MATHEMATICS 503 FALL 2014

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Exercises for the second week of classes, i.e., for the lectures of September 9 and 11.

You should practice by trying to do all the problems in Chapter 2 of the book.

PROBLEM 1. Prove that, if $L : \mathbb{R}^2 \mapsto \mathbb{R}^2$ is an $\mathbb{R}$-linear map, and we identify $\mathbb{R}^2$ with $\mathbb{C}$ in the usual way, then L has a unique decomposition $L = L_1 + L_2$, where L_1 is complex-linear and L_2 is conjugate-complex-linear. (NOTE: A map $L : \mathbb{R}^2 \mapsto \mathbb{R}^2$ is *complex linear* if, using the identification of $\mathbb{R}^2$ with $\mathbb{C}$, L satisfies $L(\lambda \cdot h) = \lambda L(h)$ whenever $h \in \mathbb{C}$, $\lambda \in \mathbb{C}$. We call L *complex conjugate-linear* if $L(\lambda \cdot h) = \bar{\lambda} L(h)$ whenever $h \in \mathbb{C}$, $\lambda \in \mathbb{C}$. It's easy to show that an $\mathbb{R}$-linear map L is complex-linear if and only if L is multiplication by a complex number—that is, there exists $\alpha \in \mathbb{C}$ such that $L(h) = \alpha \cdot h$ for all $h \in \mathbb{C}$— and L is complex conjugate-linear if and only if L is conjugation followed by multiplication by a complex number—that is, there exists $\alpha \in \mathbb{C}$ such that $L(h) = \alpha \cdot \bar{h}$ for all $h \in \mathbb{C}$.)

PROBLEM 2. Problem 10 of last week's assignment.

PROBLEM 3. Book, page 46, Problems 3 and 4.

PROBLEM 4. Book, Page 58, Problem 3.

PROBLEM 5. Give a rigorous proof that the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

converges to $\ln 2$. (NOTE: This is the series $\sum_{n=0}^{\infty} \frac{(-1)^{n+1} z^n}{n}$ evaluated for $z = 1$. The sum is well known to converge to $\ln z$ for $|z| < 1$, but the point $z = 1$ is on the boundary of the disc of convergence of the series, so it is not obvious that the series converges to $\ln 2$ for $z = 1$. HINT: Write

$$\ln(1 + x) = \int_0^x \frac{du}{1 + u},$$

expand $\frac{1}{1+u}$ as a geometric series, and justify the integration term by term, for $x = 1$. Notice that the series of the integrands does *not* converge uniformly for $0 \le u \le 1$, so you need something else to justify the integration term by term.)

PROBLEM 6. Give a rigorous proof that the series

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots$$

converges to $\frac{\pi}{4}$. (NOTE: This is the series $\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{2n+1}$ evaluated for $z = 1$. The sum is well known to converge to $\arctan z$ for $|z| < 1$, but the point $z = 1$ is on the boundary of the disc of convergence of the series, so it is not obvious that the series converges to $\arctan 1$, i.e., to $\frac{\pi}{4}$, for $z = 1$. HINT: Write

$$\arctan(x) = \int_0^x \frac{1}{1 + u^2} du,$$

expand $\frac{1}{1+u^2}$ as a geometric series, and justify the integration term by term, for $x = 1$. Notice that the series of the integrands does *not* converge uniformly for $0 \le u \le 1$, so you need something else to justify the integration term by term.)