_j) \cdot x + h_1^j(x_j))\Big) = \rho(\bar{y}_*)$, since $\rho$ is continuous, so $\lim_{j\to\infty} \Big(\rho(L_1^j(x_j) \cdot x + h_1^j(x_j))\Big) = \bar{y}_*$, and then $\lim_{j\to\infty} \|\rho(L_1^j(x_j) \cdot x + h_1^j(x_j)) - \bar{y}_*\| = 0$. It follows that

$$\|x_j - \bar{x}\| + \|L_1^j(x_j) - \bar{L}_1\| + \|h_1^j(x_j) - \bar{h}_1\| + \|\rho(L_1^j(x_j) \cdot x_j + h_1^j(x_j)) - \bar{y}_*\| \to 0\,. \quad (4.2.9)$$

Let $\bar{y}_* = (\bar{x}, \bar{L}, \bar{L} \cdot \bar{x})$. Then $\bar{y}_* \in \mathrm{Gr}(\tilde{\Psi}_{1,\varepsilon} \lceil K)$, so (4.2.9) contradicts (4.2.8). This concludes the proof that $H_j \xrightarrow{\text{igr}} \tilde{\Psi}_{1,\varepsilon} \lceil K$ as $j$ goes to $\infty$. We have thus established that the sequence $\mathbf{H}$ exists, under the assumption that *(v.1)* holds.

Next, we consider the case when *(v.2)* holds. Then $\bar{\sigma} = \bar{\varepsilon}$, so the fact that $\sigma_\varepsilon < \bar{\sigma}$ implies that $\varepsilon < \bar{\varepsilon}$, and then the map $F_1$ is single-valued on $\mathcal{Q}_{1,\varepsilon}$. Define $\varphi(x) = \{L_1 \cdot x + h_1 : (L_1, h_1) \in A_1(x)\}$ for $x \in K$. Since $L_1 \cdot x + h_1 \in F_1(x)$ whenever $x \in K$ and $(L_1, h_1) \in A_1(x)$, the hypothesis that $F_1$ is single-valued on $\mathcal{Q}_{1,\varepsilon}$ implies that $\varphi$ is a single-valued CCA map from $K$ to $X_2$, so $\varphi$ is an ordinary continuous map from $K$ to $X_2$. Since $L_1 \cdot x + h_1 \in \bar{\mathbb{B}}_{X_2}(0, \sigma_\varepsilon)$ whenever $x \in K$, and $(L_1, h_1) \in A_1(x)$, we conclude that $\varphi$ is in fact a continuous map from $K$ to $\mathcal{R}_{1,\varepsilon}$. We then define $H_j(x) = (x, L_1^j(x), h_1^j(x), \varphi(x))$ for $x \in K$, $j \in \mathbb{N}$. Then the $H_j$ are continuous maps from $K$ to $\mathcal{T}_{2,\varepsilon}$, and it is easy to see that $H_j \xrightarrow{\text{igr}} \tilde{\Psi}_{1,\varepsilon} \lceil K$ as $j \to \infty$. So the existence of $\mathbf{H}$ has also been proved when *(v.2)* holds.

We are now ready to prove (4.2.5). We do this by expressing $A$ as a composite of CCA maps as follows: $A = \Psi_{3,\varepsilon} \circ \Psi_{2,\varepsilon} \circ \tilde{\Psi}_{1,\varepsilon}$, where

1. $\mathcal{T}_{3,\varepsilon} = \mathcal{T}_{2,\varepsilon} \times Lin(X_2, X_3) \times X_3$;

2. $\Psi_{2,\varepsilon} : \mathcal{T}_{2,\varepsilon} \mapsto\!\!\!\rightarrow \mathcal{T}_{3,\varepsilon}$ is the set-valued map that sends $(x, L_1, h_1, y) \in \mathcal{T}_{2,\varepsilon}$ to the set $\Psi_{2,\varepsilon}(x, L_1, h_1, y) \overset{\text{def}}{=} \{x\} \times \{L_1\} \times \{h_1\} \times \{y\} \times \mathcal{A}_2(y)$;
3. $\mathcal{T}_{4,\varepsilon} = Lin(X_1, X_3) \times X_3$;
4. $\Psi_{3,\varepsilon} : \mathcal{T}_{3,\varepsilon} \mapsto \mathcal{T}_{4,\varepsilon}$ is the continuous single-valued map that sends $(x, L_1, h_1, y, L_2, h_2) \in \mathcal{T}_{3,\varepsilon}$ to the pair $(L_2 L_1, L_2 h_1 + h_2) \in \mathcal{T}_{4,\varepsilon}$.

It is clear that $\Psi_{2,\varepsilon}$ and $\Psi_{3,\varepsilon}$ are CCA maps, so $\mathcal{A}$ is a CCA map, and our proof for AGDQs is complete.

The proof of the statement for GDQs is exactly the same, except only for the fact in this case all the constant components $h$ of the various pairs $(L, h)$ are always equal to zero. □

***GDQs and AGDQs on manifolds.*** If $M$ and $N$ are manifolds of class $C^1$, $\bar{x}_* \in M$, $\bar{y}_* \in N$, $S \subseteq M$, and $F : M \mapsto\!\!\!\rightarrow N$, then it is possible to define sets $GDQ(F, \bar{x}_*, \bar{y}_*, S)$, $AGDQ(F, \bar{x}_*, \bar{y}_*, S)$ of compact subsets of the space $Lin(T_{\bar{x}_*}M, T_{\bar{y}_*}N)$ of linear maps from $T_{\bar{x}_*}M$ to $T_{\bar{y}_*}N$ as follows. We let $m = \dim M$, $n = \dim N$, and pick coordinate charts $M \ni x \hookrightarrow \xi(x) \in \mathbb{R}^m$, $N \ni y \hookrightarrow \eta(y) \in \mathbb{R}^n$, defined near $\bar{x}_*$, $\bar{y}_*$ and such that $\xi(\bar{x}_*) = 0$ and $\eta(\bar{y}_*) = 0$, and declare that a subset $\Lambda$ of $Lin(T_{\bar{x}_*}M, T_{\bar{y}_*}N)$ belongs to $GDQ(F, \bar{x}_*, \bar{y}_*, S)$ (resp. to $AGDQ(F, \bar{x}_*, \bar{y}_*, S)$) if the composite map $D\eta(\bar{y}_*) \circ \Lambda \circ D\xi(\bar{x}_*)^{-1}$ is in $GDQ(\eta \circ F \circ \xi^{-1}, 0, 0, \xi(S))$ (resp. in $AGDQ(\eta \circ F \circ \xi^{-1}, 0, 0, \xi(S))$). It then follows easily from the chain rule that, with this definition, *the sets $GDQ(F, \bar{x}_*, \bar{y}_*, S)$ and $AGDQ(F, \bar{x}_*, \bar{y}_*, S)$ do not depend on the choice of the charts $\xi$, $\eta$.* In other words, *the notions of GDQ and AGDQ are invariant under $C^1$ diffeomorphisms and therefore make sense intrinsically on manifolds of class $C^1$.*

The following facts about GDQs and AGDQs on manifolds are then easily verified.

**Proposition 4.15** *If $M$, $N$ are manifolds of class $C^1$, $S \subseteq M$, $\bar{x}_* \in M$, $\bar{y}_* \in N$, and $F : M \mapsto\!\!\!\rightarrow N$, then*

*(1) $GDQ(F, \bar{x}_*, \bar{y}_*, S) \subseteq AGDQ(F, \bar{x}_*, \bar{y}_*, S)$.*

*(2) If (i) $U$ is a neighborhood of $\bar{x}_*$ in $M$, (ii) the restriction $F \lceil (U \cap S)$ is a continuous everywhere defined map, (iii) $\bar{y}_* = F(\bar{x}_*)$, (iv) $F$ is differentiable at $\bar{x}_*$ in the direction of $S$, (v) $L$ is a differential of $F$ at $\bar{x}_*$ in the direction of $S$ (that is, $L$ belongs to $Lin(T_{\bar{x}_*}M, T_{\bar{y}_*}N)$ and $\lim_{x \to \bar{x}_*, x \in S} \|x - \bar{x}_*\|^{-1}\Big(F(x) - F(\bar{x}_*) - L \cdot (x - \bar{x}_*)\Big) = 0$ relative to some choice of coordinate charts about $\bar{x}_*$ and $\bar{y}_*$), then $\{L\}$ belongs to $GDQ(F, \bar{x}_*, \bar{y}_*, S)$.*

*(3) If (i) $U$ is an open neighborhood of $\bar{x}_*$ in $M$, (ii) the restriction $F \lceil U$ is a Lipschitz-continuous everywhere defined map, (iii) $F(\bar{x}_*) = \bar{y}_*$, and (iv) $\Lambda$ is the Clarke generalized Jacobian of $F$ at $\bar{x}_*$, then $\Lambda$ belongs to $GDQ(F, \bar{x}_*, \bar{y}_*, M)$.* □

**Proposition 4.16** ***(The chain rule.)*** *Assume that (I) for $i = 1, 2, 3$, $M_i$ is a manifold of class $C^1$ and $\bar{x}_{*,i} \in M_i$, and (II) for $i = 1, 2$, (II.1) $S_i \subseteq M_i$,* *(II.2)* $F_i : M_i \mapsto\!\!\!\rightarrow M_{i+1}$, *and (II.3)* $\Lambda_i \in AGDQ(F_i, \bar{x}_{*,i}, \bar{x}_{*,i+1}, S_i)$. *Assume, in addition, that either* $S_2$ *is a local quasiretract of* $M_2$ *or* $F_1$ *is single-valued on* $U \cap S_1$ *for some neighborhood* $U$ *of* $\bar{x}_{*,1}$. *Then the composite* $\Lambda_2 \circ \Lambda_1$ *belongs to* $AGDQ(F_2 \circ F_1, \bar{x}_{*,1}, \bar{x}_{*,3}, S_1)$. *If in addition* $\Lambda_i \in GDQ(F_i, \bar{x}_{*,i}, \bar{x}_{*,i+1}, S_i)$ *for* $i = 1, 2$, *then* $\Lambda_2 \circ \Lambda_1 \in GDQ(F_2 \circ F_1, \bar{x}_{*,1}, \bar{x}_{*,3}, S_1)$. □

**Proposition 4.17** ***(The product rule.)*** *Assume that, for* $i = 1, 2$, *(1)* $M_i$ *and* $N_i$ *are manifolds of class* $C^1$, *(2)* $S_i \subseteq M_i$, *(3)* $\bar{x}_{*,i} \in M_i$, *(4)* $\bar{y}_{*,i} \in N_i$, *(5)* $F_i : M_i \mapsto\!\!\!\rightarrow N_i$, *(6)* $\Lambda_i \in AGDQ(F_i, \bar{x}_{*,i}, \bar{y}_{*,i}, S_i)$. *Assume also that*

*(7)* $\bar{x}_* = (\bar{x}_{*,1}, \bar{x}_{*,2})$, $\bar{y}_* = (\bar{y}_{*,1}, \bar{y}_{*,2})$, *and* $S = S_1 \times S_2$;

*(8)* $F = F_1 \times F_2$, *where* $F_1 \times F_2$ *is the set-valued map from* $M_1 \times M_2$ *to* $N_1 \times N_2$ *that sends each point* $(x_1, x_2) \in M_1 \times M_2$ *to the subset* $F_1(x_1) \times F_2(x_2)$ *of* $N_1 \times N_2$;

*(9)* $\Lambda = \Lambda_1 \oplus \Lambda_2$, where (i) $\Lambda_1 \oplus \Lambda_2$ is the set of all linear maps $L_1 \oplus L_2$ for all $L_1 \in \Lambda_1$, $L_2 \in \Lambda_2$, (ii) $L_1 \oplus L_2$ is the map

$$T_{\bar{x}_{*,1}} M_1 \oplus T_{\bar{x}_{*,2}} M_2 \ni (v_1, v_2) \mapsto (L_1 v_1, L_2 v_2) \in T_{\bar{y}_{*,1}} N_1 \oplus T_{\bar{y}_{*,2}} N_2 \,,$$

and (iii) we are identifying $T_{\bar{x}_{*,1}} M_1 \oplus T_{\bar{x}_{*,2}} M_2$ with $T_{(\bar{x}_{*,1}, \bar{x}_{*,2})}(M_1 \times M_2)$ and $T_{\bar{y}_{*,1}} N_1 \oplus T_{\bar{y}_{*,2}} N_2$ with $T_{(\bar{y}_{*,1}, \bar{y}_{*,2})}(N_1 \times N_2)$.

*Then* $\Lambda \in AGDQ(F, \bar{x}_*, \bar{y}_*, S)$. *Furthermore, if* $\Lambda_i \in GDQ(F_i, \bar{x}_{*,i}, \bar{y}_{*,i}, S_i)$ *for* $i = 1, 2$, *then* $\Lambda \in AGDQ(F, \bar{x}_*, \bar{y}_*, S)$. □

**Proposition 4.18** ***(Locality.)*** *Assume that (1)* $M$, $N$, *are manifolds of class* $C^1$, *(2)* $\bar{x}_* \in M$, *(3)* $\bar{y}_* \in N$, *(4)* $S_i \subseteq M$, *(5)* $F_i : M \mapsto\!\!\!\rightarrow N$ *for* $i = 1, 2$, *and (6) there exist neighborhoods* $U$, $V$ *of* $\bar{x}_*$, $\bar{y}_*$, *in* $M$, $N$, *respectively, such that* $U \cap S_1 = U \cap S_2$ *and* $(U \times V) \cap \mathrm{Gr}(F_1) = (U \times V) \cap \mathrm{Gr}(F_2)$. *Then (a)* $AGDQ(F_1, \bar{x}_*, \bar{y}_*, S_1) = AGDQ(F_2, \bar{x}_*, \bar{y}_*, S_2)$, *and in addition (b)* $GDQ(F_1, \bar{x}_*, \bar{y}_*, S_1) = GDQ(F_2, \bar{x}_*, \bar{y}_*, S_2)$. □

**Remark 4.19** It is easy to exhibit maps that have GDQs at a point $\bar{x}_*$ but are not classically differentiable at $\bar{x}_*$ and do not have differentials at $\bar{x}_*$ in the sense of other theories such as Clarke's generalized Jacobians, Warga's derivate containers, or our "semidifferentials" and "multidifferentials". (A simple example is provided by the function $f : \mathbb{R} \mapsto \mathbb{R}$ given by $f(x) = x \sin 1/x$ if $x \neq 0$, and $f(0) = 0$. The set $[-1, 1]$ belongs to $GDQ(f, 0, 0, \mathbb{R})$, but is not a differential of $f$ at 0 in the sense of any of the other theories.) □

***Closedness and monotonicity.*** GDQs and AGDQs have an important *closedness property.* In order to state it, we first recall that, if $Z$ is a metric space, then (i) $Comp^0(Z)$ is the set of all compact subsets of $Z$, (ii) $Comp^0(Z)$ has a natural non-Hausdorff topology $\mathcal{T}_{Comp^0(Z)}$, defined in §3.1. In particular, if $X$ and $Y$ are FDRLSs, then $Comp^0(Lin(X, Y))$ is the set of all compact subsets of $Lin(X, Y)$. Clearly, a subset $\mathcal{O}$ of $Comp^0(Lin(X, Y))$ is open in the topology $\mathcal{T}_{Comp^0(Lin(X,Y))}$ if and only if for every $\bar{\Lambda} \in \mathcal{O}$ there exists an open subset $\Omega$ of $Lin(X, Y)$ such that

(i) $\bar{\Lambda} \subseteq \Omega$ and (ii) $\{\Lambda \in \mathit{Comp}^0(\mathit{Lin}(X,Y)) : \Lambda \subseteq \Omega\} \subseteq \mathcal{O}$. It is clear that the topology $\mathcal{T}_{\mathit{Comp}^0(\mathit{Lin}(X,Y))}$ can be entirely characterized by its convergent sequences. (That is, a subset $\mathcal{C}$ of $\mathit{Comp}^0(\mathit{Lin}(X,Y))$ is closed if and only if it is sequentially closed, i.e., such that, whenever $\{\Lambda_k\}_{k\in\mathbb{N}}$ is a sequence of members of $\mathcal{C}$ and $\Lambda \in \mathit{Comp}^0(\mathit{Lin}(X,Y))$ is such that $\Lambda_k \to \Lambda$ in the topology $\mathcal{T}_{\mathit{Comp}^0(\mathit{Lin}(X,Y))}$ as $k \to \infty$, it follows that $\Lambda \in \mathcal{C}$.)

Furthermore, convergence of sequences is easily characterized as follows.

**Fact 4.20** *Assume that $X$ and $Y$ are FDRLSs, $\{\Lambda_k\}_{k\in\mathbb{N}}$ is a sequence of members of $\mathit{Comp}^0(\mathit{Lin}(X,Y))$, and $\Lambda$ belongs to $\mathit{Comp}^0(\mathit{Lin}(X,Y))$. Then $\Lambda_k \to \Lambda$ as $k \to \infty$ in the topology $\mathcal{T}_{\mathit{Comp}^0(\mathit{Lin}(X,Y))}$ if and only if* $\lim_{k\to\infty} \sup \Big\{ \mathrm{dist}(L,\Lambda) : L \in \Lambda_k \Big\} = 0$. □

The following result is then an easy consequence of the definitions of GDQ and AGDQ.

**Fact 4.21** *If $M$, $N$ are manifolds of class $C^1$, $F : M \mapsto\!\!\!\!\to N$, $(\bar{x}_*, \bar{y}_*) \in M \times N$, $S \subseteq M$, $X = T_{\bar{x}_*}M$, and $Y = T_{\bar{y}_*}N$, then the sets $\mathit{GDQ}(F,\bar{x}_*,\bar{y}_*,S)$ and $\mathit{AGDQ}(F,\bar{x}_*,\bar{y}_*,S)$ are closed relative to the topology $\mathcal{T}_{\mathit{Comp}^0(\mathit{Lin}(X,Y))}$.* □

Fact 4.21 then implies that GDQs and AGDQs also have the following *monotonicity property.*

**Fact 4.22** *If $M$, $N$ are manifolds of class $C^1$, $F : M \mapsto\!\!\!\!\to N$, $(\bar{x}_*, \bar{y}_*) \in M \times N$, $S \subseteq M$, $\Lambda \in \mathit{AGDQ}(F,\bar{x}_*,\bar{y}_*,S)$, $\tilde{\Lambda} \in \mathit{Comp}^0(\mathit{Lin}(T_{\bar{x}_*}M, T_{\bar{y}_*}N))$, and $\Lambda \subseteq \tilde{\Lambda}$, then $\tilde{\Lambda} \in \mathit{AGDQ}(F,\bar{x}_*,\bar{y}_*,S)$. Furthermore, if $\Lambda \in \mathit{GDQ}(F,\bar{x}_*,\bar{y}_*,S)$ then $\tilde{\Lambda}$ belongs to $\mathit{GDQ}(F,\bar{x}_*,\bar{y}_*,S)$.*

*Proof.* It suffices to use Fact 4.21 and observe that, under our hypotheses, $\tilde{\Lambda}$ belongs to the closure of the set $\{\Lambda\}$ relative to $\mathcal{T}_{\mathit{Comp}^0(\mathit{Lin}(X,Y))}$. □

In addition, GDQs and AGDQs also have a monotonicity property with respect to $F$ and $S$. Precisely, the following is a trivial corollary of the definitions of GDQ and AGDQ.

**Fact 4.23** *Suppose that $M$, $N$ are manifolds of class $C^1$, $(\bar{x}_*, \bar{y}_*) \in M \times N$, $\tilde{S} \subseteq S \subseteq M$, $F : M \mapsto\!\!\!\!\to N$, $\tilde{F} : M \mapsto\!\!\!\!\to N$, and $\mathrm{Gr}(F) \subseteq \mathrm{Gr}(\tilde{F})$. Then*

$$\mathit{GDQ}(F,\bar{x}_*,\bar{y}_*,S) \subseteq \mathit{GDQ}(\tilde{F},\bar{x}_*,\bar{y}_*,\tilde{S})$$

and

$$\mathit{AGDQ}(F,\bar{x}_*,\bar{y}_*,S) \subseteq \mathit{AGDQ}(\tilde{F},\bar{x}_*,\bar{y}_*,\tilde{S})\,.$$ □

Fact 4.21 says in particular that every GDQ of a map is also a GDQ of any “larger” map. On the other hand, it is perfectly possible for the “larger” map to have smaller GDQs. For example, if $f : \mathbb{R} \mapsto \mathbb{R}$ is the function given by $f(x) = |x|$, then the interval $[-1,1]$ is a GDQ of $f$ at 0 in the direction of $\mathbb{R}$, and no proper subset of $[-1,1]$ has this property. But if we “enlarge” $f$ and consider the set-valued map $F : \mathbb{R} \mapsto\!\!\!\!\to \mathbb{R}$ given by $F(x) = [0,|x|]$, then $\{0\} \in \mathit{GDQ}(F,0,0,\mathbb{R})$.

### 4.3 The directional open mapping and transversality properties

The crucial fact about GDQs and AGDQs that leads to the maximum principle is the transversal intersection property, which is a very simple consequence of the directional open mapping theorem. We will now prove these results. As a preliminary, we need information on pseudoinverses.

***Linear (Moore-Penrose) pseudoinverses.*** If $X$, $Y$ are FDRLSs and $L \in Lin(X,Y)$, a *linear right inverse* of $L$ is a linear map $M \in Lin(Y,X)$ such that $L \cdot M = \mathbb{I}_Y$. It is clear that $L$ has a right inverse if and only if it is surjective. Let $Lin_{onto}(X,Y)$ be the set of all surjective linear maps from $X$ to $Y$. Since every $L \in Lin_{onto}(X,Y)$ has a right inverse, it is natural to ask if it is possible to choose a right inverse $I(L)$ for each $L$ in a way that depends continuously (or smoothly, or real-analytically) on $L$. One way to make this choice is to let $I(L)$ be $L^{\#}$, the "Moore-Penrose pseudoinverse" of $L$ (with respect to a particular inner product on $X$).

To define $L^{\#}$, assume $X$, $Y$ are FDRLSs and endow both $X$ and $Y$ with Euclidean inner products (although, as will become clear below, only the choice of the inner product on $X$ matters). Then every map $L \in Lin(X,Y)$ has an *adjoint* (or *transpose*) $L^\dagger \in Lin(Y,X)$, characterized by the property that $\langle L^\dagger y, x\rangle = \langle y, Lx\rangle$ whenever $x \in X$, $y \in Y$. It is then easy to see that

**Fact 4.24** *If $X$ and $Y$ are FDRLSs endowed with Euclidean inner products, then $L \in Lin_{onto}(X,Y)$ if and only if $LL^\dagger$ is invertible.* □

**Definition 4.25** *If $X$ and $Y$ are FDRLSs endowed with Euclidean inner products, and $L \in Lin_{onto}(X,Y)$, the* ***Moore-Penrose pseudoinverse*** *of $L$ is the linear map $L^{\#} \in Lin(Y,X)$ given by $L^{\#} = L^\dagger(LL^\dagger)^{-1}$, where the symbol "$^\dagger$" stands for "adjoint."* □

The following result is then a trivial consequence of the definition.

**Fact 4.26** *Suppose that $X$ and $Y$ are FDRLSs endowed with Euclidean inner products. Then $Lin_{onto}(X,Y)$ is an open subset of the space $Lin(X,Y)$, and the map $Lin_{onto}(X,Y) \ni L \mapsto L^{\#} \in Lin(Y,X)$ is real-analytic. Furthermore, the identity $LL^{\#} = \mathbb{I}_X$ holds for all $L \in Lin(X,Y)$.* □

**Remark 4.27** If $X$, $Y$, $L$ are as in Definition 4.25, $y \in Y$, $x = L^{\#}y$, and $\xi$ is any member of $L^{-1}y$, then

$$\langle \xi, x\rangle = \langle \xi, L^{\#}y\rangle = \langle \xi, L^\dagger(LL^\dagger)^{-1}y\rangle = \langle L\xi, (LL^\dagger)^{-1}y\rangle = \langle y, (LL^\dagger)^{-1}y\rangle\,.$$

In particular, the above equalities are true for $x$ in the role of $\xi$, so that $\langle x, x\rangle = \langle y, (LL^\dagger)^{-1}y\rangle$, and then $\langle \xi, x\rangle = \langle x, x\rangle$, so $\langle \xi - x, x\rangle = 0$. Therefore

$$\|\xi\|^2 = \|\xi - x + x\|^2 = \|\xi - x\|^2 + \|x\|^2 + 2\langle \xi - x, x\rangle = \|\xi - x\|^2 + \|x\|^2 \geq \|x\|^2\,.$$

It follows that *$L^{\#}y$ is the member of $L^{-1}y$ of minimum norm.* This shows, in particular, that *the map $L^{\#}$ does not depend on the choice of a Euclidean inner product on $Y$.* □

More generally, we would like to find a pseudoinverse $P$ of a given surjective map $L \in Lin(X,Y)$ that, for a given $v \in X$, has the value $v$ when applied to $Lv$. This is clearly impossible if $Lv = 0$ but $v \neq 0$, because $P0$ has to be 0. But, as we now show, it can be done as long as $Lv \neq 0$, with a $P$ that depends continuously on $L$ and $v$.

To see this, we first define $\Omega(X,Y) = \{(L,v) : L \in Lin_{onto}(X,Y)\,,\ Lv \neq 0\}$. We then fix inner products $\langle\cdot,\cdot\rangle_X$, $\langle\cdot,\cdot\rangle_Y$, on $X$, $Y$, and use $L^{\#}$ to denote, for $L \in Lin_{onto}(X,Y)$, the Moore-Penrose pseudoinverse of $L$ corresponding to these inner products. Then, for $(L,v) \in \Omega(X,Y)$, we define

$$L^{\#,v}(y) = L^{\#}(y) + \frac{\langle y, Lv\rangle_Y}{\langle Lv, Lv\rangle_Y}(v - L^{\#}Lv)\,. \tag{4.3.1}$$

Then it is clear that

**Fact 4.28** *If $(L,v)$ belongs to $\Omega(X,Y)$, then (1) $L^{\#,v}$ is a linear map from $Y$ to $X$, (2) $LL^{\#,v} = \mathbb{I}_Y$, and (3) $L^{\#,v}Lv = v$. Furthermore, the map $\Omega(X,Y) \ni (L,v) \mapsto L^{\#,v} \in Lin(Y,X)$ is real-analytic.* ☐

***Pseudoinverses on cones.*** If $X, Y$ are FDRLSs and $C$ is a convex cone in $X$, we define

$$\Sigma(X,Y,C) = \Big\{(L,y) \in Lin(X,Y) \times Y : y \in \mathrm{Int}(LC)\Big\}. \tag{4.3.2}$$

(Here "Int($LC$)" denotes the absolute interior of $LC$, i.e., the largest open subset $U$ of $Y$ such that $U \subseteq LC$.)

**Lemma 4.29** *Let $X, Y$ be FDRLSs, let $C$ be a convex cone in $X$, let $S_C$ be the linear span of $C$, and let $\overset{o}{C}$ be the interior of $C$ relative to $S_C$. Then*

*(1) $\Sigma(X,Y,C)$ is an open subset of $Lin(X,Y) \times Y$.*
*(2) There exists a continuous map $\eta_{X,Y,C} : \Sigma(X,Y,C) \mapsto X$ such that the following are true whenever $(L,y) \in \Sigma(X,Y,C)$ and $r \geq 0$:*

$$\eta_{X,Y,C}(L,y) \in \overset{o}{C} \cup \{0\}\,, \tag{4.3.3}$$

$$L\eta_{X,Y,C}(L,y) = y\,, \tag{4.3.4}$$

$$\eta_{X,Y,C}(L,ry) = r\eta_{X,Y,C}(L,y)\,. \tag{4.3.5}$$

*Proof.* We assume, as we clearly may, that $X$ and $Y$ are endowed with inner products, and we write $\Sigma = \Sigma(X,Y,C)$, $S = S_C$.

Statement (1) is trivial, because if $(\bar{L},\bar{y}) \in \Sigma$, and $m = \dim(Y)$, then we can find $m+1$ points $q_0,\ldots,q_m$ in $\mathrm{Int}(\bar{L}C)$ such that $\bar{y}$ is an interior point of the convex hull of the set $Q = \{q_0,\ldots,q_m\}$. Then we can write $q_j = \bar{L}p_j$, with $p_j \in C$, for $j = 0,\ldots,m$. If $L \in \mathrm{Lin}(X,Y)$ is close to $\bar{L}$, and $y \in Y$ is close to $\bar{y}$, then the points $q_j^L = Lp_j$ belong to $LC$, and $y$ is an interior point of their convex hull, so $y \in \mathrm{Int}(LC)$, proving (1).

For each $(\bar{L},\bar{y})\in\Sigma$, we pick a point $x_{\bar{L},\bar{y}}\in\overset{\circ}{C}$ such that $\bar{L}\cdot x_{\bar{L},\bar{y}}=\bar{y}$. (To see that such a point exists, fix a $z\in\overset{\circ}{C}$, and observe that $\bar{y}-\varepsilon\bar{L}\cdot z\in\bar{L}C$ if $\varepsilon$ is positive and small enough, because $\bar{y}\in\mathrm{Int}(\bar{L}C)$, since $(\bar{L},\bar{y})\in\Sigma$. Pick one such $\varepsilon$, write $\bar{y}-\varepsilon\bar{L}\cdot z=\bar{L}\cdot x$ for an $x\in C$, and then let $x_{\bar{L},\bar{y}}=x+\varepsilon z$. It is then clear that $\bar{L}\cdot x_{\bar{L},\bar{y}}=\bar{y}$ and $x_{\bar{L},\bar{y}}\in\overset{\circ}{C}$.) We then define a map $\mu_{\bar{L},\bar{y}}:\Sigma\mapsto X$ by letting $\mu_{\bar{L},\bar{y}}(L,y)=x_{\bar{L},\bar{y}}+(L_S)^{\#}(y-L_Sx_{\bar{L},\bar{y}})$ for $(L,y)\in\Sigma$, where $L_S$ denotes the restriction of $L$ to $S$ (so $L_S\in Lin_{onto}(S,Y)$, because $L_S\in Lin(S,Y)$ and $y\in\mathrm{Int}(LC)=\mathrm{Int}(L_SC)\subseteq\mathrm{Int}(L_SS)$, showing that $\mathrm{Int}(L_SS)\neq\emptyset$, so $L_S$ is surjective).

Then $\mu_{\bar{L},\bar{y}}$ is a continuous map from $\Sigma$ to $S$, and satisfies the identity $\mu_{\bar{L},\bar{y}}(\bar{L},\bar{y})=x_{\bar{L},\bar{y}}$. In addition, if $(L,y)\in\Sigma$, then

$$\begin{aligned}L\cdot\mu_{\bar{L},\bar{y}}(L,y)&=L\cdot x_{\bar{L},\bar{y}}+L\cdot(L_S)^{\#}\cdot(y-L\cdot x_{\bar{L},\bar{y}})\\&=L\cdot x_{\bar{L},\bar{y}}+y-L\cdot x_{\bar{L},\bar{y}}\\&=y\,.\end{aligned}$$

Since $\mu_{\bar{L},\bar{y}}(\bar{L},\bar{y})=x_{\bar{L},\bar{y}}\in\overset{\circ}{C}$, $\overset{\circ}{C}$ is a relatively open subset of $S$, and $\mu_{\bar{L},\bar{y}}$ is a continuous map from $\Sigma$ to $S$, we can pick an open neighborhood $V_{\bar{L},\bar{y}}$ of $(\bar{L},\bar{y})$ in $\Sigma$ such that $\mu_{\bar{L},\bar{y}}(L,y)\in\overset{\circ}{C}$ whenever $(L,y)\in V_{\bar{L},\bar{y}}$.

The family $\mathcal{V}=\{V_{\bar{L},\bar{y}}\}_{(\bar{L},\bar{y})\in\Sigma}$ of open sets is an open covering of $\Sigma$. So we can find a locally finite set $\mathcal{W}$ of open subsets of $\Sigma$ which is a covering of $\Sigma$ and a refinement of $\mathcal{V}$. (That is, (a) every $W\in\mathcal{W}$ is an open subset of $\Sigma$, (b) for every $W\in\mathcal{W}$ there exists $(\bar{L},\bar{y})\in\Sigma$ such that $W\subseteq V_{\bar{L},\bar{y}}$, (c) every $(L,y)\in\Sigma$ belongs to some $W\in\mathcal{W}$, and (d) every compact subset $K$ of $\Sigma$ intersects only finitely many members of $\mathcal{W}$.)

Let $\{\varphi_W\}_{W\in\mathcal{W}}$ be a continuous partition of unity subordinate to the covering $\mathcal{W}$. (That is, (a) each $\varphi_W$ is a continuous nonnegative real-valued function on $\Sigma$ such that $\mathrm{support}(\varphi_W)\subseteq W$, and (b) $\sum_{W\in\mathcal{W}}\varphi_W\equiv 1$. Recall that the *support* of a function $\psi:\Sigma\mapsto\mathbb{R}$ is the closure in $\Sigma$ of the set $\{\sigma\in\Sigma:\psi(\sigma)\neq 0\}$.) Select, for each $W\in\mathcal{W}$, a point $(\bar{L}_W,\bar{y}_W)\in\Sigma$ such that $W\subseteq V_{\bar{L}_W,\bar{y}_W}$, and define $\tilde{\eta}(L,y)=\sum_{W\in\mathcal{W}}\varphi_W(L,y)\mu_{\bar{L}_W,\bar{y}_W}(L,y)$ for $(L,y)\in\Sigma$. Then $\tilde{\eta}$ is a continuous map from $\Sigma$ to $X$. If $(L,y)\in\Sigma$, let $\mathcal{W}(L,y)$ be the set of all $W\in\mathcal{W}$ such that $\varphi_W(L,y)\neq 0$. Then $(L,y)\in W$ for every $W\in\mathcal{W}(L,y)$, so $\mathcal{W}(L,y)$ is a finite set. Clearly, $\tilde{\eta}(L,y)=\sum_{W\in\mathcal{W}(L,y)}\varphi_W(L,y)\mu_{\bar{L}_W,\bar{y}_W}(L,y)$, and $\sum_{W\in\mathcal{W}(L,y)}\varphi_W(L,y)=1$.

If $W\in\mathcal{W}(L,y)$, then $(L,y)\in W\subseteq V_{\bar{L}_W,\bar{y}_W}$, so $\mu_{\bar{L}_W,\bar{y}_W}(y,L)\in\overset{\circ}{C}$ and $L\cdot\mu_{\bar{L}_W,\bar{y}_W}(L,y)=y$. So $\tilde{\eta}(L,y)$ is a convex combination of points belonging to $\overset{\circ}{C}$, and then $\tilde{\eta}(L,y)\in\overset{\circ}{C}$. Furthermore,

$$L\cdot\tilde{\eta}(L,y)=\sum_{W\in\mathcal{W}(L,L)}\varphi_W(L,y)L\cdot\mu_{\bar{L}_W,\bar{y}_W}(L,y)=\Big(\sum_{W\in\mathcal{W}(L,y)}\varphi_W(L,y)\Big)y=y\,.$$

Hence, if we took $\eta_{X,Y,C}$ to be $\tilde{\eta}$, we would be satisfying all the required conditions, except only for the homogeneity property (4.3.5). In order to satisfy (4.3.5) as well, we define $\eta_{X,Y,C}(L,y)$, for $(L,y) \in \Sigma$, by letting

$$\eta_{X,Y,C}(L,y) = \begin{cases} \|y\|\tilde{\eta}\Big(L, \frac{y}{\|y\|}\Big) & \text{if } \ y \neq 0\,, \\ 0 & \text{if } \ y = 0\,. \end{cases}$$

(This is justified, because if $(L,y) \in \Sigma$ and $y \neq 0$ then $\Big(L.\frac{y}{\|y\|}\Big) \in \Sigma$ as well.)

Then $\eta_{X,Y,C}$ clearly satisfies (4.3.3), (4.3.4) and (4.3.5), and it is easy to verify that $\eta_{X,Y,C}$ is continuous. (Continuity at a point $(L,y)$ of $\Sigma$ such that $y \neq 0$ is obvious. To prove continuity at a point $(L,0)$ of $\Sigma$, we pick a sequence $\{(L_j,y_j)\}_{j\in\mathbb{N}}$ of members of $\Sigma$ such that $L_j \to L$ and $y_j \to 0$, and prove that $\eta_{X,Y,C}(L_j,y_j) \to 0$. If this conclusion was not true, there would exist a positive number $\varepsilon$ and an infinite subset $J$ of $\mathbb{N}$ such that

$$\|\eta_{X,Y,C}(L_j,y_j)\| \geq \varepsilon \quad \text{for all} \quad j \in J\,. \tag{4.3.6}$$

In particular, if $j \in J$ then $y_j \neq 0$, so we can define a unit vector $z_j = \frac{y_j}{\|y_j\|}$ and conclude that $(L_j,z_j) \in \Sigma$ and $\eta_{X,Y,C}(L_j,y_j) = \|y_j\|\tilde{\eta}(L_j,z_j)$. Since the $z_j$ are unit vectors, there exists an infinite subset $J'$ of $J$ such that the limit $z = \lim_{j\to\infty,j\in J'} z_j$ exists. Since $(L,0) \in \Sigma$, 0 is an interior point of the cone $LC$, so $LC = Y$ and then $z \in \mathrm{Int}(LC)$ as well. Therefore $(L,z) \in \Sigma$. Since $(L_j,z_j) \to (L,z)$ as $j \to \infty$ via values in $J'$, the continuity of $\tilde{\eta}$ on $\Sigma$ implies that $\tilde{\eta}(L_j,z_j) \to \tilde{\eta}(L,z)$ as $j \to \infty$ via values in $J'$. But then $\eta_{X,Y,C}(L_j,y_j) \to 0$ as $j \to_{J'} \infty$, because $\eta_{X,Y,C}(L_j,y_j) = \|y_j\|\tilde{\eta}(L_j,z_j)$ and $y_j \to 0$. This contradicts (4.3.6).) So $\eta_{X,Y,C}$ satisfies all our conditions, and the proof is complete. □

***The open mapping theorem.*** We are now ready to prove the open mapping theorem.

**Theorem 4.30** *Let $X,Y$ be FDNRLSs, and let $C$ be a convex cone in $X$. Let $F : X \mapsto\!\!\!\to Y$ be a set-valued map, and let $\Lambda \in AGDQ(F,0,0,C)$. Let $\bar{y} \in Y$ be such that $\bar{y} \in \mathrm{Int}(LC)$ for every $L \in \Lambda$. Then*

- (I) *there exist a closed convex cone $D$ in $X$ such that $\bar{y} \in \mathrm{Int}(D)$, and positive constants $\bar{\alpha}$, $\kappa$, having the property that*
  - (I.∗) *for every $y \in D$ such that $0 < \|y\| \leq \bar{\alpha}$ there exists an $x \in C$ such that $\|x\| \leq \kappa\|y\|$ and $y \in F(x)$.*
- (II) *Moreover, $\bar{\alpha}$ and $\kappa$ can be chosen so that*
  - (II.∗) *there exists a function $]\,0,\bar{\alpha}] \ni \alpha \mapsto \rho(\alpha) \in [0,1\,[$ such that $\lim_{\alpha\downarrow 0}\rho(\alpha) = 0$, for which, if we write $C(r) = C \cap \bar{\mathbb{B}}_X(0,r)$, then*
    - (II.∗.#) *for every $\alpha \in ]\,0,\bar{\alpha}]$ and every $y \in D$ such that $\|y\| = \alpha$ there exists a compact connected subset $Z_y$ of the product $C(\kappa\alpha) \times [\rho(\alpha),1]$ having the following properties:*

$$Z_y \cap \Big(C(\kappa\alpha) \times \{\rho(\alpha)\}\Big) \neq \emptyset\,, \qquad Z_y \cap \Big(C(\kappa\alpha) \times \{1\}\Big) \neq \emptyset\,, \tag{4.3.7}$$

$$ry \in F(x) \text{ and } \|x\| \leq \kappa r\|y\| \text{ whenever } \rho(\alpha) \leq r \leq 1 \text{ and } (x,r) \in Z_y\,. \tag{4.3.8}$$

(III) *Finally, if* $\Lambda \in GDQ(F,0,0,C)$ *then the cone* $D$ *and the constants* $\bar{\alpha}$, $\kappa$ *can be chosen so that the following stronger conclusion holds:*

(III.$*$) *if* $y \in D$ *and* $\|y\| \leq \bar{\alpha}$ *then there exists a compact connected subset* $Z_y$ *of* $C(\kappa\|y\|) \times [0,1]$ *such that* $(0,0) \in Z_y$, $Z_y \cap \Big(C(\kappa\alpha) \times \{1\}\Big) \neq \emptyset$, *and* $ry \in F(x)$ *whenever* $(x,r) \in Z_y$.

**Remark 4.31** For $\bar{y} \neq 0$, Conclusion (I) of Theorem 4.30 is the *directional open mapping property with linear rate and fixed angle for the restriction of* $F$ *to* $C$, since it asserts that there is a neighborhood $\mathcal{N}$ of the half-line $H_{\bar{y}} = \{r\bar{y} : r \geq 0\}$ in the space $\mathcal{H}_Y$ of all closed half-lines emanating from 0 in $Y$ such that, if $D_{\mathcal{N}}$ is the union of all the members of $\mathcal{N}$, then for every sufficiently small ball $\bar{\mathbb{B}}_Y(0,\alpha)$ the set $\Big(\bar{\mathbb{B}}_Y(0,\alpha) \cap D_{\mathcal{N}}\Big) \backslash \{0\}$ is contained in the image under $F$ of a relative neighborhood $\bar{\mathbb{B}}_X(0,r) \cap C$ of 0 in $C$, whose radius $r$ can be chosen proportional to $\alpha$.

For $\bar{y} = 0$, Conclusion (I) is the *punctured open mapping property with linear rate for the restriction of* $F$ *to* $C$, because in that case the cone $D$ is necessarily the whole space $Y$, and Conclusion (I) asserts that for every sufficiently small ball $\bar{\mathbb{B}}_Y(0,\alpha)$ the punctured neighborhood $\bar{\mathbb{B}}_Y(0,\alpha) \backslash \{0\}$ is contained in the image under $F$ of a relative neighborhood $\bar{\mathbb{B}}_X(0,r) \cap C$ of 0 in $C$, whose radius $r$ can be chosen proportional to $\alpha$. □

*Proof of Theorem 4.30.* It is clear that (II) implies (I), so there is no need to prove (I), and we may proceed directly to the proof of (II). Furthermore, Conclusion (III) is exactly the same as Conclusion (II), except only for the fact that in (III) $\rho(\alpha)$ is chosen to be equal to 0. So we will just prove (II), making sure that whenever we show the existence of $\rho(\alpha)$ it also follows that $\rho(\alpha)$ can be chosen to be equal to zero when $\Lambda \in GDQ(F,0,0,C)$.

Next, we observe that, once our conclusion is proved for $\bar{y} \neq 0$, its validity for $\bar{y} = 0$ follows by a trivial compactness argument. So we will asume from now on that $\bar{y} \neq 0$, and in that case it is clear that, without loss of generality, we may assume that $\|\bar{y}\| = 1$.

Let $S_C$ be the linear span of $C$, and let $\overset{o}{C}$ be the interior of $C$ relative to $S_C$. Write $\Sigma = \Sigma(X,Y,C)$ (cf. (4.3.2)). Then Lemma 4.29 tells us that $\Sigma$ is open in $Lin(X,Y) \times Y$, and there exists a continuous map $\eta_{X,Y,C} : \Sigma \mapsto X$ such that (4.3.3), (4.3.4) and (4.3.5) hold. We write $\eta = \eta_{X,Y,C}$.

Our hypothesis says that the compact set $\Lambda \times \{\bar{y}\}$ is a subset of $\Sigma$. Hence we can find numbers $\hat{\gamma}$, $\delta$, such that $\delta > 0$, $0 < \hat{\gamma} < 1$, and $\Lambda^{\delta} \times \bar{\mathbb{B}}_Y(\bar{y},\hat{\gamma}) \subseteq \Sigma$. Let $\hat{D} = \{ry : r \in \mathbb{R},\ r \geq 0,\ y \in Y,\ \|y - \bar{y}\| \leq \hat{\gamma}\}$. Then $\hat{D}$ is a closed convex cone in $Y$ and $\bar{y} \in \mathrm{Int}(\hat{D})$. Furthermore, it is clear that $\Lambda^{\delta} \times (\hat{D} \backslash \{0\}) \subseteq \Sigma$. So $\eta(L,y)$ is well defined whenever $L \in \Lambda^{\delta}$ and $y \in \hat{D} \backslash \{0\}$. In particular, $\eta(L,y)$ is defined for $(L,y) \in J$, where $J = \{(L,y) : L \in \Lambda^{\delta},\ y \in \hat{D},\ \|y\| = 1\}$, so $J$ is compact. Let $H = \{\eta(L,y) : (L,y) \in J\}$. Then $H$ is a compact subset of $\overset{o}{C}$. Pick a compact subset $\tilde{H}$ of $\overset{o}{C}$ such that $H$ is contained in the interior of $\tilde{H}$.

Since $\tilde{H}$ is a compact subset of the convex set $\overset{o}{C}$, the convex hull $\hat{H}$ of $\tilde{H}$ is also a compact subset of $\overset{o}{C}$. If $0 \notin \hat{H}$, and we define $\mathcal{C} = \{rx : r \geq 0, x \in \hat{H}\}$, then $\mathcal{C}$ is a closed convex cone in $\mathbb{R}^n$ such that

$$\mathcal{C} \subseteq \overset{o}{C} \cup \{0\} \qquad \text{and} \qquad \eta(L,y) \in \mathrm{Int}(\mathcal{C}) \ \text{ whenever } \ (L,y) \in J\,. \tag{4.3.9}$$

If $0 \in \hat{H}$, then $0 \in \overset{o}{C}$, so $C = S_C$ and then in particular $C$ is closed, so we can define $\mathcal{C} = C$, and then then $\mathcal{C}$ is a closed convex cone in $\mathbb{R}^n$ such that (4.3.9) holds. Let $\hat{\kappa} = \max\{\|\eta(L,y)\| : (y,L) \in J\}$. Then $\|\eta(L,y)\| \leq \hat{\kappa}\|y\|$ whenever $(L,y) \in \Lambda^\delta \times (\hat{D}\backslash\{0\})$. This shows that $\eta$ can be extended to a continuous map from $\Lambda^\delta \times \hat{D}$ to $\mathcal{C}$ by letting $\eta(L,0) = 0$ for $L \in \Lambda^\delta$.

Fix a $\gamma \in ]\,0, \hat{\gamma}\,[$, and let $D = \{ry : r \in \mathbb{R},\ r \geq 0,\ y \in Y,\ \|y - \bar{y}\| \leq \gamma\}$. Then $D$ is a closed convex cone in $Y$, $\bar{y} \in \mathrm{Int}(D)$, and $D \subseteq \mathrm{Int}(\hat{D}) \cup \{0\}$. More precisely, we may pick a $\tilde{\sigma}$ such that $\tilde{\sigma} > 0$ and $\bar{\mathbb{B}}_Y(y, \tilde{\sigma}\|y\|) \subseteq \hat{D}$ whenever $y \in D$. (For example, $\tilde{\sigma} = \hat{\gamma} - \gamma$ will do. A simple calculation shows that the best—i.e., largest—possible choice of $\tilde{\sigma}$ is $\tilde{\sigma} = (\hat{\gamma} - \gamma)(1 - \gamma)^{-1/2}$.) We then let $\sigma = \frac{\tilde{\sigma}}{2}$, $\kappa = \hat{\kappa}(1 + 2\sigma)$.

Fix an AGDQ modulus $\theta$ for $(F, 0, 0, C)$. For each $\varepsilon$ such that $\theta(\varepsilon)$ is finite, pick a map $A_\varepsilon \in CCA(C(\varepsilon), Lin(X,Y) \times Y)$ such that

$$\Big(x \in C(\varepsilon) \wedge (L,h) \in A_\varepsilon(x)\Big) \Rightarrow \Big(L \in \Lambda^{\theta(\varepsilon)} \wedge \|h\| \leq \theta(\varepsilon)\varepsilon \wedge L \cdot x + h \in F(x)\Big).$$

Also, observe that when $\Lambda \in GDQ(F, 0, 0, C)$ then $A_\varepsilon$ can be chosen so that all the members $(L,h)$ of $A_\varepsilon(x)$ are such that $h = 0$. In that case, we let $G_\varepsilon(x)$ be such that $A_\varepsilon(x) = G_\varepsilon(x) \times \{0\}$.

Next, fix a positive number $\bar{\varepsilon}$ such that $\theta(\bar{\varepsilon}) < \delta$ and $\theta(\bar{\varepsilon}) < \frac{\sigma}{\kappa}$. Let $\bar{\alpha} = \frac{\bar{\varepsilon}}{\kappa}$.

Fix an $\alpha$ such that $0 < \alpha \leq \bar{\alpha}$, and let $\varepsilon = \kappa\alpha$, so $0 < \varepsilon \leq \bar{\varepsilon}$. Then $\theta(\varepsilon) < \delta$ and $\theta(\varepsilon) < \frac{\sigma}{\kappa}$. Let $\mathcal{C}(\varepsilon) = \mathcal{C} \cap \bar{\mathbb{B}}_X(0, \varepsilon)$. Then $\mathcal{C}(\varepsilon)$ is a nonempty compact convex subset of $X$.

Now choose $\rho(\alpha)$—for $\alpha \in ]\,0, \bar{\alpha}]$—as follows:

$$\rho(\alpha) = \begin{cases} 0 & \text{if } \Lambda \in GDQ(F,0,0,C) \\ \frac{\kappa\theta(\kappa\alpha)}{\sigma} & \text{if } \Lambda \notin GDQ(F,0,0,,C)\,. \end{cases}$$

It is then clear that $0 \leq \rho(\alpha) < 1$, because $\theta(\kappa\alpha) \leq \theta(\kappa\bar{\alpha}) = \theta(\bar{\varepsilon}) < \frac{\sigma}{\kappa}$. Furthermore, $\rho(\alpha)$ clearly goes to 0 as $\alpha \downarrow 0$.

Fix a $y \in D$ such that $\|y\| = \alpha$. Let $Q_\varepsilon = \mathcal{C}(\varepsilon) \times [0,1]$, and define a set-valued map $H_\varepsilon : Q_\varepsilon \mapsto\!\!\!\rightarrow X$ by letting $H_\varepsilon(x,t) = x - U_\varepsilon(x,t)$ (that is, $H_\varepsilon(x,t) = \{x - \xi : \xi \in U_{\bar{\varepsilon}}(x,t)\}$) for $x \in \mathcal{C}(\varepsilon)$, $t \in [0,1]$, where, for $(x,t) \in Q_\varepsilon$,

- if $\Lambda \notin GDQ(F,0,0,C)$, then $U_\varepsilon(x,t) = \Big\{\eta(L, ty - \varphi_\varepsilon(t)h) : (L,h) \in A_\varepsilon(x)\Big\}$, where the function $\varphi_\varepsilon : [0,1] \mapsto [0,1]$ is defined by

$$\varphi_\varepsilon(t) = \frac{t}{\rho(\alpha)} \quad \text{if} \quad 0 \leq t < \rho(\alpha)\,, \qquad \varphi_\varepsilon(t) = 1 \quad \text{if} \quad \rho(\alpha) \leq t \leq 1\,;$$

- if $\Lambda \in GDQ(F,0,0,C)$, then $U_\varepsilon(x,t) = \Big\{\eta(L,ty) : L \in G_\varepsilon(x)\Big\}$.

We claim that $H_\varepsilon \in CCA(Q_\varepsilon, X)$. To see this, we first show that

$$ty - \varphi_\varepsilon(t)h \in \hat{D} \qquad \text{whenever} \quad (x,t) \in Q_\varepsilon \text{ and } (L,h) \in A_\varepsilon(x)\,. \qquad (4.3.10)$$

This conclusion is trivial if $\Lambda \in GDQ(F,0,0,C)$, because in that case $h = 0$. Now consider the case when $\Lambda \notin GDQ(F,0,0,C)$, and observe that if $x \in \mathcal{C}(\varepsilon)$ and $(L,h) \in A_\varepsilon(x)$ then $L \in \Lambda^{\theta(\varepsilon)}$ and

$$\|\varphi_\varepsilon(t)h\| \le \frac{t}{\rho(\alpha)}\|h\| \le \frac{t}{\rho(\alpha)}\theta(\varepsilon)\varepsilon = \frac{t}{\rho(\alpha)}\theta(\kappa\alpha)\kappa\|y\| = \frac{t}{\rho(\alpha)}\theta(\kappa\alpha)\kappa\|y\|$$

$$= \frac{1}{\rho(\alpha)}\frac{\theta(\kappa\alpha)\kappa}{\sigma}t\sigma\|y\| = \frac{1}{\rho(\alpha)}\rho(\alpha)t\sigma\|y\| = t\sigma\|y\|\,.$$

It follows that $ty - \varphi_\varepsilon(t)h$ belongs to the ball $\bar{\mathbb{B}}_Y(ty, t\sigma\|y\|)$, which is contained in $\hat{D}$. So $ty - \varphi_\varepsilon(t)h \in \hat{D}$, completing the proof of (4.3.10).

Next, let $\mu$ be the set-valued map with source $Q_\varepsilon$ and target $Lin(X,Y) \times Y$, such that $\mu(x,t) = \{(L, ty - \varphi_\varepsilon(t)h) : (L,h) \in A_\varepsilon(x)\}$. Then $\mu$ belongs to $CCA(Q_\varepsilon, Lin(X,Y) \times Y)$, because it is the composite of the maps

$$Q_\varepsilon \ni (x,t) \mapsto\!\!\!\rightarrow A_\varepsilon(x) \times \{t\} \subseteq Lin(X,Y) \times Y \times \mathbb{R}\,,$$

and $$Lin(X,Y) \times Y \times \mathbb{R} \ni (L,h,t) \mapsto (L, ty - \varphi_\varepsilon(t)h) \in Lin(X,Y) \times Y\,.$$

On the other hand, $\mu$ actually takes values in $\Lambda^{\theta(\varepsilon)} \times \hat{D}$. Therefore, if we let $\nu$ be the map having exactly the same graph as $\mu$, but with target $\Lambda^\delta \times \hat{D}$, then $\nu \in CCA(Q_\varepsilon, \Lambda^\delta \times \hat{D})$. (Indeed, if $\{\mu_j\}_{j\in\mathbb{N}}$ is a sequence of continuous maps from $Q_\varepsilon$ to $Lin(X,Y) \times Y$ with the property that $\mu_j \xrightarrow{\text{igr}} \mu$, and we write $\mu_j(x,t) = (L_j(x,t), \zeta_j(x,t))$, then $L_j$ will take values in $\Lambda^\delta$ if $j$ is large enough, because $\Lambda^\delta$ is a neighborhood of $\Lambda^{\theta(\varepsilon)}$. On the other hand, $\hat{D}$ is a closed convex subset of $Y$, so it is a retract of $Y$. If $\omega : Y \mapsto \hat{D}$ is a retraction, and $\nu_j(x,t) = (L_j(x,t), \omega(\zeta_j(z,t)))$, then $\{\nu_j\}_{j\in\mathbb{N}, j\ge j_*}$ is—for some $j_*$—a sequence of continuous maps from $Q_\varepsilon$ to $\Lambda^\delta \times \hat{D}$ such that $\nu_j \xrightarrow{\text{igr}} \nu$.) Now, $U_\varepsilon$ is the composite $\eta \circ \nu$, and $\eta$ is a continuous map on $\Lambda^\delta \times \hat{D}$. So $U_\varepsilon \in CCA(Q_\varepsilon, X)$, and then $H_\varepsilon \in CCA(Q_\varepsilon, X)$ as well, completing the proof that $H_\varepsilon \in CCA(Q_\varepsilon, X)$.

It is clear that

$$\text{if} \quad (x,t) \in Q_\varepsilon \quad \text{then} \quad 0 \in H_\varepsilon(x,t) \Longleftrightarrow x \in U_\varepsilon(x,t)\,.$$

We now analyze the implications of the statement "$x \in U_\varepsilon(x,t)$" in two cases.

First, suppose that $\Lambda \in GDQ(F,0,0,C)$. Then $x \in U_\varepsilon(x,t)$ if and only if $(\exists L \in G_\varepsilon(x))(x = \eta(L,ty))$. If such an $L$ exists, then $L \cdot x = L\eta(L,ty) = ty$, so $ty \in G_\varepsilon(x) \cdot x$, and then $ty \in F(x)$. Furthermore, the fact that $x = \eta(L,ty)$ implies that $\|x\| \le \hat{\kappa}t\|y\|$, so *a fortiori* $\|x\| \le \kappa t\|y\|$.

Now suppose that $\Lambda \notin GDQ(F,0,0,C)$. Then $x \in U_\varepsilon(x,t)$ if and only if $\Big(\exists (L,h) \in A_\varepsilon(x)\Big)\Big(x = \eta(L, ty - \varphi_\varepsilon(t)h)\Big)$. If such a pair $(L,h)$ exists, and $t \geq \rho(\alpha)$, then

$$L \cdot x = L\eta(L, ty - \varphi_\varepsilon(t)h) = L\eta(L, ty - h) = ty - h$$

so $L \cdot x + h = ty$, and then $ty \in F(x)$. On the other hand, the fact that $x = \eta(L, ty - \varphi_\varepsilon(t)h)$ implies that $\|x\| \leq \hat{\kappa}(t\|y\| + t\sigma\|y\|)$, since we have already established that $\|\varphi_\varepsilon(t)h\| \leq t\sigma\|y\|$. Hence $\|x\| \leq \kappa t\|y\|$.

So we have shown, in both cases, that

(A) *if* $(x,t) \in Q_\varepsilon$, $0 \in H_\varepsilon(x,t)$ *and* $\rho(\alpha) \leq t \leq 1$, *then* $ty \in F(x)$ *and* $\|x\| \leq \kappa t\|y\|$.

In addition, $H_\varepsilon$ obviously satisfies

(B) $H_\varepsilon(x,0) = \{x\}$ *whenever* $x \in \mathcal{C}(\varepsilon)$.

Next, choose a sequence $\{v_j\}_{j\in\mathbb{N}}$ of interior points of $\mathcal{C}$ such that $v_j \to 0$ as $j \to \infty$ and $\|v_j\| < \sigma\hat{\kappa}\|y\|$ for all $j$. We claim that

(C) $v_j \notin H_\varepsilon(x,t)$ *whenever* $x \in \partial\mathcal{C}(\varepsilon)$, $t \in [0,1]$, *and* $j \in \mathbb{N}$.

To see this, we first observe that the condition $v_j \in H_\varepsilon(x,t)$ is equivalent to $x \in v_j + U_\varepsilon(x,t)$. If $x \in \partial\mathcal{C}(\varepsilon)$, then either $x \in \partial\mathcal{C}$ or $\|x\| = \kappa\|y\|$. If $x \in \partial\mathcal{C}$, then $x$ cannot belong to $v_j + U_\varepsilon(x,t)$, because $U_\varepsilon(x,t) \subseteq \mathcal{C}$ and $v_j \in \mathrm{Int}(\mathcal{C})$, so $v_j + U_\varepsilon(x,t) \subseteq \mathrm{Int}(\mathcal{C})$. If $\|x\| = \kappa\|y\|$, then $x$ cannot belong to $v_j + U_\varepsilon(x,t)$ either, because if $(L,h) \in A_{\bar{\varepsilon}}(x)$ then

$$\|\eta(L, ty - \varphi_\varepsilon(t)h)\| \leq \hat{\kappa}\|ty - \varphi_\varepsilon(t)h\| \leq \hat{\kappa}\|ty\| + \hat{\kappa}\|\varphi_\varepsilon(t)h\|$$

$$\leq t\hat{\kappa}\|y\| + t\hat{\kappa}\|\sigma\|y\| = t\hat{\kappa}(1+\sigma)\|y\| \leq \hat{\kappa}(1+\sigma)\|y\|\,,$$

so $\|v_j + \eta(L,ty)\| \leq \hat{\kappa}(1+\sigma)\|y\| + \|v_j\| < \hat{\kappa}(1+\sigma)\|y\| + \hat{\kappa}\sigma\|y\| = \kappa\|y\|$.

Hence we can apply Theorem 3.8 and conclude that there exists a compact connected subset $Z$ of $\mathcal{C}(\varepsilon) \times [0,1]$ such that (i) the sets $Z \cap (\mathcal{C}(\varepsilon) \times \{0\})$ and $Z \cap (\mathcal{C}(\varepsilon) \times \{1\})$ are nonempty, and (ii) $0 \in H_\varepsilon(x,t)$ whenever $(x,t) \in Z$.

For $\beta$ such that $0 < \beta < 1 - \rho(\alpha)$, let $Z^{(\beta)}$ be the open $\beta$-neighborhood of $Z$ in $Q_\varepsilon$, so that $Z^{(\beta)} = \{q \in Q_\varepsilon : \mathrm{dist}(q,Z) < \beta\}$. Then $Z^{(\beta)}$ is a relatively open subset of $Q_\varepsilon$. It is clear that $Z^{(\beta)}$ is connected, so it is path-connected. Since $Z^{(\beta)}$ intersects both sets $\mathcal{C}(\varepsilon) \times \{0\}$ and $\mathcal{C}(\varepsilon) \times \{1\}$, there exists a continuous map $\xi : [0,1] \mapsto Z^{(\beta)}$ such that $\xi(0) \in \mathcal{C}(\varepsilon) \times \{0\}$ and $\xi(1) \in \mathcal{C}(\varepsilon) \times \{1\}$. Let

$$I = \Big\{t \in [0,1] : \xi(t) \in \mathcal{C}(\varepsilon) \times [0, \rho(\alpha) + \beta]\Big\}\,,$$

Then it is clear that $I$ is a nonempty compact subset of $[0,1]$, so $I$ has a largest element $\tau$. Then $\xi(\tau) \in \mathcal{C}(\varepsilon) \times \{\rho(\alpha) + \beta\}$, and

$$\xi(t) \in \mathcal{C}(\varepsilon) \times [\rho(\alpha) + \beta, 1] \quad \text{whenever} \quad \tau \leq t \leq 1\,. \tag{4.3.11}$$

Hence, if we define $W^\beta = \gamma([\tau, 1])$, we see that (i) $W^\beta$ is compact and connected, (ii) $W^\beta \subseteq \mathcal{C}(\varepsilon) \times [\rho(\alpha) + \beta, 1]$, (iii) $W^\beta \cap \Big(\mathcal{C}(\varepsilon) \times \{\rho(\alpha) + \beta\}\Big) \neq \emptyset$, (iv) $W^\beta \cap \Big(\mathcal{C}(\varepsilon) \times \{1\}\Big) \neq \emptyset$, and (v) $\operatorname{dist}(w, Z) \leq \beta$ whenever $w \in W^\beta$.

Let $\tilde{Z} = Z \cap (\mathcal{C}(\varepsilon) \times [\rho(\alpha), 1])$. Then $\tilde{Z}$ is a compact subset of $\mathcal{C}(\varepsilon) \times [\rho(\alpha), 1]$. If $w \in W^\beta$, then the point $z_w \in Z$ closest to $w$ is at a distance $\leq \beta$ from $w$, and must therefore belong to $\mathcal{C}(\varepsilon) \times [\rho(\alpha), 1]$, since $w \in \mathcal{C}(\varepsilon) \times [\rho(\alpha) + \beta, 1]$. It follows that $z_w \in \tilde{Z}$. Therefore

$$\operatorname{dist}(w, \tilde{Z}) \leq \beta \;\text{ whenever }\; w \in W^\beta\,. \tag{4.3.12}$$

We now use Theorem 3.7 to pick a sequence $\{\beta_j\}_{j \in \mathbb{N}}$ converging to zero, such that the sets $W^{\beta_j}$ converge in $Comp(Q_\varepsilon)$ to a compact connected set $W$. It then follows from (4.3.12) that $W \subseteq \tilde{Z}$. On the other hand, since the sets $W^{\beta_j} \cap \Big(\mathcal{C}(\varepsilon) \times \{\rho(\alpha) + \beta_j\}\Big)$ and $W^{\beta_j} \cap \Big(\mathcal{C}(\varepsilon) \times \{1\}\Big)$ are nonempty for each $j$, we can easily conclude that $W \cap \Big(\mathcal{C}(\varepsilon) \times \{\rho(\alpha)\}\Big) \neq \emptyset$ and $W \cap \Big(\mathcal{C}(\varepsilon) \times \{1\}\Big) \neq \emptyset$. Hence, if we take $Z_y$ to be the set $W$, we see that (i) $Z_y$ is compact connected, (ii) $Z_y \subseteq Z \cap \Big(\mathcal{C}(\varepsilon) \times [\rho(\alpha), 1]\Big)$, and (iii) $Z_y$ has a nonempty intersection with both $\mathcal{C}(\varepsilon) \times \{\rho(\alpha)\}$ and $\mathcal{C}(\varepsilon) \times \{1\}$.

Now, if $(x, t) \in Z_y$, we know that $0 \in H_\varepsilon(x, t)$, and then (A) implies that $ty \in F(x)$ and $\|x\| \leq \kappa t \|y\|$, since $\rho(\alpha) \leq t \leq 1$. This shows that $Z_y$ satisfies all the conditions of our statement, and completes our proof. $\square$

***Approximating multicones.*** Assume that $M$ is a manifold of class $C^1$, $S$ is a subset of $M$ and $\bar{x}_* \in S$ .Recall that "multicones" were defined on Page 5.

**Definition 4.32** *An* ***AGDQ approximating multicone to*** $S$ ***at*** $\bar{x}_*$ *is a convex multicone* $\mathcal{C}$ *in* $T_{\bar{x}_*}M$ *such that there exist an* $m \in \mathbb{Z}_+$, *a set-valued map* $F : \mathbb{R}^m \mapsto\!\!\!\!\rightarrow M$, *a convex cone* $D$ *in* $\mathbb{R}^m$, *and a* $\Lambda \in AGDQ(F, 0, \bar{x}_*, D)$, *such that* $F(D) \subseteq S$ *and* $\mathcal{C} = \{LD : L \in \Lambda\}$. *If* $\Lambda$ *can be chosen so that* $\Lambda \in GDQ(F, 0, \bar{x}_*, D)$, *then* $\mathcal{C}$ *is said to be a* ***GDQ approximating multicone to*** $S$ ***at*** $\boldsymbol{x}_*$. $\square$

***Transversality of cones and multicones.*** If $S_1$, $S_2$ are subsets of a linear space $X$, we define the *sum* $S_1 + S_2$ and the *difference* $S_1 - S_2$ by letting

$$S_1 + S_2 = \{s_1 + s_2 : s_1 \in S_1, s_2 \in S_2\}\,, \quad S_1 - S_2 = \{s_1 - s_2 : s_1 \in S_1, s_2 \in S_2\}\,.$$

**Definition 4.33** *Let* $X$ *be a FDRLS. We say that two convex cones* $C^1$, $C^2$ *in* $X$ *are* ***transversal****, and write* $C^1 \overline{\pitchfork} C^2$, *if* $C^1 - C^2 = X$. $\square$

**Definition 4.34** *Let* $X$ *be a FDRLS. We say that two convex cones* $C^1$, $C^2$ *in* $X$ *are* ***strongly transversal****, and write* $C^1 \overline{\pitchfork\!\!\!\shortmid} C^2$, *if* $C^1 \overline{\pitchfork} C^2$ *and in addition* $C^1 \cap C^2 \neq \{0\}$. $\square$

The definition of "transversality" of multicones is a straightforward extension of that of transversality of cones.

**Definition 4.35** *Let $X$ be a FDRLS. We say that two convex multicones $\mathcal{C}^1$ and $\mathcal{C}^2$ in $X$ are **transversal**, and write $\mathcal{C}^1 \overline{\pitchfork} \mathcal{C}^2$, if $C^1 \overline{\pitchfork} C^2$ for all pairs $(C^1, C^2) \in \mathcal{C}^1 \times \mathcal{C}^2$.* □

The definition of "strong transversality" for multicones requires more care. It is clear that two convex cones $C^1$, $C^2$ are strongly transversal if and only if (i) $C^1 \overline{\pitchfork} C^2$, and (ii) there exists a nontrivial linear functional $\lambda \in X^\dagger$ such that $C^1 \cap C^2 \cap \{x \in X : \lambda(x) > 0\} \neq \emptyset$. It is under this form that the definition generalizes to multicones.

**Definition 4.36** *Let $X$ be a finite-dimensional real linear space. Let $\mathcal{C}^1$, $\mathcal{C}^2$ be convex multicones in $X$. We say that $\mathcal{C}^1$ and $\mathcal{C}^2$ are **strongly transversal**, and write $\mathcal{C}^1 \overline{\overline{\pitchfork}} \mathcal{C}^2$, if (i) $\mathcal{C}^1 \overline{\pitchfork} \mathcal{C}^2$, and (ii) there exists a nontrivial linear functional $\lambda \in X^\dagger$ such that $C^1 \cap C^2 \cap \{x \in X : \lambda(x) > 0\} \neq \emptyset$ for every $(C^1, C^2) \in \mathcal{C}^1 \times \mathcal{C}^2$.* □

***The nonseparation theorem.*** If $S_1$, $S_2$ are subsets of a topological space $T$, and $\bar{s}_* \in S_1 \cap S_2$, we say that $S_1$ and $S_2$ are *locally separated* at $\bar{s}_*$ if there exists a neighborhood $U$ of $\bar{s}_*$ such that $S_1 \cap S_2 \cap U = \{\bar{s}_*\}$. If $T$ is metric, then it is clear that $S_1$ and $S_2$ are locally separated at $\bar{s}_*$ if and only if there does not exist a sequence $\{s_j\}_{j \in \mathbb{N}}$ of points of $(S_1 \cap S_2) \backslash \{0\}$ converging to $\bar{s}_*$.

**Theorem 4.37** *Let $M$ be a manifold of class $C^1$, let $S_1$, $S_2$ be subsets of $M$, and let $\bar{s}_* \in S_1 \cap S_2$. Let $\mathcal{C}_1$, $\mathcal{C}_2$ be AGDQ-approximating multicones to $S_1$, $S_2$ at $\bar{s}_*$ such that $\mathcal{C}_1 \overline{\overline{\pitchfork}} \mathcal{C}_2$. Then $S_1$ and $S_2$ are not locally separated at $\bar{s}_*$ (that is, the set $S_1 \cap S_2$ contains a sequence of points $s_j$ converging to $\bar{s}_*$ but not equal to $\bar{s}_*$). Furthermore,*

*(1) if $\xi : \Omega \mapsto \mathbb{R}^n$ is a coordinate chart of $M$, defined on an open set $\Omega$ containing $\bar{s}_*$, and such that $\xi(\bar{s}_*) = 0$, then there exist positive numbers $\bar{\alpha}$, $\kappa$, $\sigma$, and a function $\rho : ]\,0, \bar{\alpha}] \mapsto [0, 1\,[$ such that $\lim_{\alpha \downarrow 0} \rho(\alpha) = 0$, having the property that, whenever $0 < \alpha \leq \bar{\alpha}$, the set $\xi(S_1 \cap S_2 \cap \Omega)$ contains a nontrivial compact connected set $Z_\alpha$ such that $Z_\alpha$ contains points $x_-(\alpha)$, $x_+(\alpha)$, for which $\|x_-(\alpha)\| \leq \kappa \rho(\alpha) \alpha$ and $\|x_+(\alpha)\| \geq \sigma \alpha$,*

*(2) if $\mathcal{C}_1$, $\mathcal{C}_2$ are GDQ-approximating multicones to $S_1$, $S_2$ at $\bar{s}_*$. then $S_1 \cap S_2$ contains a nontrivial compact connected set $Z$ such that $\bar{s}_* \in Z$.*

In view of our definitions, Theorem 4.37 will clearly follow if we prove:

**Theorem 4.38** *Let $n_1, n_2, m$ be positive integers. Assume that, for $i = 1, 2$, (1) $C_i$ is a convex cone in $\mathbb{R}^{n_i}$, (2) $F_i : \mathbb{R}^{n_i} \mapsto\!\!\!\!\rightarrow \mathbb{R}^m$ is a set-valued map, and (3) $\Lambda_i \in AGDQ(F_i, 0, 0, C_i)$. Assume that the transversality condition*

$$L_1 C_1 - L_2 C_2 = \mathbb{R}^m \ \text{ for all } \ (L_1, L_2) \in \Lambda_1 \times \Lambda_2 \tag{4.3.13}$$

*holds, and there exists a nontrivial linear functional* $\mu : \mathbb{R}^m \mapsto \mathbb{R}$ *such that*

$$L_1C_1 \cap L_2C_2 \cap \{y \in \mathbb{R}^m : \mu(y) > 0\} \neq \emptyset \quad \text{for all} \quad (L_1, L_2) \in \Lambda_1 \times \Lambda_2 \,. \tag{4.3.14}$$

*Let* $\mathcal{I} = \Big\{ (x_1, x_2, y) \in C_1 \times C_2 \times \mathbb{R}^m : y \in F_1(x_1) \cap F_2(x_2) \Big\}$*. Then there exist positive constants* $\bar{\alpha}$*,* $\kappa$*,* $\sigma$*, and a function* $\rho : ]\,0, \bar{\alpha}] \mapsto [0, 1[$ *such that* $\lim_{\alpha \downarrow 0} \rho(\alpha) = 0$*, having the property that*

*(*) for every* $\alpha$ *for which* $0 < \alpha \leq \bar{\alpha}$ *there exist a compact connected subset* $Z_\alpha$ *of* $\mathcal{I}$*, and points* $(x_{1,\alpha,-}, x_{2,\alpha,-}, y_{\alpha,-})$*,* $(x_{1,\alpha,+}, x_{2,\alpha,+}, y_{\alpha,+})$ *of* $Z_\alpha$*, for which* $\|y_{\alpha,+}\| \geq \sigma\alpha$ *and* $\|y_{\alpha,-}\| \leq \kappa\rho(\alpha)\alpha$*.*

*Furthermore, if* $\Lambda_i \in GDQ(F_i, 0, 0, C_i)$ *for* $i = 1, 2$*, then it is possible to choose* $\rho(\alpha) \equiv 0$*.*

*Proof.* Define a set-valued map $\mathcal{F} : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^m \mapsto\!\!\!\to \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}$ by letting $\mathcal{F}(x_1, x_2, y) = (y - F_1(x_1), y - F_2(x_2), \mu(y))$ for $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$, $y \in \mathbb{R}^m$. (Precisely, this means that $\mathcal{F}(x_1, x_2, y)$ is the set of all triples $(y - y_1, y - y_2, \mu(y))$, for all $y_1 \in F_1(x_1)$, $y_2 \in F_2(x_2)$.)

Also, define a cone $C \subseteq \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^m$ by letting $C = C_1 \times C_2 \times \mathbb{R}^m$, and a subset $\mathcal{L}$ of $Lin(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^m, \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R})$ by letting $\mathcal{L}$ be the set of all linear maps $\mathcal{L}_{L_1,L_2}$, for all $(L_1, L_2) \in \Lambda_1 \times \Lambda_2$, where $\mathcal{L}_{L_1,L_2}$ is the map from $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^m$ to $\mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}$ such that

$$\begin{aligned} \mathcal{L}_{L_1,L_2}(x_1, x_2, y) = (y - L_1x_1, y - L_2x_2, \mu(y)) \\ \text{if} \quad (x_1, x_2, y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^m. \end{aligned} \tag{4.3.15}$$

It then follows immediately from the definition of AGDQs and GDQs that $\mathcal{L} \in AGDQ(\mathcal{F}, (0,0,0), (0,0), C)$, and also that $\mathcal{L} \in GDQ(\mathcal{F}, (0,0,0), (0,0), C)$ if $\Lambda_i$ is in $GDQ(F_i, 0, 0, C_i)$ for $i = 1, 2$.

Let $\bar{w}_* = (0, 0, 1)$. We want to show that the conditions of the directional open mapping theorem are satisfied, that is, that $\bar{w}_* \in \text{Int}(LC)$ whenever $L \in \mathcal{L}$. Let $L \in \mathcal{L}$, and write $L = \mathcal{L}_{L_1,L_2}$, with $L_1 \in \Lambda_1$, $L_2 \in \Lambda_2$. Using (4.3.14), find $\bar{c}_1 \in C_1$, $\bar{c}_2 \in C_2$, such that $L_1\bar{c}_1 = L_2\bar{c}_2$ and $\mu(L_1\bar{c}_1) > 0$. Let $\bar{\alpha} = \mu(L_1\bar{c}_1)$. Let $v_1, v_2 \in \mathbb{R}^m$ be arbitrary vectors. We claim that the equation

$$L(x_1, x_2, y) = (v_1, v_2, r) \tag{4.3.16}$$

has a solution $(x_1, x_2, y) \in C$ provided that $r$ is large enough. To see this, observe first that (4.3.13) implies that we can express $v_2 - v_1$ as a difference

$$v_2 - v_1 = L_1c_1 - L_2c_2 \,, \quad c_1 \in C_1 \,, \; c_2 \in C_2 \,. \tag{4.3.17}$$

Then, if we let $\tilde{y} = v_1 + L_1c_1$ (so that (4.3.17) implies that $\tilde{y} = v_2 + L_2c_2$ as well), it is clear that $L(c_1, c_2, \tilde{y}) = (\tilde{y} - L_1c_1, \tilde{y} - L_2c_2, \mu(\tilde{y})) = (v_1, v_2, \tilde{r})$, if we let $\tilde{r} \overset{\text{def}}{=} \mu(\tilde{y})$. If $r \geq \tilde{r}$, then we can choose

$$y = \tilde{y} + \frac{r - \tilde{r}}{\bar{\alpha}} \cdot L_1\bar{c}_1 \,, \quad x_1 = c_1 + \frac{r - \tilde{r}}{\bar{\alpha}} \cdot \bar{c}_1 \,, \quad x_2 = c_2 + \frac{r - \tilde{r}}{\bar{\alpha}} \cdot \bar{c}_2 \,.$$

With this choice, we have

$$\begin{aligned} y - L_1x_1 &= \tilde{y} - L_1c_1 + \frac{r-\tilde{r}}{\bar{\alpha}} \cdot L_1\bar{c}_1 - \frac{r-\tilde{r}}{\bar{\alpha}} \cdot L_1\bar{c}_1 = \tilde{y} - L_1c_1 = v_1\,, \\ y - L_2x_2 &= \tilde{y} - L_2c_2 + \frac{r-\tilde{r}}{\bar{\alpha}} \cdot L_1\bar{c}_1 - \frac{r-\tilde{r}}{\bar{\alpha}} \cdot L_2\bar{c}_2 \\ &= \tilde{y} - L_2c_2 + \frac{r-\tilde{r}}{\bar{\alpha}} \cdot L_1\bar{c}_1 - \frac{r-\tilde{r}}{\bar{\alpha}} \cdot L_1\bar{c}_1 = \tilde{y} - L_2c_2 = v_2\,, \end{aligned}$$

and $\mu(y) = \mu(\tilde{y}) + \frac{r-\tilde{r}}{\bar{\alpha}}\mu(L_1\bar{c}_1) = \tilde{r} + r - \tilde{r} = r$. It then follows that $L(x_1, x_2, y) = (v_1, v_2, r)$, and we have found our desired solution of (4.3.16).

So we have shown that for every $(v_1, v_2) \in \mathbb{R}^m \times \mathbb{R}^m$ the vector $(v_1, v_2, r)$ belongs to $L \cdot C$ if $r$ is large enough. This easily implies that the point $\bar{w}_* = (0, 0, 1)$ belongs to the interior of $L \cdot C$. (This can be proved in many ways. For example, let $E = (e_0, \ldots, e_{2m})$ be a sequence of $2m + 1$ affinely independent vectors in $\mathbb{R}^m \times \mathbb{R}^m$ such that the origin of $\mathbb{R}^m \times \mathbb{R}^m$ is an interior point of the convex hull of $E$. Then we can find an $\bar{r}$ such that $\bar{r} > 0$ and $(e_i, \bar{r}) \in LC$ whenever $r \geq \bar{r}$. It then follows that the vectors $(e_i, \bar{r})$ and $(e_i, \bar{r} + 2)$ belong to $LC$, so the vector $(0, 0, \bar{r} + 1)$ is in $\mathrm{Int}(LC)$, and then $(0, 0, 1) \in \mathrm{Int}(LC)$ as well.)

We can then apply Theorem 4.30 to the map $\mathcal{F}$ and conclude that there exist positive numbers $\bar{\alpha}$, $\kappa$, and a function $\rho : ]\,0, \bar{\alpha}] \mapsto [0, 1\,[$ such that, if $\alpha \in ]\,0, \bar{\alpha}]$ and we let $\hat{w}_*(\alpha) = \alpha\bar{w}_*$, then there exists a compact connected subset $\hat{Z}_\alpha$ of $C(\kappa\alpha) \times [\rho(\alpha), 1]$ such that $Z_\alpha$ intersects the sets $C(\kappa\alpha) \times \{\rho(\alpha)\}$ and $C(\kappa\alpha) \times \{1\}$, and the conditions

$$r\hat{w}_*(\alpha) \in \mathcal{F}(x_1, x_2, y) \quad \text{and} \quad \|x_1\| + \|x_2\| + \|y\| \leq \kappa r\alpha$$

hold whenever $((x_1, x_2, y), r) \in \hat{Z}_\alpha$ and $\rho(\alpha) \leq r \leq 1$. (Here we are writing $C(r) = \{(x_1, x_2, y) \in C : \|x_1\| + \|x_2\| + \|y\| \leq r\}$.) We let $\sigma = \|\mu\|^{-1}$.

If we now define $Z_\alpha = \Big\{(x_1, x_2, y) : (\exists r \in [\rho(\alpha), 1])\Big(((x_1, x_2, y), r) \in \hat{Z}_\alpha\Big)\Big\}$, then $Z_\alpha$ is a continuous projection of a compact connected set, so $Z_\alpha$ is compact and connected. If $(x_1, x_2, y) \in Z_\alpha$, then there is an $r \in [\rho(\alpha), 1]$ such that $((x_1, x_2, y), r) \in \hat{Z}_\alpha$, and then $(0, 0, r\alpha) \in \mathcal{F}(x_1, x_2, y)$, so in particular $0 = y - y_1 = y - y_2$ for some $y_1 \in F_1(x_1)$ and some $y_2 \in F_2(x_2)$. But then $y_1 = y_2 = y$, so $y \in F_1(x_1) \cap F_2(x_2)$, showing that $(x_1, x_2, y) \in \mathcal{I}$. So $Z_\alpha \subseteq \mathcal{I}$, as desired.

Finally, we must show that $Z_\alpha$ contains points $(x_{1,\alpha,-}, x_{2,\alpha,-}, y_{\alpha,-})$ and $(x_{1,\alpha,+}, x_{2,\alpha,+}, y_{\alpha,+})$ for which $\|y_{\alpha,-}\| \leq \kappa\rho(\alpha)\alpha$ and $\|y_{\alpha,+}\| \geq \sigma\alpha$. Let $((x_{1,\alpha,-}, x_{2,\alpha,-}, y_{\alpha,-}), r_{\alpha,-})$ and $((x_{1,\alpha,+}, x_{2,\alpha,+}, y_{\alpha,+}), r_{\alpha,+})$ be members of $\hat{Z}_\alpha \cap (C(\kappa\alpha) \times \{\rho(\alpha)\})$ and $\hat{Z}_\alpha \cap (C(\kappa\alpha) \times \{1\})$, respectively. Then $r_{\alpha,-} = \rho(\alpha)$, and $(0, 0, \rho(\alpha)\alpha) = (0, 0, r_{\alpha,-}\alpha) = r_{\alpha,-}\hat{w}_*(\alpha) \in \mathcal{F}(x_{1,\alpha,-}, x_{2,\alpha,-}, y_{\alpha,-})$, from which it follows that $\|y_{\alpha,-}\| \leq \kappa r_{\alpha,-}\alpha$. On the other hand, $r_{\alpha,+} = 1$, and then $(0, 0, \alpha) = (0, 0, r_{\alpha,+}\alpha) = r_{\alpha,+}\hat{w}_*(\alpha) \in \mathcal{F}(x_{1,\alpha,+}, x_{2,\alpha,+}, y_{\alpha,+})$, from which it follows that $\mu(y_{\alpha,+}) = \alpha$, so that $\alpha = \mu(y_{\alpha,+}) \leq \|\mu\|\,\|y_{\alpha,+}\|$, and then $\|y_{\alpha,+}\| \geq \sigma\alpha$. □

## 5 Variational generators

### 5.1 Linearization error and weak GDQs

Assume that $X$ and $Y$ are FDNRLSs, $S \subseteq X$, $F : S \mapsto Y$, and $\bar{x}_* \in X$. Recall that a linear map $L : X \mapsto Y$ is said to be a *differential* of $F$ at $\bar{x}_*$ in the direction of $S$ if the *linearization error* $E^{lin}_{F,L,\bar{x}_*}(h) = F(\bar{x}_* + h) - F(\bar{x}_*) - L \cdot h$ is $o(\|h\|)$ as $h \to 0$ via values such that $\bar{x}_* + h \in S$.

**Remark 5.1** The precise meaning of the sentence "$E^{lin}_{F,L,\bar{x}_*}(h)$ is $o(\|h\|)$ as $h \to 0$ via values such that $\bar{x}_* + h \in S$" is:

- *There exists a function* $\theta \in \boldsymbol{\Theta}$ *(cf. §4, page 23) having the property that* $\|E^{lin}_{F,\Lambda,\bar{y}_*}(\bar{x}_*, h)\| \le \theta(\|h\|)\|h\|$ *for every* $h$ *such that* $\bar{x}_* + h \in S$. □

A natural generalization of that, when $\Lambda$ is a set of linear maps, $F$ is set-valued, and we have picked a point $\bar{y}_* \in Y$ to play the role of $F(\bar{x}_*)$, is obtained by defining the linearization error via the formula

$$E^{lin}_{F,\Lambda,\bar{x}_*,\bar{y}_*}(h) \overset{\text{def}}{=} \inf \Big\{ \|y - \bar{y}_* - L \cdot h\| : y \in F(\bar{x}_* + h),\, L \in \Lambda \Big\} . \qquad (5.1.1)$$

**Definition 5.2** *Assume that* $X$ *and* $Y$ *are FDNRLSs,* $(\bar{x}_*, \bar{y}_*) \in X \times Y$, $F : X \mapsto\!\!\!\!\to Y$, *and* $S \subseteq X$. *A* ***weak GDQ*** *of* $F$ *at* $(\bar{x}_*, \bar{y}_*)$ *in the direction of* $S$ *is a compact set* $\Lambda$ *of linear maps from* $X$ *to* $Y$ *such that the linearization error* $E^{lin}_{F,\Lambda,\bar{x}_*,\bar{y}_*}(h)$ *is* $o(\|h\|)$ *as* $h \to 0$ *via values such that* $\bar{x}_* + h \in S$. □

In other words, a weak GDQ is just the same as a classical differential, except for the fact that, since the map $F$ is set-valued and the "differential" $\Lambda$ is a set, we compute the linearization error by choosing the $y \in F(\bar{x}_* + h)$ and the linear map $L \in \Lambda$ that give the smallest possible error.

We will write $WGDQ(F, \bar{x}_*, \bar{y}_*, S)$ to denote the set of all weak GDQs of $F$ at $(\bar{x}_*, \bar{y}_*)$ in the direction of $S$.

The following trivial observations will be important, so we state them explicitly. (The second assertion is true because the infimum of the empty subset of $[0, +\infty]$ is $+\infty$.)

**Fact 5.3** *Assume that* $X$ *and* $Y$ *are FDNRLSs,* $(\bar{x}_*, \bar{y}_*) \in X \times Y$, $F : X \mapsto\!\!\!\!\to Y$, *and* $S \subseteq X$. *Then*

- *If* $\Lambda \in WGDQ(F, \bar{x}_*, \bar{y}_*, S)$, $\Lambda' \in Comp^0(X, Y)$, *and* $\Lambda \subseteq \Lambda'$, *then* $\Lambda' \in WGDQ(F, \bar{x}_*, \bar{y}_*, S)$.
- $\emptyset \in WGDQ(F, \bar{x}_*, \bar{y}_*, S)$ *if and only if* $\bar{x}_* \notin \text{Closure}(S)$. □

We recall that the *distance* $\text{dist}(S, S')$ between two subsets $S, S'$ of a metric space $(M, d_M)$ is defined by $\text{dist}(S, S') = \inf\{d_M(s, s') : s \in S,\, s' \in S'\}$. It follows that $\text{dist}(S, S') \ge 0$, and also that $\text{dist}(S, S') < +\infty$ if and only if both $S$ and $S'$ are nonempty. Furthermore, the linearization error $E^{lin}_{F,\Lambda,\bar{x}_*,\bar{y}_*}(h)$ defined in (5.1.1) is exactly equal to $\text{dist}(\bar{y}_* + \Lambda \cdot h, F(\bar{x}_* + h))$.

The following two propositions are rather easy to prove, but we find it convenient to state them explicitly, because they will be the key to the notion of "variational generator" in GDQ theory.

**Proposition 5.4** *Suppose* $X, Y$ *are FDNRLSs,* $F : X \mapsto\!\!\!\!\rightarrow Y$*,* $S \subseteq X$*,* $(\bar{x}_*, \bar{y}_*)$ *belongs to* $X \times Y$*, and* $\Lambda$ *is a compact set of linear maps from* $X$ *to* $Y$*. Then the following three conditions are equivalent:*

*(1)* $\Lambda \in WGDQ(F, \bar{x}_*, \bar{y}_*, S)$*;*

*(2) there exist a positive number* $\bar{\delta}_*$ *and a family* $\{\kappa^\delta\}_{0<\delta\leq\bar{\delta}_*}$ *of positive numbers such that* $\lim_{\delta\downarrow 0}\kappa^\delta = 0$*, having the property that*

$$\operatorname{dist}\Big(\bar{y}_*+\Lambda\cdot h, F(\bar{x}_*+h)\Big) \leq \delta\kappa^\delta \quad \text{whenever} \quad \|h\|\leq\delta\leq\bar{\delta}_* \text{ and } \bar{x}_*+h\in S; \tag{5.1.2}$$

*(3) if* $\{h_j\}_{j\in\mathbb{N}}$ *is a sequence in* $X$ *such that* $\lim_{j\to\infty} h_j = 0$ *and* $\bar{x}_* + h_j \in S$ *for all* $j$*, then there exist (i) a sequence* $\{L_j\}_{j\in\mathbb{N}}$ *of members of* $\Lambda$ *(ii) a sequence* $\{y_j\}_{j\in\mathbb{N}}$ *for which* $y_j \in F(\bar{x}_* + h_j)$ *for each* $j$*, (iii) a sequence* $\{r_j\}_{j\in\mathbb{N}}$ *of positive numbers such that* $\|y_j - \bar{y}_* - L_j \cdot h_j\| \leq r_j\|h_j\|$ *for all* $j \in \mathbb{N}$ *and* $\lim_{j\to\infty} r_j = 0$*.* □

**Proposition 5.5** *Let* $X, Y, F, S, \bar{x}_*, \bar{y}_*$ *be as in Proposition 5.4. Then*

- *If* $\Lambda \in AGQD(F, \bar{x}_*, \bar{y}_*, S)$ *it follows that* $\Lambda \in WGQD(F, \bar{x}_*, \bar{y}_*, S)$*.*
- *If* $\Lambda$ *belongs to* $WGQD(F, \bar{x}_*, \bar{y}_*, S)$*,* $\Lambda$ *is convex, and the restriction* $F\lceil S$ *is upper semicontinuous with closed convex values, then it follows that* $\Lambda \in GQD(F, \bar{x}_*, \bar{y}_*, S)$*.* □

## 5.2 GDQ variational generators

For a set-valued map $F : X \times \mathbb{R} \mapsto\!\!\!\!\rightarrow Y$, we write $F_x$, $F^t$, if $x \in X$, $t \in \mathbb{R}$, to denote the partial maps $F_x : \mathbb{R} \mapsto\!\!\!\!\rightarrow Y$, $F^t : X \mapsto\!\!\!\!\rightarrow Y$, given by

$$F_x(s) = F(x, s) \text{ and } F^t(u) = F(u, t) \text{ if } s \in \mathbb{R},\ u \in X\,.$$

For a subset $S$ of $X \times \mathbb{R}$, we write $S_x$, $S^t$, if $x \in X$, $t \in \mathbb{R}$, to denote the sections $S_x \subseteq \mathbb{R}$, $S^t \subseteq X$, given by $S_x = \{s \in \mathbb{R} : (x, s) \in S\}$ and $S^t = \{u \in X : (u, t) \in S\}$.

We would like to define the notion of "variational generator" as follows, assuming that:

(VGA1) *$X$ and $Y$ are FDNRLSs,* $a, b \in \mathbb{R}$*, and* $a \leq b$;

(VGA2) $\xi_* \in C^0([a, b]\,; X)$ *and* $\sigma_*$ *is a ppd single-valued function from* $[a, b]$ *to* $Y$;

(VGA3) $S \subseteq X \times \mathbb{R}$;

(VGA4) $F : X \times \mathbb{R} \mapsto\!\!\!\!\rightarrow Y$ *is a set-valued map.*

*Tentative definition*: Assume that (VGA1,2,3,4) hold. A *GDQ variational generator of* $F$ *along* $(\xi_*, \sigma_*)$ *in the direction of* $S$ is a set-valued map $\Lambda : [a, b] \mapsto\!\!\!\!\rightarrow Lin(X, Y)$ such that, for every $t \in [a, b]$, the set $\Lambda(t)$ is a weak GDQ of $F^t$ at $(\xi_*(t), \sigma_*(t))$ in the direction of $S^t$. □

The trouble with this definition is twofold:

- First of all, there at least two natural ways to define the "linearization error" at a particular time $t$, because we could
  (1) use the "fixed time error" $h \mapsto E^{lin}_{F,\Lambda,\xi_*,\sigma_*}(h,t) \stackrel{\mathrm{def}}{=} E^{lin}_{F^t,\Lambda(t),\xi_*(t),\sigma_*(t)}(h)$, where $E^{lin}_{F^t,\Lambda(t),\xi_*(t),\sigma_*(t)}(h)$ is obtained by applying Formula (5.1.1) to the map $F^t$, so that
  $$E^{lin}_{F,\Lambda,\xi_*,\sigma_*}(h,t) = \mathrm{dist}(\sigma_*(t) + \Lambda(t)\cdot h, F(\xi_*(t)+h,t))\,; \tag{5.2.1}$$
  (2) work instead with a "robust" version of the error, in which we try to approximate $F(\xi_*(t+s)+h,t+s) - \sigma_*(t+s)$ by $\Lambda(t)\cdot h$ not just for $s=0$ but also for $s$ in some neighborhood of 0; this leads to defining
  $$E^{lin,rob}_{F,\Lambda,\xi_*,\sigma_*}(h,s,t) = \mathrm{dist}(\sigma_*(t+s) + \Lambda(t)\cdot h, F(\xi_*(t+s)+h,t+s))\,. \tag{5.2.2}$$
- Second, once we have settled on which form of the error to use, this will lead to introducing functions $t \mapsto \kappa^\delta(t)$, $t \mapsto \kappa^{\delta,s}(t)$ such that $\|E^{lin}_{F,\Lambda,\xi_*,\sigma_*}(h,t)\| \le \delta\kappa^\delta(t)$ and $\|E^{lin,rob}_{F,\Lambda,\xi_*,\sigma_*}(h,s,t)\| \le \delta\kappa^{\delta,s}(t)$ whenever $\|h\| \le \delta$, and require that these functions "go to zero." However, when functions are involving, "going to zero" can mean many different things, since the convergence could be, for example, pointwise, in $L^1$, or uniform.

It follows that, in principle, there are at least twice as many reasonable notions of "variational generators" as there are notions of convergence of functions, since for each convergence notion we can require that the convergence take place for the fixed-time error or for the robust one.

It turns out, however, that of all these possible notions of "variational generator," only two will be important to us. So we will define these two notions and ignore all the others.

***$L^1$ fixed-time GDQ variational generators.*** Let us assume that $X$, $Y$, $a$, $b$, $\xi_*$, $\sigma_*$, $S$, $F$ are such that (VGA1,2,3,4) hold.

**Definition 5.6** *An $L^1$* ***fixed-time GDQ variational generator of the map*** *$F$* ***along*** *$(\xi_*,\sigma_*)$* ***in the direction of the set*** *$S$ is a set-valued map $\Lambda : [a,b] \mapsto\!\!\!\rightarrow Lin(X,Y)$ such that,*

- *there exist a positive number $\bar\delta$ and a family $\{\kappa^\delta\}_{0<\delta\le\bar\delta}$ of measurable functions $\kappa^\delta : [a,b] \mapsto [0,+\infty]$ such that $\lim_{\delta\downarrow 0}\int_a^b \kappa^\delta(t)\,dt = 0$ and, in addition, $\mathrm{dist}(\sigma_*(t) + \Lambda(t)\cdot h, F(\xi_*(t)+h,t)) \le \delta\kappa^\delta(t)$ whenever $h \in X$, $t \in [a,b]$, $(\xi_*(t)+h,t) \in S$, and $\|h\| \le \delta$.* □

We will write $VG^{L^1,ft}_{GDQ}(F,\xi_*,\sigma_*,S)$ to denote the set of all $L^1$ fixed-time GDQ variational generators of $F$ along $(\xi_*,\sigma_*)$ in the direction of $S$.

***Pointwise robust GDQ variational generators.*** Again, let us assume that $X$, $Y$, $a$, $b$, $\xi_*$, $\sigma_*$, $S$, $F$ are such that (VGA1,2,3,4) hold.

**Definition 5.7** *A* ***pointwise robust GDQ variational generator of the map*** $F$ ***along*** $(\xi_*, \sigma_*)$ ***in the direction of the set*** $S$ *is a set-valued map* $\Lambda : [a, b] \mapsto\!\!\!\to Lin(X, Y)$ *such that,*

- *there exist* $\bar{\delta} > 0$, $\bar{s} > 0$, *and a family* $\{\kappa^{\delta,s}\}_{0<\delta\leq\bar{\delta},0<s\leq\bar{s}}$ *of functions* $\kappa^{\delta,s} : [a, b] \mapsto [0, +\infty]$, *such that (i)* $\lim_{\delta\downarrow 0,s\downarrow 0} \kappa^{\delta,s}(t) = 0$ *for every* $t \in [a, b]$ *and (ii)* $\mathrm{dist}(\sigma_*(t+s)+\Lambda(t)\cdot h, F(\xi_*(t+s)+h, t+s)) \leq \delta\kappa^{\delta,s}(t)$ *whenever* $h \in X$, $\|h\| \leq \delta$, $t \in [a, b]$, $t+s \in [a, b]$, *and* $(\xi_*(t+s)+h, t+s) \in S$. □

We write $VG^{pw,rob}_{GDQ}(F, \xi_*, \sigma_*, S)$ to denote the set of all pointwise robust GDQ variational generators of $F$ along $(\xi_*, \sigma_*)$ in the direction of $S$.

### 5.3 Examples of variational generators

We now prove four propositions giving important examples of variational generators.

***Clarke generalized Jacobians.*** Recall that $\partial_x f(q, t)$ denotes the Clarke generalized Jacobian (cf. Definition 2.9) at $x = q$ of the map $x \mapsto f(x, t)$.

**Proposition 5.8** *Assume that* $X$, $Y$ *are FDNRLSs, and* $f$ *is a single-valued ppd map from* $X \times \mathbb{R}$ *to* $Y$, *whose domain contains a tube* $\mathcal{T}^X(\xi_*, \bar{\delta})$ *about a continuous curve* $\xi_* : [a, b] \mapsto X$. *Assume that each partial map* $t \mapsto f(x, t)$ *is measurable and each partial map* $x \mapsto f(x, t)$ *is Lipschitz with a Lipschitz constant* $C(t)$ *such that the function* $C(\cdot)$ *is integrable. Let* $Z = Lin(X, Y)$, *and define* $\Lambda(t) = \partial_x f(\xi_*(t), t)$ *and* $\sigma_*(t) = f(\xi_*(t), t)$ *for* $t \in [a, b]$. *Then* $\Lambda$ *is an integrably bounded measurable set-valued function from* $[a, b]$ *to* $Z$ *with a.e. nonempty compact convex values, and* $\Lambda$ *is an* $L^1$ *fixed-time variational GDQ of* $f$ *along* $(\xi_*, \sigma_*)$ *in the direction of* $X \times [a, b]$.

*Proof.* To begin with, we observe that the bound $\|L\| \leq C(t)$ holds for every $t \in [a, b]$ and every $L \in \Lambda(t)$, so $\Lambda$ is integrably bounded. Furthermore, $\Lambda$ clearly has compact convex a.e. nonempty values. A somewhat tedious but elementary argument proves that $\Lambda$ is measurable.

Now, let $\kappa^\delta(t)$ denote the maximum of the distances $\mathrm{dist}(L, \Lambda(t))$ for all $L \in \Lambda^{(\delta)}(t)$, where $\Lambda^{(\delta)}(t)$ is the closed convex hull of the set of all the differentials $Df^t(x)$ for all $x \in \mathcal{D}^t_\delta$, and $\mathcal{D}^t_\delta$ is the set of all points $x$ in the the open ball $\mathbb{B}_X(\xi_*(t), \delta)$ such that $f^t$ is differentiable at $x$. Then $\kappa^\delta$ is easily seen to be measurable, and such that $\lim_{\delta\downarrow 0} \kappa^\delta(t) = 0$ for every $t$. Furthermore, if $\|h\| \leq \delta$, then the equality $f(\xi_*(t) + h, t) - f(\xi_*(t), t) = \tilde{L} \cdot h$ holds for some $\tilde{L} \in \Lambda^{(\delta)}(t)$, and we can pick $L \in \Lambda(t)$ such that $\|\tilde{L} - L\| \leq \kappa^\delta(t)$, and conclude that

$$f(\xi_*(t) + h, t) - f(\xi_*(t), t) - L \cdot h = (\tilde{L} - L) \cdot h \,,$$

from which it follows that $\|f(\xi_*(t) + h, t) - f(\xi_*(t), t) - L \cdot h\| \leq \delta\kappa^\delta(t)$.

On the other hand, it is clear that $\kappa^\delta(t) \leq 2C(t)$. So the functions $\kappa^\delta$ converge pointwise to zero and are bounded by a fixed integrable function. Hence $\lim_{\delta\downarrow 0}\int_a^b \kappa^\delta(t)\,dt = 0$, and our proof is complete. □

***Michel-Penot subdifferentials.*** Recall that if $f : X\times\mathbb{R} \hookrightarrow \mathbb{R}$ then $\partial^o_x f(q,t)$ is the Michel-Penot subdifferential (cf. Definition 2.11) at $x = q$ of the function $x \mapsto f(x,t)$, and that the notion of *epimap* was defined in §2.1, page 5.

**Proposition 5.9** *Let $X$ be a FDNRLS, and let $f$ be a single-valued ppd map from $X\times\mathbb{R}$ to $\mathbb{R}$, whose domain contains a tube $\mathcal{T}^X(\xi_*,\bar\delta)$ about a continuous curve $\xi_* : [a,b] \mapsto X$. Assume that each partial map $t \mapsto f(x,t)$ is measurable and each partial map $x \mapsto f(x,t)$ is Lipschitz with a Lipschitz constant $C(t)$ such that the function $C(\cdot)$ is integrable. Let $\Lambda(t) = \partial^o_x f(\xi_*(t),t)$, and let $\sigma_*(t) = f(\xi_*(t),t)$. Let $F$ be the epimap of $f$. Then $\Lambda$ is an integrably bounded measurable set-valued function with a.e. nonempty compact convex values, and $\Lambda$ is an $L^1$ fixed-time variational GDQ of $F$ along $(\xi_*,\sigma_*)$ in the direction of $X\times[a,b]$.*

*Proof.* To begin with, we observe, as in the previous proof, that (i) the bound $\|L\| \leq C(t)$ holds for every $t \in [a,b]$ and every $L \in \Lambda(t)$, so $\Lambda$ is integrably bounded, and (ii) $\Lambda$ clearly has compact convex a.e. nonempty values.

Next, we prove that $\Lambda$ is measurable. For this purpose, we need to review how the Michel-Penot subdifferential $\Lambda(t)$ is defined: for each $t \in [a,b]$, let $f^t$ be the function $\bar{\mathbb{B}}_X(\xi_*(t),\bar\delta) \ni x \mapsto f(x,t) \in \mathbb{R}$; extend $f^t$ to all of $X$ by defining it in an arbitrary fashion outside $\bar{\mathbb{B}}_X(\xi_*(t),\bar\delta)$; for $x,h \in X$, define $d^o f^t(x,h) = \sup_{k\in X}\limsup_{t\downarrow 0} t^{-1}\Big(f(x+t(k+h))-f(x+tk)\Big)$, so that, for each $x \in X$, the function $X \ni h \mapsto df(x,h) \in [-\infty,+\infty]$ is convex and positively homogeneous; then $\Lambda(t)$ is the set of all linear functionals $\omega \in X^\dagger$ such that $d^o f^t(\xi_*(t),h) \geq \langle\omega,h\rangle$ whenever $h \in X$.

We define the support function $\sigma_\Lambda$ using (2.1.4), with $\mathbb{R}$ in the role of $Y$, and $X^\dagger = Lin(X,\mathbb{R})$ in the role of $X$, so $\sigma_\Lambda$ is a function on $[a,b]\times X$. The measurability of $\Lambda$ will follow if we prove that the function $[a,b] \ni t \mapsto \sigma_\Lambda(t,\bar h)$ is measurable for each $\bar h \in X$.

Fix an $\bar h \in X$ and a $t \in [a,b]$. If $\omega \in \Lambda(t)$, then $\langle\omega,\bar h\rangle \leq d^o f^t(\xi_*(t),\bar h)$. Therefore $\sigma_\Lambda(t,\bar h) \leq d^o f^t(\xi_*(t),\bar h)$. We will prove that the opposite inequality is also true. Define $E = \{(h,r) \in X\times\mathbb{R} : r \geq d^o f^t(\xi_*(t),h)\}$, Then $E$ is the epigraph of the function $X \ni h \mapsto d^o f^t(\xi_*(t),h) \in \mathbb{R}$, which is everywhere finite, convex, and positively homogeneous. In particular, $E$ is a closed convex cone in $X\times\mathbb{R}$ with nonempty interior. If we let $\bar r = d^o f^t(\xi_*(t),\bar h)$, then the point $(\bar h,\bar r)$ belongs to the boundary of $E$. Hence the Hahn-Banach theorem implies that there exists a linear functional $\Omega \in (X\times\mathbb{R})^\dagger\backslash\{0\}$ such that $0 = \Omega(\bar h,\bar r) \leq \Omega(h,r)$ for all $(h,r) \in E$. Then there exist a linear functional $\omega : X \mapsto \mathbb{R}$ and a real number $\omega_0$ such that $\Omega(h,r) = -\omega(h) + \omega_0 r$ for all $(h,r) \in X\times\mathbb{R}$, and $(\omega,\omega_0) \neq (0,0)$. Clearly, $\omega_0 \geq 0$, because $0 = -\omega(\bar h) + \omega_0\bar r \leq -\omega(\bar h) + \omega_0(\bar r+1)$. Furthermore, $\omega_0 \neq 0$, because if

$\omega_0 = 0$ then $\omega(\bar{h}) = -\Omega(\bar{h}, \bar{r}) = 0$, and then the inequality $\Omega(\bar{h}, \bar{r}) \leq \Omega(h, r)$ implies $0 = -\omega(\bar{h}) \leq -\omega(h)$ for all $h \in X$, so $\omega = 0$ as well. So we may assume that $\omega_0 = 1$, and then $0 = -\omega(\bar{h}) + \bar{r} \leq -\omega(h) + r$ for all $(h, r) \in E$. Hence $\omega(h) \leq r$ for all $(h, r) \in E$, so in particular $\omega(h) \leq d^o f^t(\xi_*(t), h)$ for all $h \in X$. It follows that $\omega \in \Lambda(t)$. On the other hand, the fact that $-\omega(\bar{h}) + \bar{r} = 0$ tells us that $\omega(\bar{h}) = d^o f^t(\xi_*(t), \bar{h})$. Hence $\sigma_\Lambda(t, \bar{h}) \geq d^o f^t(\xi_*(t), \bar{h})$.

It follows that $\sigma_\Lambda(t, \bar{h}) = d^o f^t(\xi_*(t), \bar{h})$ for all $\bar{h} \in X$. This implies the desired measurability of the function $[a, b] \ni t \mapsto \sigma_\Lambda(t, \bar{h}) \in \mathbb{R}$, because $[a, b] \ni t \mapsto d^o f^t(\xi_*(t), \bar{h})$ is clearly measurable.

Now fix $t \in [a, b]$. For $h \in \mathbb{R}^n$ such that $\|h\| \leq \bar{\delta}$, let

$$\hat{\theta}^t(h) = \min\{f(\xi_*(t) + h, t) - \sigma_*(t) - \omega \cdot h : \omega \in \Lambda(t)\}\,. \tag{5.3.1}$$

If in addition $h \neq 0$, write $\theta^t(h) = \frac{\hat{\theta}^t(h)}{\|h\|}$. We claim that $\limsup_{h \to 0, h \neq 0} \theta^t(h) \leq 0$. Indeed, if this was not so there would exist a positive $\varepsilon$ and a sequence $\{h_j\}_{j \in \mathbb{N}}$ converging to zero and such that $h_j \neq 0$ and $\theta^t(h_j) \geq \varepsilon$ for all $j$. Then $f(\xi_*(t) + h_j, t) - f(\xi_*(t), t) - \omega \cdot h_j \geq \varepsilon \|h_j\|$ for all $j$ and all $\omega \in \Lambda(t)$. Let $\tau_j = \|h_j\|$, $w_j = \frac{h_j}{\tau_j}$, so $\|w_j\| = 1$. By passing to a subsequence, if necessary, assume that the limit $w = \lim_{j \to \infty} w_j$ exists. Let $e_j = w_j - w$, so $e_j \to 0$. Then $h_j = \tau_j w_j = \tau_j(w + e_j)$, so $f(\xi_*(t) + \tau_j(w + e_j), t) - f(\xi_*(t), t) - \omega \cdot h_j \geq \varepsilon \tau_j$ for all $j \in \mathbb{N}$ and all $\omega \in \Lambda(t)$.

It follows that $\limsup_{j \to \infty} \tau_j^{-1} \Big( f(\xi_*(t) + \tau_j(w + e_j), t) - f(\xi_*(t), t) - \omega \cdot h_j \Big) \geq \varepsilon$ if $\omega \in \Lambda(t)$. But $f(\xi_*(t) + \tau_j(w + e_j), t) - f(\xi_*(t) + \tau_j w, t) \leq C(t) \tau_j \|e_j\|$. Hence $\limsup_{j \to \infty} \tau_j^{-1} \Big( f(\xi_*(t) + \tau_j w, t) - f(\xi_*(t), t) - \omega \cdot h_j \Big) \geq \varepsilon$, and then we find that $\limsup_{j \to \infty} \tau_j^{-1} \Big( f(\xi_*(t) + \tau_j w, t) - f(\xi_*(t), t) \Big) \geq \varepsilon + \omega \cdot w$, from which it follows that $\limsup_{\tau \downarrow 0} \tau^{-1} \Big( f(\xi_*(t) + \tau w, t) - f(\xi_*(t), t) \Big) \geq \varepsilon + \omega \cdot w$. So we have shown that $d^o f^t(\xi_*(t), w) \geq \varepsilon + \omega \cdot w$ for all $\omega \in \Lambda(t)$. But this is impossible, because we already know that $d^o f^t(\xi_*(t), w) = \sigma_\Lambda(t, w)$, so $d^o f^t(\xi_*(t), w) = \omega \cdot w$ for some $\omega \in \Lambda(t)$. This proves our claim that $\limsup_{h \to 0} \theta^t(h) \leq 0$.

Now define $\kappa^\delta(t) = \max \Big( 0, \sup\{\theta^t(h) : \|h\| \leq \delta\} \Big)$. Then the functions $\kappa^\delta$ are measurable and nonnegative, and converge pointwise to zero. In addition, they clearly satisfy $\kappa^\delta \leq 2C(t)$, since (5.3.1) implies that $\hat{\theta}(h) \leq 2C(t)\|h\|$. Therefore $\lim_{\delta \downarrow 0} \int_a^b \kappa^\delta(t)\, dt = 0$.

Given $t \in [a, b]$ and $h \in X$ such that $\|h\| \leq \delta$, we can pick $\omega \in \Lambda(t)$ such that $f(\xi_*(t) + h, t) - \sigma_*(t) - \omega \cdot h = \hat{\theta}^t(h)$, and then

$$f(\xi_*(t) + h, t) - \sigma_*(t) - \omega \cdot h = \|h\| \theta^t(h) \leq \|h\| \kappa^\delta(t) \leq \delta \kappa^\delta(t)\,.$$

It then follows that we can pick a real number $r \in F(\xi_*(t) + h, t)$ such that $|r - \sigma_*(t) - \omega \cdot h| \leq \delta \kappa^\delta(t)$. (Indeed, if $f(\xi_*(t) + h, t) - \sigma_*(t) - \omega \cdot h \geq 0$, we may pick $r = f(\xi_*(t) + h, t)$, and if $f(\xi_*(t) + h, t) - \sigma_*(t) - \omega \cdot h < 0$ pick

$r = \sigma_*(t) + \omega \cdot h$.) But then $\mathrm{dist}\Big(F(\xi_*(t)+h,t), \sigma_*(t) + \Lambda(t)\cdot h\Big) \le \delta\kappa^\delta(t)$, since $\sigma_*(t) + \omega \cdot h \in \sigma_*(t) + \Lambda(t)\cdot h$ and $r \in F(\xi_*(t)+h,t)$. This completes our proof. □

***Classical differentials.*** If $(M, d_M)$, $(N, d_N)$ are metric spaces, and $\bar{x}_* \in M$, a map $F : M \hookrightarrow N$ is *calm* at $\bar{x}_*$ if there exist positive constants $C, \bar{\delta}$ such that $x \in \mathrm{Do}(F)$ and $d_N(F(x), F(\bar{x}_*) \le C d_M(x, \bar{x}_*)$ whenever $d_M(x, \bar{x}_*) \le \delta$. If $a, b \in \mathbb{R}$, $a < b$, and $\xi_* : [a,b] \mapsto M$ is continuous, then a ppd map $F : M \times [a,b] \hookrightarrow N$ is *integrably calm* along $\xi_*$ if there exist a positive constant $\bar{\delta}$ and an integrable function $C : [a,b] \mapsto [0,+\infty]$ such that, for almost all $t \in [a,b]$, the following two conditions are satisfied whenever $d_M(x, \xi_*(t)) \le \delta$: (i) $(x,t) \in \mathrm{Do}(F)$, and (ii) $d_N(F(x,t), F(\xi_*(t),t)) \le C(t) d_M(x, \xi_*(t))$. Then the following is easily proved.

**Proposition 5.10** *Assume that $X, Y$ are FDNRLSs, and $f$ is a single-valued ppd map from $X \times \mathbb{R}$ to $Y$ whose domain contains a tube $\mathcal{T}^X(\xi_*, \bar{\delta})$ about a continuous curve $\xi_* : [a,b] \mapsto X$. Assume that each partial map $t \mapsto f(x,t)$ is measurable. Assume in addition that*

- *for each $t$ the map $x \mapsto f(x,t)$ is differentiable at $\xi_*(t)$,*
- *$f$ is integrably calm along $\xi_*$.*

*Let $\sigma_*(t) = f(\xi_*(t),t)$, and let $\Lambda(t) = \{D_x f(\xi_*(t),t)\}$. Then $\Lambda$ is an integrable single-valued map. Furthermore, $\Lambda$ is an $L^1$ fixed-time variational GDQ of $f$ along $(\xi_*, \sigma_*)$ in the direction of $X \times [a,b]$.* □

***The set-valued maps $\partial^> g$.*** We are going to assume that

(A) *$X$ is a FDNRLS, $\xi_* \in C^0([a,b], X)$, $\bar{\delta} > 0$, and $T = \mathcal{T}^X(\xi_*, \bar{\delta})$.*
(B) *$g : T \mapsto \mathbb{R}$ is a single-valued everywhere defined function such that (i) $g(\xi_*(t),t) \le 0$ for all $t \in [a,b]$, and (ii) each partial map $x \mapsto g(x,t)$ is Lipschitz on $\{x \in X : \|x - \xi_*(t)\| \le \bar{\delta}\}$, with a Lipschitz constant $C$ which is independent of $t$ for $t \in [a,b]$.*

We define $Av_g = \{(x,t) \in \mathcal{T}^X(\xi_*, \bar{\delta}) : g(x,t) > 0\}$, so $Av_g$ is the domain of the constraint indicator map $\chi_g^{co}$ (cf. §2.1, page 5).

**Remark 5.11** For an optimal control problem with an inequality state space constraint $g(x,t) \le 0$, $Av_g$ is the *set to be avoided*, that is, the set of points $(x,t)$ such that any trajectory $\xi$ for which $(\xi(t),t)$ is one of these points, for some $t$, fails to be admissible. □

We define $\partial_x^> g(\bar{x},t)$ to be the convex hull of the set of all limits $\lim_{j\to\infty} \omega_j$, for all sequences $\{(x_j, t_j, \omega_j)\}_{j\in\mathbb{N}}$ such that $\lim_{j\to\infty}(x_j,t_j) \to (\bar{x},t)$ and, for all $j$, (1) $(x_j,t_j) \in Av_g$, (2) the function $x \mapsto g(x,t_j)$ is differentiable at $x_j$, and (3) $\omega_j = \nabla_x g(x_j,t_j)$.

We let $K$ be the set of all $t \in [a,b]$ such that $(\xi_*(t),t)$ belongs to the closure of $Av_g$. Then $K$ is compact.

**Remark 5.12** The set $K$ could be empty. (This happens if and only if the closure of $Av_g$ does not contain any point of the form $(\xi_*(t), t)$, $t \in [a, b]$.) $\square$

**Proposition 5.13** *Assume that* $X$, $a$, $b$, $\xi_*$, $\bar{\delta}$, $T = \mathcal{T}^X(\xi_*, \bar{\delta})$, $g$, $C$ *are such that (A), (B) hold, and* $Av_g$, $\partial_x^> g$, $K$ *are defined as above. Let* $\sigma_*(t) = 0$ *for* $t \in [a, b]$*, and define* $\varLambda(t) = \partial_x^> g(\xi_*(t), t)$ *for* $t \in [a, b]$*. Then*

*(1)* $\varLambda$ *is an upper semicontinuous set-valued map with compact convex values;*
*(2)* $K = \{t \in [a, b] : \varLambda(t) \neq \emptyset\}$*;*
*(3)* $\varLambda$ *is a pointwise robust GDQ variational generator of* $\chi_g^{co}$ *along* $(\xi_*, \sigma_*)$ *in the direction of* $Av_g$*.*

*Proof.* The desired conclusions do not depend on the choice of a norm on $X$, so we will assume that the norm on $X$ is Euclidean. For each $t \in [a, b]$, let $g^t$ denote the function $x \mapsto g(x, t)$, with domain $B^t = \bar{\mathbb{B}}_X(\xi_*(t), \bar{\delta})$, and let $D^t$ be the set of points $x \in B^t$ such that $g^t$ is differentiable at $x$. Then $D^t$ is a subset of full measure of $B^t$.

Let us show that $\varLambda$ is upper semicontinuous and has compact convex values. The convexity of the sets $\varLambda(t)$ is clear from the definition of $\varLambda$. We will prove that the graph of $\varLambda$ is compact, from which it will follow that $\varLambda$ is upper semicontinuous and has compact values.

First, we observe that every member $(t, \omega)$ of $\mathrm{Gr}(\varLambda)$ is the limit of a sequence $\{(t_j, \omega_j)\}_{j \in \mathbb{N}}$ such that $\|\omega_j\| \leq C$ for all $j$. Therefore $\|\omega\| \leq C$ whenever $t \in [a, b]$ and $\omega \in \varLambda(t)$.

Now, take a sequence $\{(t_j, \omega_j)\}_{j \in \mathbb{N}}$ of points in $\mathrm{Gr}(\varLambda)$. Then $\|\omega_j\| \leq C$ for all $j$, so we may find an infinite subset $J$ of $\mathbb{N}$ such that the sequence $\{(t_j, \omega_j)\}_{j \in J}$ converges to a limit $(t, \omega) \in [a, b] \times X^\dagger$. We need to show that $\omega \in \varLambda(t)$. For each $j \in J$, the covector $\omega_j$ is a convex combination $\sum_{k=0}^n \alpha_{j,k}\omega_{j,k}$, where $\alpha_{j,k} \geq 0$, $\sum_{k=0}^n \alpha_{j,k} = 1$, and $\omega_{j,k} = \lim_{\ell \to \infty} \omega_{j,k,\ell}$, with $x_{j,k,\ell} \in D^{t_{j,k,\ell}}$, $g(x_{j,k,\ell}, t_{j,k,\ell}) > 0$, $\omega_{j,k,\ell} = \frac{\partial g}{\partial x}(x_{j,k,\ell}, t_{j,k,\ell})$, and $\lim_{\ell \to \infty}(x_{j,k,\ell}, t_{j,k,\ell}) = (\xi_*(t_j), t_j)$. Pick an infinite subset $J'$ of $J$ such that the limits $\tilde{\omega}_k = \lim_{j \to \infty, j \in J'} \omega_{j,k}$ and $\tilde{\alpha}_k = \lim_{j \to \infty, j \in J'} \alpha_{j,k}$ exist. Then $\tilde{\alpha}_k \geq 0$, $\sum_{k=0}^n \tilde{\alpha}_k = 1$, and $\sum_{k=0}^n \tilde{\alpha}_k\tilde{\omega}_k = \omega$. Therefore the conclusion that $\omega \in \varLambda(t)$ will follow if we show that $\tilde{\omega}_k \in \varLambda(t)$ for each $k$. For $j \in J'$, $k \in \{0, \ldots, n\}$, pick $\ell(j, k) \in \mathbb{N}$ such that

$$\|\hat{\omega}_{j,k} - \omega_{j,k}\| + \|\hat{x}_{j,k} - \xi_*(t_j)\| + |\hat{t}_{j,k} - t_j| \leq 2^{-j},$$

where $\hat{\omega}_{j,k} = \omega_{j,k,\ell(j,k)}$, $\hat{x}_{j,k} = x_{j,k,\ell(j,k)}$, $\hat{t}_{j,k} = t_{j,k,\ell(j,k)}$. Then $\tilde{\omega}_k = \lim_{j \to \infty, j \in J'} \hat{\omega}_{j,k}$, with $\hat{\omega}_{j,k} \in \partial_x g(\hat{x}_{j,k}, \hat{t}_{j,k})$, $g(\hat{x}_{j,k}, \hat{t}_{j,k}) > 0$, and $\lim_{j \to \infty}(\hat{x}_{j,k}, \hat{t}_{j,k}) = (\xi_*(t), t)$. Therefore $\tilde{\omega}_k \in \varLambda(t)$ for each $k$, and then $\omega \in \varLambda(t)$, completing the proof that $\varLambda$ is upper semicontinuous and has compact values. So we have proved (1).

Now let us prove (2). Fix a $t \in K$. Then there exist, for $j \in \mathbb{N}$, pairs $(\tilde{x}_j, t_j) \in S_g$ such that $g(\tilde{x}_j, t_j) > 0$ and $\|\tilde{x}_j - \xi_*(t)\| + |t_j - t| < 2^{-j}$. Pick

$x_j \in D^{t_j}$ such that $g(x_j,t_j)>0$ and $\|x_j-\tilde{x}_j\|<2^{-j}$. Let $\omega_j=\frac{\partial g}{\partial x}(x_j,t_j)$. Then $\|\omega_j\|\le C$ for all $j$. Therefore, we can pick an infinite subset $J$ of $\mathbb{N}$ such that $\omega=\lim_{j\to\infty,j\in J}\omega_j$ exists. Then $\omega\in\Lambda(t)$, so $\Lambda(t)\neq\emptyset$. Next, fix a $t\in[a,b]\backslash K$. Then no sequence $\{(x_j,t_j,\omega_j)_{j\in\mathbb{N}}$ of the kind specified in the definition of $\partial_x^> g$ exists, so $\partial_x^> g(\bar{x},t)$ is empty, that is, $\Lambda(t)=\emptyset$. This completes the proof of (2).

We now prove (3). We take a point $\bar{t}\in[a,b]$, a sequence $\{(t_j,h_j)\}_{j\in\mathbb{N}}$ of points of $S_g$ such that $\lim_{j\to\infty}h_j=0$, and $\lim_{j\to\infty}t_j=\bar{t}$, and show that

$$\lim_{j\to\infty}\mu_j=0,\ \text{where}\ \mu_j=\frac{\rho_j}{\|h_j\|},\ \rho_j=\mathrm{dist}(\chi_g^{co}(\xi_*(t_j)+h_j,t_j),\Lambda(\bar{t})\cdot h_j).\quad(5.3.2)$$

Write $x_j=\xi_*(t_j)+h_j$, $\bar{x}=\xi_*(\bar{t})$ (so that $\lim_{j\to\infty}x_j=\bar{x}$). Suppose (5.3.2) is not true. Then we can pick an infinite subset $J$ of $\mathbb{N}$ and an $\varepsilon\in\mathbb{R}$ such that $\varepsilon>0$ and $\mu_j\ge\varepsilon$ for all $j\in J$. Fix a $j\in J$. Then $g(x_j,t_j)>0$. Let $\gamma_j=g(x_j,t_j)$, and use $\Sigma_j$ to denote the sphere $\{h\in X:\|h\|=\|h_j\|\}$. (Recall that $h_j\neq0$, so $\Sigma_j$ is a true sphere, not reduced to a point.) For $h\in X\backslash\{0\}$, let $\sigma_h$ denote the segment $\{\xi_*(t_j)+sh:0\le s\le1\}$. It then follows from Fubini's theorem and Rademacher's theorem that the function $g^{t_j}$ is differentiable at almost all points of $\sigma_h$ (that is, $\xi_*(t_j)+sh\in D^{t_j}$ for almost all $s\in[0,1]$) for almost all $h\in\Sigma_j$. Therefore we can pick $\tilde{h}_j\in\Sigma_j$ such that, if we let $\tilde{x}_j=\xi_*(t_j)+\tilde{h}_j$, then $\|\tilde{h}_j-h_j\|\le(2C)^{-1}\gamma_j$ and $\xi_*(t_j)+s\tilde{h}_j\in D^{t_j}$ for almost all $s\in[0,1]$. Therefore $\|\tilde{x}_j-x_j\|\le(2C)^{-1}\gamma_j$ and $g(x_j,t_j)-g(\tilde{x}_j,t_j)\le C\|x_j-\tilde{x}_j\|\le\frac{\gamma_j}{2}$ from which it follows (since $g(x_j,t_j)=\gamma_j$) that $g(\tilde{x}_j,t_j)\ge\frac{\gamma_j}{2}$. Clearly,

$$g(\tilde{x}_j,t_j)=g(\xi_*(t_j),t_j)+\Big(\int_0^1\frac{\partial g}{\partial x}(\xi_*(t_j)+s\tilde{h}_j,t_j)\,ds\Big)\cdot\tilde{h}_j\,.$$

Since $g(\xi_*(t_j),t_j)\le0$, and $g(\tilde{x}_j,t_j)\ge\frac{\gamma_j}{2}$, we conclude that

$$\Big(\int_0^1\frac{\partial g}{\partial x}(\xi_*(t_j)+s\tilde{h}_j,t_j)\,ds\Big)\cdot\tilde{h}_j\ge\frac{\gamma_j}{2}\,.$$

We claim that we can pick $s_j\in[0,1]$ such that the three conditions

$$\xi_*(t_j)+s_j\tilde{h}_j\in D^{t_j},\ g(\xi_*(t_j)+s_j\tilde{h}_j,t_j)>0,\ \frac{\partial g}{\partial x}(\xi_*(t_j)+s_j\tilde{h}_j,t_j)\cdot\tilde{h}_j\ge\frac{\gamma_j}{2}\quad(5.3.3)$$

hold. To see this, let $\eta(s)=g(\xi_*(t_j)+s\tilde{h}_j,t_j)-g(\xi_*(t_j),t_j)$ for $s\in[0,1]$, so $\eta(0)=0$, $\eta(1)>0$, $\eta$ is Lipschitz, and $\dot{\eta}(s)=\frac{\partial g}{\partial x}(\xi_*(t_j)+s\tilde{h}_j,t_j)\cdot\tilde{h}_j$ for almost all $s\in[0,1]$. Let $\tau=\sup\{s\in[0,1]:\eta(s)\le0\}$. Then $\tau<1$, $\eta(\tau)=0$, $\eta(1)\ge\frac{\gamma_j}{2}$, and $\eta(s)>0$ for $s>\tau$. Therefore, there exists an $s$ such that $\tau<s<1$, $\xi_*(t_j)+s\tilde{h}_j\in D^{t_j}$, and $\dot{\eta}(s)\ge\frac{\gamma_j}{2}$ (because if such an $s$ did not exist it would follow that $\dot{\eta}(s)<\frac{\gamma_j}{2}$ for all $s\in]\tau,1[$ such that $\xi_*(t_j)+s\tilde{h}_j\in D^{t_j}$, i.e., that $\dot{\eta}(s)<\frac{\gamma_j}{2}$ for almost all $s\in[\tau,1]$, and then $\int_\tau^1\dot{\eta}(s)\,ds<\frac{\gamma_j}{2}$, so $\eta(1)-\eta(\tau)<\frac{\gamma_j}{2}$, contradicting the fact that $\eta(\tau)=0$ and $\eta(1)\ge\frac{\gamma_j}{2}$). This $s$ is our desired $s_j$, and the claim is proved.

Now let $\hat{h}_j = s_j\tilde{h}_j$,k $\hat{x}_j = \xi_*(t_j) + \hat{h}_j$, $\omega_j = \frac{\partial g}{\partial x}(\hat{x}_j, t_j)$. Then the sequence $\{\omega_j\}_{j\in J_3}$ is bounded (because $\|\omega_j\| \le C$) so we may find an infinite subset $J'$ of $J$ such that $\omega = \lim_{j\to\infty, j\in J'} \omega_j$ exists. It then follows from the definition of $\Lambda$ that $\omega \in \Lambda(\bar{t})$. Then, if $j \in J'$, we have

$$\omega \cdot h_j = (\omega - \omega_j) \cdot h_j + \omega_j \cdot (h_j - \tilde{h}_j) + \omega_j \cdot \tilde{h}_j . \tag{5.3.4}$$

It follows from (5.3.3) that $\omega_j \cdot \tilde{h}_j \ge \frac{\gamma_j}{2}$, while on the other hand we also have $|\omega_j \cdot (h_j - \tilde{h}_j)| \le \frac{\gamma_j}{2}$, since $\|\tilde{h}_j - h_j\| \le (2C)^{-1}\gamma_j$ and $\|\omega_j\| \le C$. Then (5.3.4) allows us to conclude that

$$\omega \cdot h_j \ge -\|\omega - \omega_j\| \cdot \|h_j\| . \tag{5.3.5}$$

Since $\chi_g^{co}(\xi_*(t_j) + h_j, t_j) = [0, +\infty[$, and $\omega \cdot h_j$ belongs to $\Lambda(\bar{t}) \cdot h_j$, (5.3.5) implies that the distance $\rho_j$ between the sets $\Lambda(\bar{t}) \cdot h_j$ and $\chi_g^{co}(\xi_*(t_j) + h_j, t_j)$ is not greater than $\|\omega - \omega_j\| \cdot \|h_j\|$. Hence $\mu_j \le \|\omega - \omega_j\|$. Therefore $\lim_{j\to\infty, j\in J'} \mu_j = 0$. But this contradicts the facts that $J' \subseteq J$ and $\mu_j \ge \varepsilon$ for all $j \in J$. This contradiction concludes our proof. □

# 6 Discontinuous vector fields

In this section we will study classes of discontinuous vector fields $f$ that have good properties, such as local existence of trajectories, local Cellina approximability of flow maps, and differentiability of the flow maps $(t, s, x) \mapsto \Phi^f(t, s, x)$ at points $(\bar{t}, \bar{t}, \bar{x})$. (The *flow map* of a ppd time-varying vector field was defined in §2.1, page 6.)

These classes have already been studied in great detail in [24], so here we will just limit ourselves to presenting the relevant definitions, referring the reader to [24] for the proofs.

## 6.1 Co-integrably bounded integrally continuous maps.

The goal of this subsection is to define (i) the class of "co-IBIC" time-varying maps $K \ni (x, t) \mapsto f(x, t) \in Y$, where $X, Y$ are FDNRLSs and $K$ is a compact subset of $X \times \mathbb{R}$, and (ii) the lower semicontinuous analogue of the co-IBIC condition—called "co-ILBILSC,"—in the case when $Y = \mathbb{R}$. (The two abbreviations "co-IBIC" and "co-ILBILSC" stand, respectively, for "co-integrably bounded and integrally continuous" and "co-integrably lower bounded and integrally lower semicontinuous.")

The co-IBIC class will be interesting when $Y = X$, i.e., when $f$ is a time-varying vector field on $X$. Roughly speaking the co-IBIC condition is the minimum requirement that has to be satisfied so that local existence of trajectories can be proved using the Schauder fixed point theorem. For a time-varying vector field $f : X \times \mathbb{R} \mapsto X$, and an initial condition $(\bar{t}, \bar{x})$, one would like to prove existence of a trajectory $\xi$ of $f$, defined on some interval $[\bar{t}-\varepsilon, \bar{t}+\varepsilon]$, by finding a fixed point of the map

$$\varXi_{\bar{t},\varepsilon,\bar{x}} \ni \xi \mapsto \mathcal{I}(\xi) \in \mathcal{Z} \quad \text{such that} \quad \mathcal{I}(\xi)(t) = \bar{x} + \int_{\bar{t}}^{t} f(\xi(s), s)\, ds\,,$$

where $\mathcal{Z} = C^0([\bar{t}-\varepsilon, \bar{t}+\varepsilon], X)$, and $\varXi_{\bar{t},\varepsilon,\bar{x}}$ is the set of all $\xi \in C^0([\bar{t}-\varepsilon, \bar{t}+\varepsilon], X)$ for which $\xi(\bar{t}) = \bar{x}$. To guarantee the existence of a fixed point, one needs $\mathcal{I}$ to map $\varXi_{\bar{t},\varepsilon,\bar{x}}$ continuously into a compact convex subset of $\varXi_{\bar{t},\varepsilon,\bar{x}}$.

Traditionally, this is done—if, for example, $f$ is continuous with respect to $x$ for each $t$ and measurable with respect to $t$ for each $x$—by assuming that a bound $\|f(x,t)\| \le k(t)$ is satisfied for all $x, t$, where the function $k : \mathbb{R} \mapsto [0, +\infty]$ is locally integrable. (Naturally, it suffices to assume that a function $k_J$ exists for every compact subset $J$ of $X$.) In that case, the functions $\mathcal{I}(\xi)$, for $\xi \in \varXi_{\bar{t},\varepsilon,\bar{x}}$, are absolutely continuous with derivatives $\dot{\xi}(t)$ bounded in norm by $k(t)$, and the Ascoli-Arzelà theorem guarantees the desired compactness, while the continuity of the map follows from the Lebesgue dominated convergence theorem.

Here we will consider a much large class of time-varying vector fields, and in particular we will not require that $f(x,t)$ be continuous with respect to $x$. The main condition is going to be the continuity of the map $\mathcal{I}$. We will still want to assume the existence of the integral bounds $k$, and the continuity of the integral map will only be assumed on the set of absolutely continuous arcs $\xi$ whose derivatives are bounded by the same function $k$. That is, we will single out, for each compact subset $S$ of $X \times \mathbb{R}$, the set $\mathrm{Arc}\,(S)$ of all arcs $\xi : I \mapsto X$, defined on a $\xi$-dependent compact interval $I$, and such that $(\xi(t), t) \in S$ for all $t \in I$, and the subset $\mathrm{Arc}_{\,k}(S)$ of $\mathrm{Arc}\,(S)$ consisting of all absolutely continuous $\xi \in \mathrm{Arc}\,(S)$ such that $\|\dot{\xi}(t)\| \le k(t)$ for almost all $t$. This leads us to the concept of "co-IBIC" time-varying ppd vector fields, that is, maps $f : X \times \mathbb{R} \hookrightarrow X$ such that, on a given compact subset $S$ of $X \times \mathbb{R}$, satisfy a bound $\|f(x,t)\| \le k(t)$ and also give rise to a continuous integral map $\mathcal{I}$ on $\mathrm{Arc}_{\,k'}(S)$, *with the integrable functions $k$ and $k'$ equal to each other.*

Finally, we point out that, for the integral map to be continuous, an obvious prerequisite is that it be well defined. If $\xi \in \mathrm{Arc}\,(S)$, and $\mathrm{Do}(\xi) = I$, then of course the map $I \ni t \mapsto f(\xi(t), t)$ will be bounded by an integrable function of $t$ as long as $f$ satisfies a bound $\|f(x,t)\| \le k(t)$. But in addition we have to make sure that the map is measurable, and this will require that $f$ be measurable with respect to $(x,t)$ in some appropriate sense. This is why our discussion will begin with the definition of "essential Borel×Lebesgue measurability."

***Measurability conditions.*** If $X$ is a FDNRLS, we use $\mathcal{B}o(X)$, $\mathcal{L}eb(X)$, $\mathcal{BL}eb(X, \mathbb{R})$, to denote, respectively, the Borel and Lebesgue $\sigma$-algebras of subsets of $X$, and the product $\sigma$-algebra $\mathcal{B}o(X) \otimes \mathcal{L}eb(\mathbb{R})$. We let $\mathcal{N}(X, \mathbb{R})$ denote the set of all subsets $S$ of $X \times \mathbb{R}$ such that $\varPi_X(S)$ is a Lebesgue-null subset of $\mathbb{R}$, where $\Pi_X$ is the canonical projection $X \times \mathbb{R} \ni (x,t) \to t \in \mathbb{R}$. Finally, we use $\mathcal{BL}^e(X,\mathbb{R})$ to denote the $\sigma$-algebra of subsets of $X \times \mathbb{R}$ generated by $\mathcal{BL}eb(X,\mathbb{R}) \cup \mathcal{N}(X,\mathbb{R})$. It is then clear that the relations $\mathcal{B}o(X \times \mathbb{R}) \subset \mathcal{BL}eb(X,\mathbb{R}) \subset \mathcal{BL}^e(X,\mathbb{R})$ hold, and both inclusions are strict.

**Definition 6.1** *Let $X, Y$ be FDNRLSs, let $f$ be a ppd map from $X \times \mathbb{R}$ to $Y$, and let $K$ be a compact subset of $X \times \mathbb{R}$.*

- *We say that $f$ is **essentially Borel×Lebesgue measurable on** $K$, or $\mathcal{BL}^e(X,\mathbb{R})$**-measurable on** $K$, if $K \subseteq \mathrm{Do}(f)$ and $f^{-1}(U) \cap K$ belongs to $\mathcal{BL}^e(X,\mathbb{R})$ for all open subsets $U$ of $Y$.*
- *We use $\mathcal{M}_{\mathcal{BL}^e}(X \times \mathbb{R}, K, Y)$ to denote the set of ppd maps from $X \times \mathbb{R}$ to $Y$ that are $\mathcal{BL}^e(X,\mathbb{R})$-measurable on $K$.*

***Integrable boundedness.*** Assume that $X, Y$ are FDNRLSs, $f$ is a ppd map from $X \times \mathbb{R}$ to $Y$, and $K$ is a compact subset of $X \times \mathbb{R}$.

- *An **integrable bound** for $f$ on the set $K$ is an integrable function $\mathbb{R} \ni t \to \varphi(t) \in [0,+\infty]$ such that $\|f(x,t)\| \le \varphi(t)$ for all $(x,t) \in K$.*
- *If $Y = \mathbb{R}$, an **integrable lower bound** for $f$ on $K$ is an integrable function $\mathbb{R} \ni t \to \varphi(t) \in [0,+\infty]$ such that $f(x,t) \ge -\varphi(t)$ for all $(x,t) \in K$.*
- *We call $f$ **integrably bounded** (IB)—resp. **integrably lower bounded** (ILB)—on $K$ if $f$ is $\mathcal{BL}^e(X,\mathbb{R})$-measurable on $K$ and there exists an integrable bound—resp. an integrable lower bound—for $f$ on $K$.*
- *We write $\mathcal{IB}(X \times \mathbb{R}, K, Y)$, $\mathcal{ILB}(X \times \mathbb{R}, K, \mathbb{R})$ to denote, respectively, the sets of (i) all ppd maps from $X \times \mathbb{R}$ to $Y$ that are IB on $K$, and (ii) all ppd maps from $X \times \mathbb{R}$ to $\mathbb{R}$ that are ILB on $K$.* □

***Spaces of arcs.*** If $S \subseteq X \times \mathbb{R}$, and $I$ is a nonempty compact interval, we write $\mathrm{Arc}\,(I,S)$ to denote the set of all curves $\xi \in C^0(I;X)$ such that $(\xi(t),t) \in S$ for all $t \in I$. If $k : \mathbb{R} \mapsto \mathbb{R}_+ \cup \{+\infty\}$ is a locally integrable function, then $\mathrm{Arc}_k(I,S)$ will denote the set of all $\xi \in \mathrm{Arc}\,(I,S)$ such that $\xi$ is absolutely continuous and $\|\dot{\xi}(t)\| \le k(t)$ for almost all $t \in I$. We then write $\mathrm{Arc}\,(S)$, $\mathrm{Arc}_k(S)$ to denote, respectively, the union of the sets $\mathrm{Arc}\,(J,S)$ and the union of the $\mathrm{Arc}_k(J,S)$, taken over all nonempty compact subintervals $J$ of $\mathbb{R}$. It is then easy to show that

**Fact 6.2** *If $X, Y$ are FDNRLSs, $K \subseteq X \times \mathbb{R}$ is compact, $\xi \in \mathrm{Arc}\,(K)$, and $f$ belongs to $\mathcal{M}_{\mathcal{BL}^e}(X \times \mathbb{R}, K, Y)$, then the function $\mathrm{Do}(\xi) \ni t \mapsto f(\xi(t),t) \in Y$ is measurable.* □

The sets $\mathrm{Arc}\,(S)$ are metric spaces, with the distance $d(\xi,\xi')$ between two members $\xi : [a,b] \mapsto X$, $\xi' : [a',b'] \mapsto X$ of $\mathrm{Arc}\,(S)$ defined by

$$d(\xi,\xi') = |a-a'| + |b-b'| + \sup\{\|\tilde{\xi}(t) - \tilde{\xi}'(t)\| : t \in \mathbb{R}\}$$

where, for any continuous map $\gamma : [\alpha,\beta] \mapsto X$, $\tilde{\gamma}$ denotes the extension of $\gamma$ to $\mathbb{R}$ which is identically equal to $\gamma(\alpha)$ on $]-\infty,\alpha]$ and to $\gamma(\beta)$ on $[\beta,+\infty[$. Clearly, then

**Fact 6.3** *If $X$ is a FDNRLS and $S \subseteq X \times \mathbb{R}$, then*

*(1) if $\{\xi_j\}_{j\in\mathbb{N}}$ is a sequence of members of $\mathrm{Arc}\,(S)$, with domains $[a_j, b_j]$, and $\xi \in \mathrm{Arc}\,(S)$ has domain $[a, b]$, then $\{\xi_j\}_{j\in\mathbb{N}}$ converges to $\xi$ if and only if (a) $\lim_{j\to\infty} a_j = a$, (b) $\lim_{j\to\infty} b_j = b$, and (c) $\lim_{j\to\infty} \xi_j(t_j) = \xi(t)$ whenever $\{t_j\}_{j\in\mathbb{N}}$ is a sequence such that $t_j \in [a_j, b_j]$ for each $j$ and $\lim_{j\to\infty} t_j = t \in [a, b]$,*

*(2) if $S$ is compact, and $k : \mathbb{R} \mapsto \mathbb{R}_+ \cup \{+\infty\}$ is locally integrable, then $\mathrm{Arc}\,_k(S)$ is compact.* □

***Integral continuity.*** If $X, Y$ are FDNRLSs, $K \subseteq X \times \mathbb{R}$ is compact, and $f \in \mathcal{IB}(X \times \mathbb{R}, K, Y)$, then it is convenient to define a real-valued ***integral map*** $\mathcal{I}_{f,K} : \mathrm{Arc}\,(K) \mapsto \mathbb{R}$, by letting $\mathcal{I}_{f,K}(\xi) = \int_{\mathrm{Do}(\xi)} f(\xi(s), s)\, ds$ for every $\xi \in \mathrm{Arc}\,(K)$. If $\mathcal{S} \subseteq \mathrm{Arc}\,(K)$, we call $f$ ***integrally continuous*** (abbr. IC) ***on*** $\mathcal{S}$ if $\mathcal{I}_{f,K} \lceil \mathcal{S}$ is continuous. If $f \in \mathcal{ILB}(X \times \mathbb{R}, K, \mathbb{R})$, then $\mathcal{I}_{f,K}$ is still well defined as a map into $\mathbb{R} \cup \{+\infty\}$, and we call $f$ ***integrally lower semicontinuous*** (abbr. ILSC) ***on*** $\mathcal{S}$ if $\mathcal{I}_{f,K} \lceil \mathcal{S}$ is lower semicontinuous.

We will be particularly interested in maps $f$ that, for some integrable function $k$, are both integrably bounded with integral bound $k$ and integrally continuous on $\mathrm{Arc}\,_k(K)$.

**Definition 6.4** *If $X$, $Y$ are FDNRLSs, $K$ is a compact subset of $X \times \mathbb{R}$, and $f : X \times \mathbb{R} \hookrightarrow Y$, we call $f$ **co-IBIC** ("co-integrably bounded and integrally continuous") **on** $K$ if $f \in \mathcal{IB}(X \times \mathbb{R}, K, Y)$ and there exists an integrable bound $k : \mathbb{R} \mapsto [0, +\infty]$ for $f$ on $K$ such that $f$ is integrally continuous on $\mathrm{Arc}\,_k(K)$. If $f : X \times \mathbb{R} \hookrightarrow \mathbb{R}$, we call $f$ **co-ILBILSC** ("co-integrably bounded and integrally lower semicontinuous") **on** $K$ if $f \in \mathcal{ILB}(X \times \mathbb{R}, K, \mathbb{R})$ and there exists an integrable lower bound $k : \mathbb{R} \mapsto [0, +\infty]$ for $f$ on $K$ such that $f$ is integrally lower semicontinuous on $\mathrm{Arc}\,_k(K)$.* □

### 6.2 Points of approximate continuity

Suppose that $X$ and $Y$ are FDNRLSs, $f$ is a ppd map from $X \times \mathbb{R}$ to $Y$, and $(\bar{x}_*, \bar{t}_*) \in X \times \mathbb{R}$. A *modulus of approximate continuity* (abbr. MAC) for $f$ near $(\bar{x}_*, \bar{t}_*)$ is a function $]\,0, +\infty\,[ \times \mathbb{R} \ni (\beta, r) \mapsto \psi(\beta, r) \in ]\,0, +\infty\,]$ such that

(MAC.1) *the function $\mathbb{R} \ni r \mapsto \psi(\beta, r) \in ]\,0, +\infty\,]$ is measurable for each $\beta \in ]\,0, +\infty\,[$,*

(MAC.2) $\lim_{(\beta,\rho)\to(0,0),\beta>0,\rho>0} \frac{1}{\rho} \int_{-\rho}^{\rho} \psi(\beta, r)\, dr = 0$,

(MAC.3) *there exist positive numbers $\beta_*$, $\rho_*$, such that*

  (MAC.3.a) *$f(x, t)$ is defined whenever $\|x - \bar{x}_*\| \le \beta_*$ and $|t - \bar{t}_*| \le \rho_*$,*

  (MAC.3.b) *the inequality $\|f(x, t) - f(\bar{x}_*, \bar{t}_*)\| \le \psi(\beta, t - \bar{t}_*)$ holds whenever $\beta \in \mathbb{R}$, $x \in X$, $t \in \mathbb{R}$ are such that $\|x - \bar{x}_*\| \le \beta \le \beta_*$ and $|t - \bar{t}_*| \le \rho_*$.*

**Definition 6.5** *A **point of approximate continuity** (abbr. PAC) for $f$ is a point $(\bar{x}_*, \bar{t}_*) \in X \times \mathbb{R}$ having the property that there exists a MAC for $f$ near $(\bar{x}_*, \bar{t}_*)$.* □

An important example of a class of maps with many points of approximate continuity is given by the following corollary of the well-known Scorza-Dragoni theorem.

**Proposition 6.6** *Suppose* $X$, $Y$ *are FDNRLSs,* $\Omega$ *is open in* $X$, $a, b \in \mathbb{R}$, $a < b$, *and* $f : \Omega \times [a,b] \mapsto Y$ *is such that*

- *the partial map* $[a,b] \ni t \mapsto f(x,t) \in Y$ *is measurable for every* $x \in \Omega$,
- *the partial map* $\Omega \ni x \mapsto f(x,t) \in Y$ *is continuous for every* $t \in [a,b]$, *and*
- *there exists an integrable function* $[a,b] \ni t \mapsto k(t) \in [0,+\infty]$ *such that the bound* $\|f(x,t)\| \le k(t)$ *holds whenever* $(x,t) \in \Omega \times [a,b]$.

*Then there exists a subset* $G$ *of* $[a,b]$ *for which* $\mathrm{meas}([a,b]\backslash G) = 0$, *such that every* $(\bar{x}_*, \bar{t}) \in \Omega \times G$ *is a point of approximate continuity of* $f$. □

Another important example of maps with many PACs is given by the following result, proved in [24].

**Proposition 6.7** *Suppose that* $X$ *and* $Y$ *are FDNRLSs,* $a, b \in \mathbb{R}$, $a < b$, *and* $F : X \times [a,b] \mapsto\!\!\!\rightarrow Y$ *is an almost lower semicontinuous set-valued map with closed nonempty values such that for every compact subset* $K$ *of* $X$ *the function* $[a,b] \ni t \mapsto \sup\{\min\{\|y\| : y \in F(x,t)\} : x \in K\}$ *is integrable. Then there exists a subset* $G$ *of* $[a,b]$ *such that* $\mathrm{meas}([a,b]\backslash G) = 0$, *having the property that, whenever* $x_* \in X$, $t_* \in G$, $v_* \in F(x_*, t_*)$, *and* $K \subseteq X$ *is compact, there exists a map* $K \times [a,b] \ni (x,t) \mapsto f(x,t) \in F(x,t)$ *which is co-IBIC on* $K \times [a,b]$ *and such that* $(x_*, t_*)$ *is a PAC of* $f$ *and* $f(x_*, t_*) = v_*$. □

## 7 The maximum principle

We consider a *fixed time-interval optimal control problem with state space constraints*, of the form

$$\text{minimize} \qquad \varphi(\xi(b)) + \int_a^b f_0(\xi(t), \eta(t), t)\, dt$$

$$\text{subject to} \begin{cases} \xi(\cdot) \in W^{1,1}([a,b], X) \quad \text{and} \quad \dot{\xi}(t) = f(\xi(t), \eta(t), t) \ \text{ a.e.}, \\ \xi(a) = \bar{x}_* \quad \text{and} \quad \xi(b) \in S\,, \\ g_i(\xi(t), t) \le 0 \ \text{ for } \ t \in [a,b]\,, \ \ i = 1, \dots, m\,, \\ h_j(\xi(b)) = 0 \ \text{ for } j = 1, \dots, \tilde{m}\,, \\ \eta(t) \in U \ \text{ for all } t \in [a,b] \quad \text{and} \quad \eta(\cdot) \in \mathcal{U}\,, \end{cases}$$

and a *reference trajectory-control pair* $(\xi_*, \eta_*)$.

***The technical hypotheses.*** We will make the assumption that the data 14-tuple $\mathcal{D} = (X, m, \tilde{m}, U, a, b, \varphi, f_0, f, \bar{x}_*, \mathbf{g}, \mathbf{h}, S, \mathcal{U})$ satisfies:

(H1) $X$ *is a normed finite-dimensional real linear space,* $\bar{x}_* \in X$, *and* $m, \tilde{m}$ *are nonnegative integers*;

(H2) $U$ *is a set,* $a, b \in \mathbb{R}$ *and* $a < b$*;*

(H3) *$f_0$, $f$ are ppd functions from $X \times U \times \mathbb{R}$ to $\mathbb{R}$, $X$, respectively;*
(H4) *$\mathbf{g} = (g_1, \dots, g_m)$ is an $m$-tuple of ppd functions from $X \times \mathbb{R}$ to $\mathbb{R}$;*
(H5) *$\mathbf{h} = (h_1, \dots, h_{\tilde{m}})$ is an $\tilde{m}$-tuple of ppd functions from $X$ to $\mathbb{R}$;*
(H6) *$\varphi$ is a ppd function from $X$ to $\mathbb{R}$;*
(H7) *$S$ is a subset of $X$;*
(H8) *$\mathcal{U}$ is a set of ppd functions from $\mathbb{R}$ to $U$ such that the domain of every $\eta \in \mathcal{U}$ is a nonempty compact interval.*

Given such a $\mathcal{D}$, a *controller* is a ppd function $\eta : \mathbb{R} \hookrightarrow U$ whose domain is a nonempty compact interval. (Hence (H8) says that $\mathcal{U}$ is a set of controllers.) An *admissible controller* is a member of $\mathcal{U}$. If $\alpha, \beta \in \mathbb{R}$ and $\alpha \leq \beta$, then we use $W^{1,1}([\alpha, \beta], X)$ to denote the space of all absolutely continuous maps $\xi : [\alpha, \beta] \mapsto X$. A *trajectory* for a controller $\eta : [\alpha, \beta] \mapsto U$ is a map $\xi \in W^{1,1}([\alpha, \beta], X)$ such that, for almost every $t \in [\alpha, \beta]$, $(\xi(t), \eta(t), t)$ belongs to $\mathrm{Do}(f)$ and $\dot{\xi}(t) = f(\xi(t), \eta(t), t)$. A *trajectory-control pair* (abbr. TCP) is a pair $(\xi, \eta)$ such that $\eta$ is a controller and $\xi$ is a trajectory for $\eta$. The *domain* of a TCP $(\xi, \eta)$ is the domain of $\eta$, which is, by definition, the same as the domain of $\xi$. A TCP $(\xi, \eta)$ is *admissible* if $\eta \in \mathcal{U}$.

A TCP $(\xi, \eta)$ with domain $[\alpha, \beta]$ is *cost-admissible* if

- *$(\xi, \eta)$ is admissible*;
- *the function $[\alpha, \beta] \ni t \mapsto f_0(\xi(t), \eta(t), t)$ is a.e. defined, measurable, and such that $\int_\alpha^\beta \min\Big(0, f_0(\xi(t), \eta(t), t)\Big)\, dt > -\infty$*;
- *the terminal point $\xi(\beta)$ belongs to the domain of $\varphi$.*

It follows that if $(\xi, \eta)$ is cost-admissible then the number

$$J(\xi, \eta) = \varphi(\xi(\beta)) + \int_\alpha^\beta f_0(\xi(t), \eta(t), t)\, dt$$

—called the *cost* of $(\xi, \eta)$—is well defined and belongs to $]-\infty, +\infty]$.

A TCP $(\xi, \eta)$ with domain $[\alpha, \beta]$ is *constraint-admissible* if it satisfies all our state space constraints, that is, if

(CA1) $\xi(\alpha) = \bar{x}_*$,
(CA2) $(\xi(t), t) \in \mathrm{Do}(g_i)$ *and* $g_i(\xi(t), t) \leq 0$ *if* $t \in [\alpha, \beta]$ *and* $i \in \{1, \dots, m\}$,
(CA3) $\xi(\beta) \in S \cap \Big(\cap_{j=1}^{\tilde{m}} \mathrm{Do}(h_j)\Big)$ *and* $h_j(\xi(\beta)) = 0$ *for* $j = 1, \dots, \tilde{m}$.

For the data tuple $\mathcal{D}$, we use $ADM(\mathcal{D})$ to denote the set of all cost-admissible, constraint-admissible TCPs $(\xi, \eta)$, and $ADM_{[a,b]}(\mathcal{D})$ to denote the set of all $(\xi, \eta) \in ADM(\mathcal{D})$ whose domain is $[a, b]$.

The hypothesis on the reference TCP $(\xi_*, \eta_*)$ is that it is a cost-minimizer in $ADM_{[a,b]}(\mathcal{D})$, i.e., an admissible, cost admissible, constraint-admissible TCP with domain $[a, b]$ that minimizes the cost in the class of all admissible, cost-admissible, constraint-admissible, TCP's with domain $[a, b]$. That is,

(H9) *The pair $(\xi_*, \eta_*)$ satisfies $(\xi_*, \eta_*) \in ADM_{[a,b]}(\mathcal{D})$, $J(\xi_*, \eta_*) < +\infty$, and $J(\xi_*, \eta_*) \leq J(\xi, \eta)$ for all pairs $(\xi, \eta) \in ADM_{[a,b]}(\mathcal{D})$.*

To the data $\mathcal{D}$, $\xi_*$, $\eta_*$ as above, we associate the *cost-augmented dynamics* $\mathbf{f} : X \times U \times \mathbb{R} \hookrightarrow \mathbb{R} \times X$ defined by

$$\mathrm{Do}(\mathbf{f}) = \mathrm{Do}(f_0) \cap \mathrm{Do}(f), \;\text{ and }\; \mathbf{f}(z) = (f_0(z), f(z)) \;\text{ for }\; z = (x, u, t) \in \mathrm{Do}(\mathbf{f})\,.$$

We also define the *epi-augmented dynamics* $\check{\mathbf{f}} : X \times U \times \mathbb{R} \mapsto\!\!\!\!\twoheadrightarrow \mathbb{R} \times X$, given, for each $z = (x, u, t) \in X \times U \times \mathbb{R}$, by

$$\check{\mathbf{f}}(z) = [f_0(z), +\infty[\times\{f(z)\} \quad\text{if}\quad z \in \mathrm{Do}(\mathbf{f}), \qquad \check{\mathbf{f}}(z) = \emptyset \quad\text{if}\quad z \notin \mathrm{Do}(\mathbf{f})\,.$$

We will also use the constraint indicator maps $\chi^{co}_{g_i} : X \times \mathbb{R} \mapsto\!\!\!\!\twoheadrightarrow \mathbb{R}$, for $i = 1, \ldots, m$, and the epimap $\check{\varphi} : X \mapsto\!\!\!\!\twoheadrightarrow \mathbb{R}$. (These two notions were defined in §2.1.)

For $i \in \{1, \ldots, m\}$, we let

$$\begin{aligned} &\sigma^{\mathbf{f}}_*(t) = \mathbf{f}(\xi_*(t), \eta_*(t), t) \;\text{ and }\; \sigma^{g_i}_*(t) = g_i(\xi_*(t), t) \quad \text{if }\; t \in [a, b],\\ &Av_{g_i} = \{(x, t) \in X \times [a, b] : g_i(x, t) > 0\} \end{aligned}$$

(so the $Av_{g_i}$ are the "sets to be avoided"). We then define $K_i$ to be the set of all $t \in [a, b]$ such that $(\xi_*(t), t)$ belongs to the closure of $Av_{g_i}$. Then $K_i$ is obviously a compact subset of $[a, b]$.

We now make technical hypotheses on $\mathcal{D}$, $\xi_*$, $\eta_*$, and five new objects called $\Lambda^{\mathbf{f}}$, $\Lambda^{\mathbf{g}}$, $\Lambda^{\mathbf{h}}$, $\Lambda^{\varphi}$, and $C$. To state these hypotheses, we let $\mathcal{U}_{c,[a,b]}$ denote the set of all constant $U$-valued functions defined on $[a, b]$, and define $\mathcal{U}_{c,[a,b],*} = \mathcal{U}_{c,[a,b]} \cup \{\eta_*\}$ . The technical hypotheses are then as follows.

(H10) *For each* $\eta \in \mathcal{U}_{c,[a,b],*}$. *there exist a positive number* $\delta_\eta$ *such that*

 (H10.a) $\mathbf{f}(x, \eta(t), t)$ *is defined for all* $(x, t)$ *in the tube* $\mathcal{T}^X(\xi_*, \delta_\eta)$;

 (H10.b) *the time-varying vector field* $\mathcal{T}^X(\xi_*, \delta_\eta) \ni (x, t) \mapsto f(x, \eta(t), t)$ *is co-IBIC on* $\mathcal{T}^X(\xi_*, \delta_\eta)$;

 (H10.c) *the time-varying function* $\mathcal{T}^X(\xi_*, \delta_\eta) \ni (x, t) \mapsto f_0(x, \eta(t), t) \in \mathbb{R}$ *is co-ILBILSC on* $\mathcal{T}^X(\xi_*, \delta_\eta)$.

(H11) *The number* $\delta_{\eta_*}$ *can be chosen so that (i) each function* $g_i$ *is defined on* $\mathcal{T}^X(\xi_*, \delta_{\eta_*})$, *and (ii) for each* $i \in \{1, \ldots, m\}$, $t \in [a, b]$, *the set* $\{x \in X : g_i(x, t) > 0, \|x - \xi_*(t)\| \le \delta_{\eta_*}\}$ *is relatively open in the ball* $\{x \in X : \|x - \xi_*(t)\| \le \delta_{\eta_*}\}$.

(H12) $\Lambda^{\mathbf{f}}$ *is a measurable integrably bounded set-valued map from* $[a, b]$ *to* $X^\dagger \times \boldsymbol{L}(X)$ *with compact convex values such that* $\Lambda^{\mathbf{f}}$ *belongs to* $VG^{L^1,ft}_{GDQ}(\check{\mathbf{f}}, [a, b], \xi_*, \sigma^{\mathbf{f}}_*, X \times \mathbb{R})$.

(H13) $\Lambda^{\mathbf{g}}$ *is an* $\hat{m}$*-tuple* $(\Lambda^{g_1}, \ldots, \Lambda^{g_{\hat{m}}})$ *such that, for each* $i \in \{1, \ldots, \hat{m}\}$, $\Lambda^{g_i}$ *is an upper semicontinuous set-valued map from* $[a, b]$ *to* $X^\dagger$ *with compact convex values, such that* $\Lambda^{g_i} \in VG^{pw,rob}_{GDQ}(\chi^{co}_{g_i}, \xi_*, \sigma^{g_i}_*, Av_{g_i})$.

(H14) $\Lambda^{\mathbf{h}}$ *is a generalized differential quotient of* $\mathbf{h}$ *at* $(\,\xi_*(b), \mathbf{h}(\xi_*(b)))$ *in the direction of* $X$.

(H15) $\Lambda^{\varphi}$ *is a generalized differential quotient of the epifunction* $\check{\varphi}$ *at the point* $(\xi_*(b), \varphi(\xi_*(b)))$ *in the direction of* $X$.

(H16) $C$ *is a limiting Boltyanskii approximating cone of* $S$ *at* $\xi_*(b)$.

Our last hypothesis will require the concept of an *equal-time interval-variational neighborhood* (abbr. ETIVN) of a controller $\eta$. We say that a set $\mathcal{V}$ of controllers is an ETIVN of a controller $\eta$ if

- *for every* $n \in \mathbb{Z}_+$ *and every* $n$*-tuple* $\mathbf{u} = (u_1, \ldots, u_n)$ *of members of* $U$*, there exists a positive number* $\varepsilon = \varepsilon(n, \mathbf{u})$ *such that whenever* $\eta' : \mathrm{Do}(\eta) \mapsto U$ *is a map obtained from* $\eta$ *by first selecting an* $n$*-tuple* $\mathbf{I} = (I_1, \ldots, I_n)$ *of pairwise disjoint subintervals of* $\mathrm{Do}(\eta)$ *with the property that* $\sum_{j=1}^{n} \mathrm{meas}(I_j) \leq \varepsilon$*, and then substituting the constant value* $u_j$ *for the value* $\eta(t)$ *for every* $t \in I_j$*,* $j = 1, \ldots, n$*, it follows that* $\eta' \in \mathcal{U}$*.*

We will then assume

(H17) *The class* $\mathcal{U}$ *is an equal-time interval-variational neighborhood of* $\eta_*$*.*

We define the *Hamiltonian* to be the function $H_\alpha : X \times U \times X^\dagger \times \mathbb{R} \hookrightarrow \mathbb{R}$ given by $H_\alpha(x, u, p, t) = p \cdot f(x, u, t) - \alpha f_0(x, u, t)$, so $H_\alpha$ depends on the real parameter $\alpha$.

***The main theorem.*** The following is our version of the maximum principle.

**Theorem 7.1** *Assume that the data* $\mathcal{D}$*,* $\xi_*$*,* $\eta_*$*,* $\Lambda^{\mathbf{f}}$*,* $\Lambda^{\mathbf{g}}$*,* $\Lambda^{\mathbf{h}}$*,* $\Lambda^{\varphi}$*,* $C$ *satisfy Hypotheses (H1) to (H17). Let* $I$ *be the set of those indices* $i \in \{1, \ldots, m\}$ *such that* $K_i$ *is nonempty. Then there exist*

1. *a covector* $\bar{\pi} \in X^\dagger$*, a nonnegative real number* $\pi_0$*, and an* $\tilde{m}$*-tuple* $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_{\tilde{m}})$ *of real numbers,*
2. *a measurable selection* $[a, b] \ni t \mapsto (L_0(t), L(t)) \in X^\dagger \times \boldsymbol{L}(X)$ *of the set-valued map* $\Lambda^{\mathbf{f}}$*,*
3. *a family* $\{\nu_i\}_{i \in I}$ *of nonnegative additive measures* $\nu_i \in bvadd([a, b], \mathbb{R})$ *such that* $\mathrm{support}(\nu_i) \subseteq |\Lambda_i|$ *for every* $i \in I$*,*
4. *a family* $\{\gamma_i\}_{i \in I}$ *of pairs* $\gamma_i = (\gamma_i^-, \gamma_i^+)$ *such that* $\gamma_i^- : |\Lambda^{g_i}| \mapsto X^\dagger$ *and* $\gamma_i^+ : |\Lambda^{g_i}| \mapsto X^\dagger$ *are measurable selections of* $\Lambda^{g_i}$*, and* $\gamma_i^-(t) = \gamma_i^+(t)$ *for all* $t$ *in the complement of a finite or countable set,*
5. *a member* $L^{\mathbf{h}} = (L^{h_1}, \ldots, L^{h_{\tilde{m}}}) \in (X^\dagger)^{\tilde{m}}$ *of* $\Lambda^{\mathbf{h}}$ *and a member* $L^{\varphi}$ *of* $\Lambda^{\varphi}$*,*

*having the property that, if we let* $\pi : [a, b] \mapsto X^\dagger$ *be the unique solution of the adjoint Cauchy problem*

$$\begin{cases} d\pi(t) = (-\pi(t) \cdot L(t) + \pi_0 L_0(t))dt + \sum_{i \in I} d\mu_i(t) \\ \pi(b) = \bar{\pi} - \sum_{j=1}^{\tilde{m}} \lambda_j L_j^{\mathbf{h}} - \pi_0 L^{\varphi} \end{cases}$$

*(where* $\mu_i \in bvadd(\Lambda^{g_i})$ *is the finitely additive* $X^\dagger$*-valued measure such that* $d\mu_i = \gamma_i \cdot d\nu_i$*, defined in Page 10), then the following conditions are true:*

I. *the* ***Hamiltonian maximization condition****: the inequality*

$$H_{\pi_0}(\xi_*(\bar{t}), \eta_*(\bar{t}), \pi(\bar{t})) \geq H_{\pi_0}(\xi_*(\bar{t}), u, \pi(\bar{t}))$$

*holds whenever* $u \in U$*,* $\bar{t} \in [a, b]$ *are such that* $(\xi_*(\bar{t}), \bar{t})$ *is a point of approximate continuity of both augmented vector fields* $(x, t) \mapsto \mathbf{f}(x, u, t)$ *and* $(x, t) \mapsto \mathbf{f}(x, \eta_*(t), t)$*,*

*II. the* ***transversality condition:*** $-\bar{\pi} \in C^{\dagger}$,
*III. the* ***nontriviality condition:*** $\|\bar{\pi}\| + \pi_0 + \sum_{j=1}^{\tilde{m}} |\lambda_j| + \sum_{i\in I} \|\nu_i\| > 0.$

**Remark 7.2** The adjoint equation satisfied by $\pi$ can be written in integral form, incorporating the terminal condition at $b$. The result is the formula

$$\pi(t) = \bar{\pi} - \sum_{j=1}^{\tilde{m}} \lambda_j L_j^{\mathbf{h}} - \pi_0 L^{\varphi} + \int_t^b \Big(\pi(s)\cdot L(s) - \pi_0 L_0(s)\Big) ds - \sum_{i\in I} \int_{[t,b]} \gamma^i(s)\, d\nu_i(s),$$

from which it follows, in particular, that $\pi(b) = \bar{\pi} - \sum_{j=1}^{\tilde{m}} \lambda_j L_j^{\mathbf{h}} - \pi_0 L^{\varphi}$ . □

**Remark 7.3** The adjoint covector $\pi$ can also be expressed using (2.3.3). This yields $\pi(t) = \pi(b) - \int_t^b M_L(s,t)^{\dagger}\Big(\pi_0 L_0(s)\, ds + \sum_{i\in I} d(\gamma^i \cdot \nu_i)(s)\Big)$, where $\pi(b) = \bar{\pi} - \sum_{j=1}^{\tilde{m}} \lambda_j L_j^{\mathbf{h}} - \pi_0 L^{\varphi}$, and $M_L$ is the fundamental solution of the equation $\dot{M} = L \cdot M$. □

## References

1. Berkovitz, L. D., *Optimal Control Theory.* Springer-Verlag, New York, 1974.
2. Browder, F. E., "On the fixed point index for continuous mappings of locally connected spaces." *Summa Brazil. Math.* **4**, 1960, pp. 253–293.
3. Cellina, A., "A theorem on the approximation of compact multivalued mappings." *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* (8) **47** (1969), pp. 429–433.
4. Cellina, A., "A further result on the approximation of set-valued mappings." *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* (8) **48** (1970), pp. 412–416.
5. Cellina, A., "Fixed points of noncontinuous mappings." *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* (8) **49** (1970), pp. 30–33.
6. Cellina, A., "The role of approximation in the theory of multivalued mappings." In *1971 Differential Games and Related Topics (Proc. Internat. Summer School, Varenna, 1970)*, North-Holland, Amsterdam, 1970, pp. 209–220.
7. Clarke, F. H., "The Maximum Principle under minimal hypotheses." *SIAM J. Control Optim.***14**, 1976, pp. 1078–1091.
8. Clarke, F. H., *Optimization and Nonsmooth Analysis.* Wiley Interscience, New York, 1983.
9. Conway, J. B., *A Course in Functional Analysis.* Grad. Texts in Math. **96**, Springer-Verlag, New York, 1990.
10. Fan, K., "Fixed point and minimax theorems in locally convex topological linear spaces." *Proc. Nat. Acad. Sci. U.S.* **38**, 1952, pp. 121–126.
11. Glicksberg, I. L., "A further generalization of the Kakutani fixed point theorem, with application to Nash equilibrium points." *Proc. Amer. Math. Soc.* **3**, 1952, pp. 170–174.
12. Halkin, H., "Necessary conditions for optimal control problems with differentiable or nondifferentiable data." In *Mathematical Control Theory,* Lect. Notes in Math. **680**, Springer-Verlag, Berlin, 1978, pp. 77–118.

13. Kakutani, S., “A generalization of Brouwer’s fixed point theorem.” *Duke Math. J.* **7**, 1941, pp. 457–459.
14. Leray, J., and J. Schauder, “Topologie et équations fonctionnelles.” *Ann. Sci. École Norm. Sup.* **51**, 1934, pp. 45–78.
15. Pontryagin, L. S., V. G. Boltyanskii, R. V. Gamkrelidze and E. F. Mischenko, *The Mathematical Theory of Optimal Processes.* Wiley, New York, 1962.
16. Sussmann, H. J., “A strong version of the Łojasiewicz maximum principle.” In *Optimal Control of Differential Equations,* N. H. Pavel Ed., Marcel Dekker Inc., New York, 1994, pp. 293–309.
17. Sussmann, H. J., *An introduction to the coordinate-free maximum principle.* In *Geometry of Feedback and Optimal Control*, B. Jakubczyk and W. Respondek Eds., M. Dekker, Inc., New York, 1997, pp. 463-557.
18. Sussmann, H. J., “Some recent results on the maximum principle of optimal control theory.” In *Systems and Control in the Twenty-First Century,* C. I. Byrnes, B. N. Datta, D. S. Gilliam and C. F. Martin Eds., Birkhäuser, Boston, 1997, pp. 351–372.
19. Sussmann, H. J., “Multidifferential calculus: chain rule, open mapping and transversal intersection theorems.” In *Optimal Control: Theory, Algorithms, and Applications*, W. W. Hager and P. M. Pardalos Eds., Kluwer, 1998, pp. 436–487.
20. Sussmann, H. J., “A maximum principle for hybrid optimal control problems.” In *Proc. 38th IEEE Conf. Decision and Control, Phoenix, AZ, Dec. 1999.* IEEE publications, New York, 1999, pp. 425–430.
21. Sussmann, H.J., “Résultats récents sur les courbes optimales.” In *15$^e$ Journée Annuelle de la Société Mathémathique de France (SMF)*, Publications de la SMF, Paris, 2000, pp. 1-52.
22. Sussmann, H.J., “New theories of set-valued differentials and new versions of the maximum principle of optimal control theory.” In *Nonlinear Control in the year 2000*, A. Isidori, F. Lamnabhi-Lagarrigue and W. Respondek Eds., Springer-Verlag, London, 2000, pp. 487-526.
23. Sussmann, H.J., “Set-valued differentials and the hybrid maximum principle.” In *Proc. 39th IEEE Conf. Decision and Control, Sydney, Australia, Dec. 12-15, 2000*, IEEE publications, New York, 2000.
24. Sussmann, H. J., “Needle variations and almost lower semicontinuous differential inclusions.” *Set-valued analysis*, Vol. 10, Issue 2-3, June - September 2002, pp. 233-285.
25. Sussmann, H. J., “Combining high-order necessary conditions for optimality with nonsmoothness.” In *Proceedings of the 43rd IEEE 2004 Conference on Decision and Control (Paradise Island, the Bahamas, December 14-17, 2004)*, IEEE Publications, New York, 2004.
26. Warga, J., “Fat homeomorphisms and unbounded derivate containers.” *J. Math. Anal. Appl.* **81**, 1981, pp. 545–560.
27. Warga, J., “Controllability, extremality and abnormality in nonsmooth optimal control.” *J. Optim. Theory Applic.* **41**, 1983, pp. 239–260.
28. Warga, J., “Optimization and controllability without differentiability assumptions.” *SIAM J. Control and Optimization* **21**, 1983, pp. 837–855.
29. Warga, J., “Homeomorphisms and local $C^1$ approximations.” *Nonlinear Anal. TMA* **12**, 1988, pp. 593–597.