MATHEMATICS 503 FALL 2014

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Exercises for the first week of classes, i.e., for the lectures of September 2 and 4.

You should practice by doing (or trying to do) the problems in Chapter 1 of the book. (Most of them are easy and short.) And, in addition, here is a list of harder problems.

PROBLEM 1. Prove that if $f : \mathbb{C} \mapsto \mathbb{R}$ is a continuous function such that $\lim_{|z|\to\infty} f(z) = +\infty$, then f has a minimum (that is, there exists $z_0 \in \mathbb{C}$ such that $f(z_0) \le f(z)$ for all $z \in \mathbb{C}$). (NOTE: "$\lim_{|z|\to\infty} f(z) = +\infty$" means "for every real number R there exists a real number S such that $f(z) > R$ whenever $|z| > S$.")

PROBLEM 2. Prove that if $P(z)$ is a nonconstant complex polynomial (that is, P is a function from $\mathbb{C}$ to $\mathbb{C}$ such that

$$P(z) = a_0 + a_1 z + \ldots + a_n z^n \qquad \text{for all } z \in \mathbb{C}\,, \tag{1}$$

where n is an integer such that $n > 0$, $a_0, a_1, \ldots, a_n$ are complex numbers, and $a_n \neq 0$), then $\lim_{|z|\to\infty} |P(z)| = +\infty$.

PROBLEM 3. Prove that if $P(z)$ is a nonconstant complex polynomial as in Problem 2, then the real-valued function $f : \mathbb{C} \mapsto \mathbb{R}$ given by $f(z) = |P(z)|$ for $z \in \mathbb{C}$ cannot have a minimum at a point z_0 such that $P(z_0) \neq 0$. (HINT: Assume f has a minimum at z_0 and $u_0 = P(z_0) \neq 0$. Let $Q(h) = P(z_0 + h) - P(z_0)$, so Q is a nonconstant complex polynomial in h such that $Q(0) = 0$. Show that there exists $h_0 \in \mathbb{C}$ such that $h_0 \neq 0$ having the property that the curve $[0, \infty) \ni r \mapsto Q(rh_0)$ points in the direction of $-u_0$ at $r = 0$, in the sense that $\lim_{r\downarrow 0} \frac{Q(rh_0)}{r^\nu} = -u_0$ for some positive integer ν. Conclude from this that $|P(z_0 + rh_0)| < |P(z_0)|$ for small enough positive r.)

PROBLEM 4. Using the results of Problems 1,2 and 3,

1. Prove the *Fundamental Theorem of Algebra*: If P is a nonconstant complex polynomial, then there exists a $z_0 \in \mathbb{C}$ such that $P(z_0) = 0$.
2. Explain where the proof proposed here does not work to prove a "fundamental theorem of algebra for the real field": for a nonconstant real polynomial $P(x) = a_0 + a_1 x + \ldots + a_n x^n$, with $a_0, a_1, \ldots, a_n$ real, $n > 0$, $a_n \neq 0$, there exists a real x_0 such that $P(x_0) = 0$.

PROBLEM 5. Let

$$g(z, w) = \frac{z - w}{1 - z\bar{w}}\,.$$

1. Prove that
   a. $|g(z, w)| \le 1$ if $\bar{z} \cdot w \neq 1$, $|z| \le 1$ and $|w| \le 1$,
   b. $|g(z, w)| < 1$ if $\bar{z} \cdot w \neq 1$, $|z| < 1$ and $|w| < 1$,
   b. $|g(z, w)| = 1$ if $\bar{z} \cdot w \neq 1$, $|z| \le 1$, $|w| \le 1$, and either $|z| = 1$ or $|w| = 1$.

   HINT: Prove first the identity

$$1 - |g(z, w)|^2 = \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - \bar{z}w|^2}\,.$$

2. Conclude from the results of Part 1 that, if $w \in \mathbb{D}$ then the transformation G_w given by $G_w(z) = g(z, w)$ maps $\mathbb{D}$ into $\mathbb{D}$ and $\partial\mathbb{D}$ to $\partial\mathbb{D}$. (Here $\mathbb{D}$ is the open unit disc $\{z \in \mathbb{C} : |z| < 1\}$, and $\partial\mathbb{D}$ is the unit circle $\{z \in \mathbb{C} : |z| = 1\}$.)
3. Prove that, as a map from $\mathbb{D}$ to $\mathbb{D}$, G_w is one-to-one and onto (provided that $w \in \mathbb{D}$), and show that the inverse map $G_w^{-1} : \mathbb{D} \mapsto \mathbb{D}$ is of the form G_u for some $u \in \mathbb{D}$ that you should determine explicitly.

PROBLEM 6. An *ordered field* is a field F endowed with a distinguished subset P (called "the set of positive members of F") such that

1. If $a, b \in P$ then $a + b \in P$ and $a \cdot b \in P$,
2. If $a \in F$ then one and only one of the following holds: $a \in P$, $-a \in P$, $a = 0$.

Verify that $\mathbb{R}$, with $P = \{x \in \mathbb{R} : x > 0\}$, is an ordered field, and prove that there is no choice of a subset P of $\mathbb{C}$ that makes $\mathbb{C}$ an ordered field. (NOTE: This means that you cannot, and should not, talk about a complex number being "larger" than another complex number, or about a complex-valued function having a maximum or a minimum.)

PROBLEM 7. An integer n is a *2-square* if it is the sum of two squares of integers, i.e., if there exist integers u, v such that $n = u^2 + v^2$. Prove that the product of two 2-squares is a 2-square. (Example: $5 = 2^2 + 1^2$, $13 = 3^2 + 2^2$, and $5 \cdot 13 = 65 = 8^2 + 1^2$.) (NOTE: If you are wondering what this problem about integer arithmetic is doing in Math 503, believe me, there is a reason.)

PROBLEM 8. The *sum* of two subsets S_1, S_2 of $\mathbb{C}$ (or of any vector space), is the set $S_1 + S_2$ given by

$$S_1 + S_2 = \{z_1 + z_2 : z_1 \in S_1,\ z_2 \in S_2\}\,.$$

Prove or disprove each of the following statements:

1. If K_1 and K_2 are compact then $K_1 + K_2$ is compact.
2. If C_1 and C_2 are closed then $C_1 + C_2$ is closed.
3. If U_1 and U_2 are open then $U_1 + U_2$ is open.
4. If U_1 is open and U_2 is an arbitray set then $U_1 + U_2$ is open.

PROBLEM 9. A *lattice point* (in the complex plane $\mathbb{C}$) is a point $x + iy$ of $\mathbb{C}$ such that x and y are integers. Let $\nu(R)$ be the number of lattice points belonging to the open disc $D(R) = \{z \in \mathbb{C} : |z| < R\}$. Find an asymptotic formula for $\nu(R)$. More precisely, find a simple polynomial $p(R)$ such that $\lim_{R \to +\infty} \frac{\nu(R)}{p(R)} = 1$.

PROBLEM 10. *This problem has a long introduction, and you should read it carefully. You will find near the end the list of things you are asked to prove.*

A function $f : U \mapsto \mathbb{R}^m$, defined on an open subset U of $\mathbb{R}^n$, is said to be *differentiable* at a point x_* of U if here exists a linear map $L : \mathbb{R}^n \mapsto \mathbb{R}^m$ such that

$$\lim_{h \to 0} \frac{\|f(x_* + h) - f(x_*) - Lh\|}{\|h\|} = 0\,.$$

(Here, if $v = (v_1, \ldots, v_d) \in \mathbb{R}^d$, the *norm* $\|v\|$ is the Euclidean norm[1] of v, that is, the number $\sqrt{\sum_{j=1}^d v_j^2}$.) It is well known that L is unique. The map L is the

[1]But, for those who are familiatr with these things, everything we are doing would work equally well with any other norm, because all norms on $\mathbb{R}^n$ are equivalent.

differential of f at x_*, and we use $Df(x_*)$ or $df(x_*)$ to denote it[2]. It is also well known that if f is differentiable at x_* then the partial derivatives $\frac{\partial f}{\partial x_j}(x_*)$ exist, and satisfy

$$\frac{\partial f}{\partial x_j}(x_*) = df(x_*).e_j \, ,$$

where e_j is the vector $(0, \ldots, 0, 1, 0, \ldots, 0)$ with the 1 in he j-th place (that is: $e_j = (\delta_j^1, \ldots, \delta_j^n)$, where δ_j^k is Kronecker's delta, defined by $\delta_j^j = 1$ and $\delta_j^k = 0$ if $j \neq k$). This implies that, if $h \in \mathbb{R}^n$,

$$\begin{aligned} df(x_*) \cdot h &= df(x_*)(\sum_{j=1}^{n} h_j e_j) \\ &= \sum_{j=1}^{n} (df(x_*) \cdot e_j) h_j \\ &= \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(x_*) h_j \, . \end{aligned}$$

In particular, the differential dx_k of the function x_k is given by

$$dx_k(x_*) \cdot h = h_k \, ,$$

because $\frac{\partial x_k}{\partial x_j} = \delta_j^k$. Hence

$$df(x_*) \cdot h = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(x_*) \cdot (dx_k(x_*) \cdot h) \, ,$$

which can be written, omitting the arguments x_*, h as

$$df = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} dx_k \, .$$

In the case when $n = m = 2$, we may identify $\mathbb{R}^2$ with $\mathbb{C}$ in the usual way, and then a function $f : U \mapsto \mathbb{R}^2$ as before can be regarded as a function from an open subset of $\mathbb{C}$ to $\mathbb{C}$. For such a function, the notion of differentiability defined above will be referred to as *real differentiablity*. And, if $f : U \mapsto \mathbb{C}$ is real-differentiable at a point $x_* \in U$, then, at x_*,

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \, .$$

On the other hand, $z = x + iy$ and $\bar{z} = x - iy$ are $\mathbb{C}$-valued functions on U, whose differentials are given by $dz = dx + idy$ and $d\bar{z} = dx - idy$.

Prove that if f is real-differentiable at x_* then, at x_*,

$$df = \frac{\partial f}{\partial z} \cdot dz + \frac{\partial f}{\partial \bar{z}} \cdot d\bar{z},$$

[2] This notation is consistent with, for example, the one used for differentials in differential geometry. For us, if U is open in $\mathbb{R}^n$, and $f : U \mapsto \mathbb{R}^m$, then $df(x_*)$ is a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$. In the special case when $m = 1$, $df(x_*)$ is a linear map from $\mathbb{R}^n$ to $\mathbb{R}$, i.e., a linear functional on $\mathbb{R}^n$. And $\mathbb{R}^n$ is naturally identified with the tangent space to U at x_*, so $df(x_*)$ is a member of the dual of that tangent space.

where $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \bar{z}}$ are defined by

$$\frac{\partial f}{\partial z} = \frac{1}{2}\Big(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\Big), \qquad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\Big(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\Big).$$

Then **prove** that the following conditions are equivalent:

a. f is complex-differentiable[3].
b. f is real-differentiable at x_* and the differential $df(x_*)$ is a complex-linear map[4].
c. f is real-differentiable at x_* and $\frac{\partial f}{\partial \bar{z}}(x_*) = 0$.
d. f is real-differentiable at x_* and, if we write $f = u + iv$ with u, v real-valued, then the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\,, \ \ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

hold at x_*.

Conclude that,

1. *If U is open in $\mathbb{C}$ and $f : U \mapsto \mathbb{C}$, then f is holomorphic if and only if f is real-differentiable at every point of U, and*

$$\frac{\partial f}{\partial \bar{z}} \equiv 0\,.$$

2. *If U is open in $\mathbb{C}$, $f : U \mapsto \mathbb{C}$, and f is holomorphic, then*

$$f' \equiv \frac{\partial f}{\partial z}\,.$$

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[3]We say that f is *complex-differentiable* at x_* if the limit $\lim_{h \to 0} \frac{f(x_*+h)-f(x_*)}{h}$ exists (where, of course, h goes to zero through complex values).

[4]Recall that $df(x_*)$ is an $\mathbb{R}$-linear map from $\mathbb{R}^2$ to $\mathbb{R}^2$, and $\mathbb{R}^2$is identified with $\mathbb{C}$, so $df(x_*)$ is a map from $\mathbb{C}$ to $\mathbb{C}$.