MATHEMATICS 503 FALL 2014

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Exercises for the seventh and eighth weeks of classes, i.e., for the lectures of October 14, 16, 21 and 23.

You should practice by trying to do as many problems as you can from Chapter III of the book.

PROBLEM 1. Book, Page 118, Problem 1. (NOTE: what the book calls "path" is the same as what I called "arc" in class. And a "closed path" is what I called a "loop.")

PROBLEM 2. Book, Page 119, Problems 2 and 3.

PROBLEM 3. If U is an open subset of $\mathbb{C}$, and a, b are real numbers such that $a < b$, we use $C^0([a,b],U)$ to denote the set of all continuous functions $\gamma : [a,b] \mapsto U$. Also, if p, q are any two points of U, we use $C^0_{p,q}([a,b],U)$ to denote the set of all $\gamma \in C^0([a,b],U)$ such that $\gamma(a) = p$ and $\gamma(b) = q$. Finally, we use $C^0_{loop}([a,b],U)$ to denote the set of all $\gamma \in C^0([a,b],U)$ such that $\gamma(a) = \gamma(b)$. Prove that

(i) homotopy in U is an equivalence relation on $C^0([a,b],U)$,
(ii) fixed-endpoint homotopy in U is an equivalence relation on $C^0_{p,q}([a,b],U)$,
(iii) loop-homotopy in U is an equivalence relation on $C^0_{loop}([a,b],U)$.

PROBLEM 4. Prove that if U is an open connected subset of $\mathbb{C}$, $a, b \in \mathbb{R}$, and $a < b$, then any two arcs $\gamma, \delta \in C^0([a,b],U)$ are homotopic. (NOTE: This proves that "homotopy", as opposed to fixed-endpoint homotopy and loop-homotopy, is a very stupid, useless condition.)

PROBLEM 5. An open subset U of $\mathbb{C}$ is *simply connected* if any two loops $\gamma_1, \gamma_2 \in C^0_{loop}([0,1],U)$ are loop-homotopic in U. Prove that if U is simply connected then U is holomorphically simply connected, in the sense of Problem 7 of the previous homework assignment. (It will be proved later, maybe in this course, or in Math 504, that the converse is true. But this depends on the Riemann Mapping Theorem, which is a much harder result than all the stuff we have been doing so far.)

PROBLEM 6. If $\gamma : [0,1] \mapsto U$ is a loop, then we can define a map $\hat{\gamma} : \mathbb{S}^1 \mapsto U$ by letting $\hat{\gamma}(e^{2\pi i t}) = \gamma(t)$ for $0 \le t \le 1$. (Here $\bar{\mathbb{D}}$ is the closed disc $\{z \in \mathbb{C} : |z| \le 1\}$, and $\mathbb{S}^1$ is the unit circle $\{z \in \mathbb{C} : |z| = 1\}$.) Prove that if $\gamma \in C^0_{loop}([0,1],U)$ then

(i) the corresponding map $\hat{\gamma} : \mathbb{S}^1 \mapsto U$ is continuous,
(ii) γ is loop-homotopic in U to a point if and only if the map $\hat{\gamma} : \mathbb{S}^1 \mapsto U$ can be extended to a continuous map $\Gamma : \bar{\mathbb{D}} \mapsto U$.

PROBLEM 7. Let f be the map from $\mathbb{S}^1$ to $\mathbb{C}\backslash\{0\}$ given by $f(z) = z$ for $z \in \mathbb{S}^1$.

(i) Prove, using the Cauchy Integral Theorem and the result of Problem 6, that f cannot be extended to a continuous map $F : \bar{\mathbb{D}} \mapsto \mathbb{C}\backslash\{0\}$. (NOTE: An extension of f to a map $F : \bar{\mathbb{D}} \mapsto \mathbb{C}$ obviously exists, since it suffices to define F by $F(z) = z$ for $z \in \bar{\mathbb{D}}$. So the key point here is that we are asking for a map F into $\mathbb{C}\backslash\{0\}$, i.e., a map F that never takes the value 0.)

(ii) Deduce from (i) the *no retraction theorem* in dimension 2, that is, the statement that there does not exist a continuous map $G : \bar{\mathbb{D}} \mapsto \mathbb{S}^1$ such that $G(z) = z$ for $z \in \mathbb{S}^1$.

PROBLEM 8. The 2-dimensional *Brouwer fixed point theorem* says that every continuous map from $\bar{\mathbb{D}}$ to $\bar{\mathbb{D}}$ has a fixed point. (That is, if $f : \bar{\mathbb{D}} \mapsto \bar{\mathbb{D}}$ is continuous, then $(\exists z \in \bar{\mathbb{D}})f(z) = z$.) Deduce the 2-dimensional Brouwer fixed point theorem from the no retraction theorem in dimension 2. (Hint: suppose $f : \bar{\mathbb{D}} \mapsto \bar{\mathbb{D}}$ has no fixed points. Define $F(z)$, for $z \in \bar{\mathbb{D}}$, to be the point in $\mathbb{S}^1$ obtained by moving along the straight line from $f(z)$ to z until you hit $\mathbb{S}^1$. Write an explicit formula for F, use the formula to show that F is continuous, and observe that F is a map whose non-existence was proved in Problem 7. Observe also that the key fact that makes it possible to construct F is the fact that f has no fixed points, since the definition of $F(z)$ doesn't make sense if $f(z) = z$.)

REMARK: The previous series of problems provides a proof of the 2-dimensional Brouwer fixed-point theorem based on the fact that $\int_{\partial\bar{\mathbb{D}}} \frac{dz}{z} = 2\pi i$, together with the fact that if you deform a loop continuously without going through 0, the integral of $\frac{dz}{z}$ along that loop doesn't change. This suggests a possible strategy for proving the n-dimensional Brouwer fixed-point theorem: find an object ω that can be integrated along closed hypersurfaces (that is, compact $n-1$-dimensional submanifolds of $\mathbb{R}^n$), and is such that $\int_\sigma \omega$ doesn't change if you deform σ continuosuly, and $\int_{\partial\bar{\mathbb{D}}^n} \omega \neq 0$. (Here $\bar{\mathbb{D}}^n$ is the closed n-dimensional unit ball $\{x \in \mathbb{R}^n : \|x\| \leq 1\}$, and $\partial\bar{\mathbb{D}}^n$ is the boundary of $\bar{\mathbb{D}}^n$, i.e., the unit sphere $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$. Naturally, $\|x\|$ is the Euclidean norm of x, given by $\|x\| = \sqrt{\sum_{j=1}^n x_j^2}$.) It turns out that the strategy works, as long as ω is a properly chosen differential form of order $n - 1$.