MATHEMATICS 503 FALL 2014

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Exercises for the fourth week of classes, i.e., for the lectures of September 23 and 25.

You should practice by trying to do all the problems in Chapter 2 of the book and starting to work on the problems of Chapter 3.

PROBLEM 1. Construct a function $f : \mathbb{R} \mapsto \mathbb{R}$ such that

(1) f is of class C^∞,

and

(2) the set of zeros of f (that is, the set $Z(f) = \{x \in \mathbb{R} : f(x) = 0\}$) consists of a point $x_\infty \in \mathbb{R}$ together with the points of a sequence $\{x_n\}_{n=1}^\infty$ such that $\lim_{n\to\infty} x_n = x_\infty$ and $x_n \neq x_\infty$ for every $n \in \mathbb{N}$. (Here $\mathbb{N}$ is the set of natural numbers, i.e., the set of positive integers.)

Recall that the concept of "function of class C^∞" (of one or several real variables, with domain an open subset U of $\mathbb{R}^n$, and with values in $\mathbb{R}^m$) is defined as follows: first one defines inductively the classes $C^k(U, \mathbb{R}^m)$, for $k \in \mathbb{N} \cup \{0\}$, by letting $C^0(U, \mathbb{R}^m)$ be the class of all continuous functions from U to $\mathbb{R}^m$ and then, if $k \in \mathbb{N} \cup \{0\}$ and we have already defined the class $C^k(U, \mathbb{R}^m)$, we let $C^{k+1}(U, \mathbb{R}^m)$ be the class of all functions $f : U \mapsto \mathbb{R}^m$ such that all the partial derivatives $f_j(x) \stackrel{\text{def}}{=} \frac{\partial f}{\partial x_j}(x)$ exist for every $x \in U$ and the f_j belong to $C^k(U, \mathbb{R}^m)$. Then we define the class $C^\infty(U, \mathbb{R}^m)$ by declaring a function $f : U \mapsto \mathbb{R}^m$ to belong to $C^\infty(U, \mathbb{R}^m)$ if it is in $C^k(U, \mathbb{R}^m)$ for every $k \in \mathbb{N}$. (REMARK: *What is this real-variables problem doing in a Math 503 homework?* ANSWER: this is part of a series of examples, some of which have already been discussed in class, showing how "everywhere differentiability", and even "being of class C^∞", for real functions, is completely different from analyticity, even though, for complex functions, "having a complex derivative everywhere" is equivalent to real analyticity.)

PROBLEM 2. Show that, if I_1, I_2 are two disjoint compact intervals of the real line, and $f_1 : I_1 \mapsto \mathbb{R}$, $f_2 : I_2 \mapsto \mathbb{R}$ are functions of class C^∞ on I_1, I_2. then there exists a function $f : \mathbb{R} \mapsto \mathbb{R}$ which is of class C^∞ and is such that $f \equiv f_1$ on I_1 and $f \equiv f_2$ on I_2. (The precise definition of "function of class C^∞ on a closed subset C of $\mathbb{R}$" is as follows: f is a function of class C^∞ on C if there exist an open subset U of $\mathbb{R}$ and a function $g : U \mapsto \mathbb{R}$ of class C^∞ such that $C \subseteq U$ and $f(x) = g(x)$ for all $x \in C$.)

PROBLEM 3. As you know from your Calculus courses, the Mean Value Theorem says that

(MVT) If $a, b \in \mathbb{R}$, $a < b$, and $f : [a, b] \mapsto \mathbb{R}$ is a function such that the limit $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$ exists for every $x \in]a, b[$. then there exists an $x \in]a, b[$ such that

$$f'(x) = \frac{f(b) - f(a)}{b - a}\,.$$

Consider the following analogue of (MVT) for complex-valued functions

(ℂMVT) If $a, b \in \mathbb{R}$, $a < b$, and $f : [a, b] \mapsto \mathbb{C}$ is a function such that the limit $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$ exists for every $x \in]a, b[$, then there exists an $x \in]a, b[$ such that

$$f'(x) = \frac{f(b) - f(a)}{b - a} .$$

Prove that (ℂMVT)] is false.

PROBLEM 4. Prove that, if U is an open connected subset of the complex plane $\mathbb{C}$, and $f : U \mapsto \mathbb{C}$ is a holomorphic function such that $f'(z) = 0$ for all $z \in U$, then f is a constant. (You are allowed–and advised—to use the Mean Value Theorem, but *be careful not to use (ℂMVT)!!*) *Make sure your proof makes it very clear where the connectedness of U is used.*

PROBLEM 5. An open subset U of $\mathbb{C}$ is *path-connected* if

(PC) whenever z_1, z_2 are points of U, there exists a continuous map $\gamma : [0, 1] \mapsto U$ such that $\gamma(0) = z_1$ and $\gamma(1) = z_2$.

Prove that if $U \subseteq \mathbb{C}$ and U is open, then U is connected if and only if it is path connected.

PROBLEM 6. **Prove** that, if $\{U_\alpha\}_{\alpha \in A}$ is a family of open subsets of $\mathbb{C}$ such that each U_α is connected and $\bigcap_{\alpha \in A} U_\alpha \neq \emptyset$, then the union $\bigcup_{\alpha \in A} U_\alpha$ is connected.

PROBLEM 7. The *exponential function* $\mathbb{C} \ni z \mapsto e^z \in \mathbb{C}$ is defined by the following power series formula:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} . \tag{1}$$

(Recall that $0! = 1$.)
The series of the right-hand side of (1) has an infinite radius of convergence, and this implies that the series converges everywhere, the "exponential function"—i.e., the function $z \mapsto e^z$—is continuous and, even more strongly, it is holomorphic and its derivative is equal to the exponential function itself. (Most of these facts were proved in class, or are proved in the book, or follow easily from things we have proved in class or are proved in the book. I am not asking you to prove them here, but you should know how to prove them, because who knows what questions may appear an exam?) **Prove** the identity

$$e^{z_1} e^{z_2} = e^{z_1 + z_2} \qquad \text{for all} \quad z_1, z_2 \in \mathbb{C} .$$

using power series multiplication, as follows: let $E_N(z)$ be the partial sum of the series of (1), for n from 0 to N. Prove that $E_N(z_1)E_N(z_2) - E_N(z_1 + z_2)$ goes to zero as $N \to \infty$, by (a) multiplying out the sums $E_N(z_1)$, $E_N(z_2)$, so as to get a double sum, (b) writing out $E_N(z_1 + z_2)$ and expanding all the powers of $z_1 + z_2$ using the Binomial Theorem, so as to get another double sum, (c) observing that the two double sums are of exactly the same terms, but over different sets of indices, and (d) subtracting the two double sums and getting a bound for the difference.

PROBLEM 7. **Prove** that, if U is an open connected subset of $\mathbb{C}$, $f : U \mapsto \mathbb{C}$ is a continuous function, and g_1, g_2 are two branches of $\ln f$ on U, then there exists an integer k such that $g_1(z) = g_2(z) + 2k\pi i$ for all $z \in U$. (Recall that a *branch of* $\ln f$ *on* U is a continuous function $g : U \mapsto \mathbb{C}$ such that $e^{g(z)} = f(z)$ for all $z \in U$.) *Make sure your proof makes it very clear where the connectedness of U is used.*