Steinhaus Sets for Finite Configurations

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S. Gao, A. W. Miller and W. A. R. Weiss,

Steinhaus sets and Jackson sets,

in: Advances in Logic, 127-145. Contemporary Mathematics 425, American Mathematical Society, RI, 2007.

Given $A \subseteq \mathbb{R}^2$, a Steinhaus set for A is a subset B of $\mathbb{R}^2$ such that any isometric copy A' of A in $\mathbb{R}^2$ meets B at exactly one point.

In symbols, we write $A \perp B$ if B is a Steinhaus set for A, that is, if $A' \cong_i A$ implies $|A' \cap B| = 1$.

Fact
If $A \perp B$ then $B \perp A$.

The Steinhaus Lattice Problem

Is there a Steinhaus set for the lattice $\mathbb{Z}^2$ in $\mathbb{R}^2$?

Jackson-Mauldin: Yes.

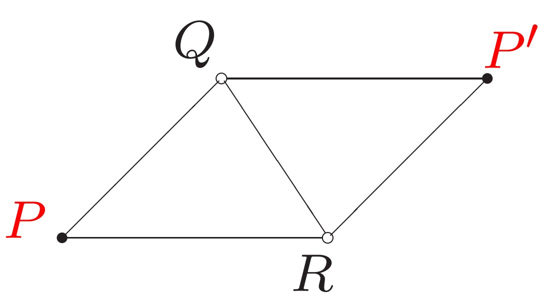
S. Jackson and R. D. Mauldin,
On a lattice problem of H. Steinhaus,
J. Amer. Math. Soc. 15 (2002), 817-856.

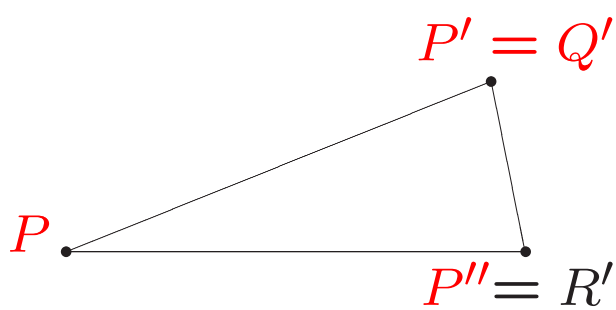
S. Jackson and R. D. Mauldin,
Survey of the Steinhaus tiling problem,
Bull. Symbolic Logic 9 (2003), no. 3, 335–361.

Problem

For which sets $A \subseteq \mathbb{R}^2$ do there exist Steinhaus sets?

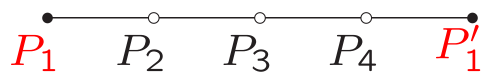
Jackson: No 3-point sets have Steinhaus sets.



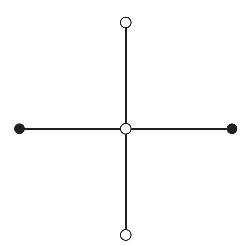


Examples

(1) The set $\{(0,0),(1,0),(2,0),(3,0)\}$ does not have a Steinhaus set.

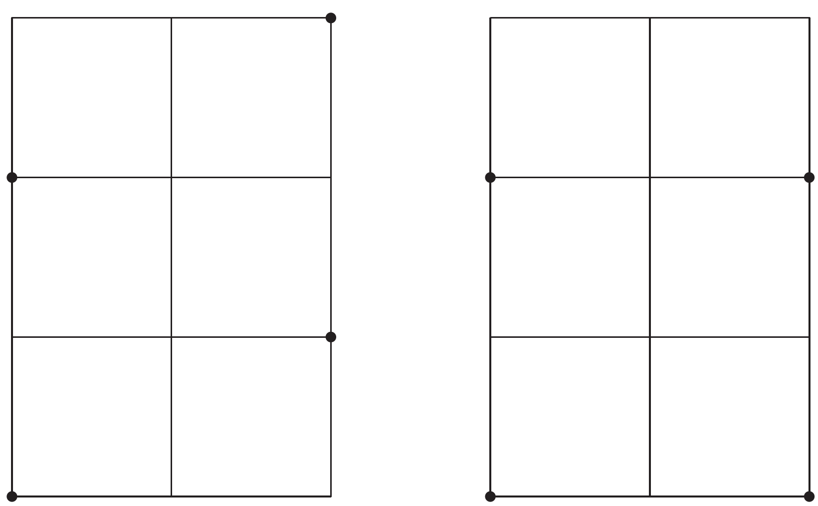


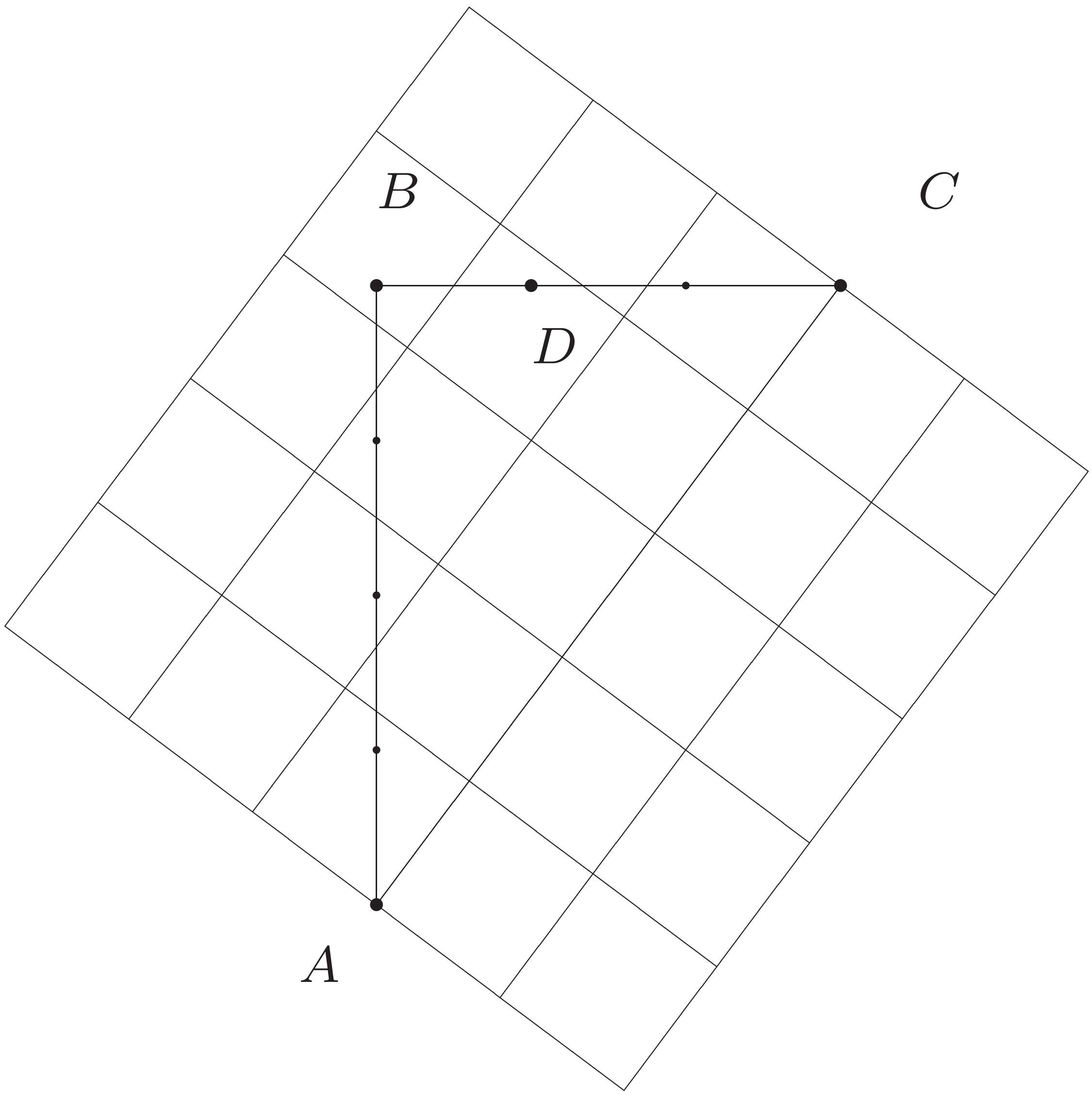
(2) The set $\{(0,0),(1,0),(0,1),(0,-1)\}$ does not have a Steinhaus set.



Proposition

The set of vertices of a square does not have a Steinhaus set.




B
C
D
A

Conjecture
No finite sets have Steinhaus sets.

A set $A \subseteq \mathbb{R}^2$ is Jackson if it does not have a Steinhaus set.

Conjecture (rephrased)
Every finite set is Jackson.

Proposition
The set of vertices of a $1 \times x$ rectangle, where x^2 is rational, is Jackson.

Proposition
For any trapezoid T and $\epsilon > 0$ there exists a trapezoid T' with corresponding vertices within ϵ of the vertices of T such that the set of vertices of T' is Jackson.

Question 3.10 of [GMW]
Is every four point collinear set ϵ-close to a four-point collinear set which is Jackson?

Proposition
A finite set $A \subseteq \mathbb{R}^2$ is Jackson iff there exists a finite set $F \subseteq \mathbb{R}^2$ such that "A is Jackson in F", i.e., for every set $B \subseteq F$ there is $A' \cong_i A$, $A' \subseteq F$, with $|A' \cap B| \neq 1$.

Proposition
Consider a finite set $A \subseteq \mathbb{R}$:

$$0 = a_0 < a_1 < \ldots < a_n.$$

If the numbers

$$a_1 - a_0, a_2 - a_1, \ldots, a_n - a_{n-1}$$

are linearly independent over $\mathbb{Q}$, then there is an $\mathbb{R}$-Steinhaus set for A (i.e., A is not Jackson in $\mathbb{R}$).

Proposition

Let $A \subseteq \mathbb{Z}$ be finite:

$$0 = a_0 < a_1 < \ldots < a_n.$$

Let $d = \max\{a_{i+1} - a_i : i < n\}$. If A has an $\mathbb{R}$-Steinhaus set, then it has an R-Steinhaus set with a period $\leq d2^d$.

Corollary

Let $\mathcal{J}$ be the collection of all $\mathbb{R}$-Jackson subsets of $\mathbb{Z}$. Then $\mathcal{J}$ is computable.

Problem

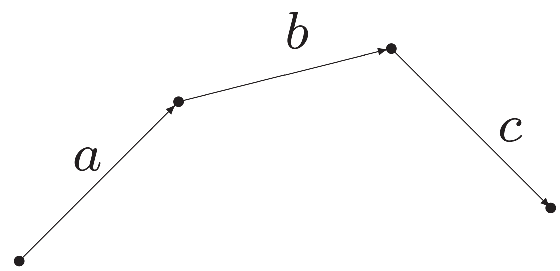
Is $\mathcal{J}$ polynomial time computable?

Proposition (Xuan)
If $A \subseteq \mathbb{Z}$ with $|A| = 4$, then there is a positive integer d such that every $\mathbb{R}$-Steinhaus set of A has period d.

Corollary (Xuan)
Every 4 point collinear subset of $\mathbb{R}^2$ with rational distances among the points is Jackson.

Let a, b, c be planar vectors, and

$$A = \{0, a, a+b, a+b+c\}.$$



Let S be a Steinhaus set for A. Then

$$S, S+a, S+a+b, S+a+b+c$$

form a partition of $\mathbb{R}^2$. Represent this partition by a coloring of the points as follows:

$$\begin{array}{cccc} S & S+a & S+a+b & S+a+b+c \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 2 & 3 & 4 \end{array}$$

Consider

$$\begin{aligned}\varphi \;:\quad \mathbb{Z}^3 &\longrightarrow \mathbb{R}^2\\ (l,m,n) &\longmapsto la+mb+nc\end{aligned}$$

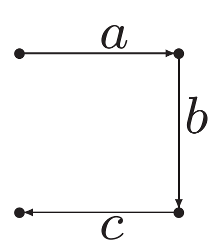
The partition on $\mathbb{R}^2$ induces a partition on $\mathbb{Z}^3$. This gives a 4-coloring κ of $\mathbb{Z}^3$ with the properties:

$$\begin{aligned}&\kappa(l,m,n)=1\\ \iff\quad &\kappa(l+1,m,n)=2\\ \iff\quad &\kappa(l+1,m+1,n)=3\\ \iff\quad &\kappa(l+1,m+1,n+1)=4.\end{aligned}$$

Proposition

If $0 \in S$ then at least two of sets $(a+c)\mathbb{Z}, (a-c)\mathbb{Z}$, and $2b\mathbb{Z}$ are subsets of S.

In the case of the unit square



Problem

Are all 4 point collinear sets Jackson?