MATHEMATICS 503 FALL 2014

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Exercises for the fifth week of classes, i.e., for the lectures of September 30 and October 2.

You should practice by trying to do as many problems as you can from Chapter 3 of the book.

PROBLEM 1. We use $\mathbb{H}_+$ to denote the upper half-plane. (That is, $\mathbb{H}_+ = \{x + iy \in \mathbb{C} : y > 0\}$.) **Prove** that the biholomorphic maps from $\mathbb{H}_+$ onto $\mathbb{H}_+$ are exactly the maps of the form $f(z) = \frac{az+b}{cz+d}$, with a, b, c, d real numbers such that $ad - bc = 1$. (HINT: Fix a biholomorphic map Φ from $\mathbb{H}_+$ onto the unit disc $\mathbb{D}$. Then the desired maps are exactly the maps $\Phi^{-1} \circ \Psi \circ \Phi$, for Ψ a biholomorphic map from $\mathbb{D}$ onto $\mathbb{D}$.)

PROBLEM 2. If U, V are open subsets of $\mathbb{C}$, a map $f : U \mapsto V$ is *proper* if the preimage of every compact set under f is compact (that is, if for every compact subset K of V the set $f^{-1}(K) = \{z \in U : f(z) \in K\}$ is compact). **Prove** that if $f : U \mapsto V$ is continuous and proper, and V is connected, then f maps U onto V.

PROBLEM 3. Let U, V be open subsets of $\mathbb{C}$, and let $f : U \mapsto V$ be an analytic proper map. **Prove** that

1. The preimage $f^{-1}(w)$ of every point $w \in V$ is a finite set.
2. If U is connected, then the "size" of the preimage $f^{-1}(w)$ is the same for all $w \in V$. Here the "size" of a set is the usual size *counting multiplicities*. Precisely (and using the obvious fact that the preimage of w is the set of zeros of the function $z \mapsto f(z) - w$), the *multiplicity* of a point $z_0 \in U$ as a zero of the function $z \mapsto f(z) - f(z_0)$ is the positive integer $\nu(z_0)$ such that, for some function g analytic near z_0 and such that $g(z_0) \neq 0$, the equality $f(z) - f(z_0) = (z - z_0)^{\nu(z_0)} g(z)$ holds for z near z_0. The "size" of the set $f^{-1}(w)$ is the sum of the multiplicities $\nu(z_0)$ of all the points z_0 such that $f(z_0) = w$.

PROBLEM 4. Give an example of U, V, open subsets of $\mathbb{R}$, with V connected, and a proper real analytic function $f : U \mapsto V$, such that the "size" (defined counting multiplicities, as in Problem 3) of the preimage $f^{-1}(w)$ is not constant as w varies in V.