MATHEMATICS 503 FALL 2014

Instructor: H. J. Sussmann

Exercises for the ninth week of classes, i.e., for the lectures of October 29 and 30.

You should practice by trying to do as many problems as you can from Chapter III and Section 2 of Chapter V of the book.

PROBLEM 1. Prove that if U is an open connected subset of $\mathbb{C}$, then for every pair (p,q) of points of U there exists an arc $\gamma : [0,1] \mapsto U$ which is of class C^∞ and such that $\gamma(0) = p$ and $\gamma(1) = q$. (NOTE: One can even choose γ to be a polynomial.)

PROBLEM 2. Let U be an open subset of $\mathbb{C}$, and let a, b be real numbers such that $a < b$. Prove that, for holomorphic functions $f : U \mapsto U$, and continuous paths $\gamma : [a,b] \mapsto U$, the integral $\int_\gamma f(z)dz$ depends continuously on γ and f. Precisely, this means the following:

(I) If $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of holomorphic functions on U that converges uniformly on every compact subset of U to a holomorphic function $f : U \mapsto \mathbb{C}$, and $\{\gamma_n\}_{n\in\mathbb{N}}$ is a sequence of continuous U-valued arcs defined on $[a,b]$ that converges uniformly to an arc $\gamma : [a,b] \mapsto U$, then

$$\lim_{n\to\infty} \int_{\gamma_n} f_n(z)dz = \int_\gamma f(z)dz\,.$$

(II) If P is a metric space, $U \times P \ni (z,p) \mapsto f(z,p) \in \mathbb{C}$ is a continuous function which is a holomorphic function of z for each fixed $p \in P$, and $[a,b] \times P \ni (t,p) \mapsto \gamma(t,p) \in U$ is a continuous map, then the function $P \ni p \mapsto \int_{\gamma(\cdot,p)} f(z,p)dz$ is continuous. (Here $\gamma(\cdot,p)$ is the arc $[a,b] \ni t \mapsto \gamma(t,p)$.)

You should first prove that (I) and (II) are equivalent, and then prove one of them, whichever one you prefer.

REMARK. It will be proved in a few days that if $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of holomorphic functions on U that converges uniformly on compact sets to a function f, then f is holomorphic. So the assumption that f is holomorphic in (I) isn't really needed.

PROBLEMS 3 to 12. Book, pages 164-165, Problems 4, 5, 6, 7, 8, 9, 10, 11, 12, 13.