UNIFORMLY RECURRENT SUBGROUPS OF INDUCTIVE LIMITS OF FINITE ALTERNATING GROUPS

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Abstract. We classify the uniformly recurrent subgroups of inductive limits of finite alternating groups.

1. Introduction

A simple locally finite group $G$ is said to be an $L(\mathrm{Alt})$-group if we can express $G = \bigcup_{i \in \mathbb{N}} G_i$ as the union of a strictly increasing chain of finite alternating groups $G_i = \mathrm{Alt}(\Delta_i)$. Here we allow arbitrary embeddings $G_i \hookrightarrow G_{i+1}$. In this paper, we will classify the uniformly recurrent subgroups of the $L(\mathrm{Alt})$-groups.

Let $G$ be a countably infinite group and let $\text{Sub}G$ be the compact space of subgroups $H \leq G$. Then $G$ acts as a group of homeomorphisms of $\text{Sub}G$ via the conjugation action, $H \mapsto gHg^{-1}$. Following Glasner-Weiss [5], a subset $X \subseteq \text{Sub}G$ is said to be a uniformly recurrent subgroup or URS if $X$ is a minimal $G$-invariant closed subset of $\text{Sub}G$. For example, if $N \trianglelefteq G$ is a normal subgroup, then the singleton set $\{N\}$ is a URS of $G$. In this paper, these singleton URSs will be called trivial. Examples of nontrivial URSs arise as the stabilizer URSs of minimal actions. (It is an open question whether every URS of every countable group $G$ can be realized as the stabilizer URS of a suitably chosen minimal $G$-action.) The definition of the stabilizer URS of an arbitrary minimal action is a little subtle. (See Glasner-Weiss [5, Section 1].) However, the stabilizer URSs which arise in our setting are easily described as follows.

Suppose that $\mathrm{Alt}(\Delta)$, $\mathrm{Alt}(\Omega)$ are finite alternating groups and that $|\Omega| = s|\Delta|$. Then an embedding $\varphi: \mathrm{Alt}(\Delta) \to \mathrm{Alt}(\Omega)$ is said to be strictly diagonal if $\varphi(\mathrm{Alt}(\Delta))$ acts via its natural permutation representation on each of its orbits in $\Omega$. The $L(\mathrm{Alt})$-group $G = \bigcup_{i \in \mathbb{N}} G_i$ is said to be the strictly diagonal limit of the finite alternating groups $G_i = \mathrm{Alt}(\Delta_i)$ if every embedding $G_i \hookrightarrow G_{i+1}$ is strictly diagonal. In this case, let $s_0 = |\Delta_0|$ and let $s_{i+1} = |\Delta_{i+1}|/|\Delta_i|$ be the number of natural $G_i$-orbits on $\Delta_{i+1}$. Then we can suppose that each

$$\Delta_i = s_0 \times s_1 \times \cdots \times s_i$$

and that the embedding $G_i \hookrightarrow G_{i+1}$ is defined by

$$g \cdot (\ell_0, \ldots, \ell_i, \ell_{i+1}) = (g \cdot (\ell_0, \ldots, \ell_i), \ell_{i+1}).$$

Equip $\Delta = \prod_{i \geq 0} s_i$ with the usual product topology. Then $G$ acts as a group of homeomorphisms of the compact space $\Delta$ via

$$g \cdot (\ell_0, \ldots, \ell_i, \ell_{i+1}, \ell_{i+2}, \cdots) = (g \cdot (\ell_0, \ldots, \ell_i), \ell_{i+1}, \ell_{i+2}, \cdots), \quad g \in G_i,$$
and it is clear that every $G$-orbit is dense in $\Delta$. Thus $G \acts \Delta$ is a minimal $G$-action. Let $f : \Delta \to \text{Sub}_G$ be the $G$-equivariant map defined by

$$x \mapsto G_x = \{ g \in G \mid g(x) = x \}$$

and let $X_\Delta = f(\Delta)$. Then $f$ is a continuous injection and it follows that $X_\Delta$ is a URS of $G$. As expected, $X_\Delta$ is called the stabilizer URS of the minimal action $G \acts \Delta$.

**Remark 1.1.** Of course, we can also define $X_\Delta$ directly as the set of subgroups $H \in \text{Sub}_G$ such that for every $i \geq 0$, there exists a point $x_i \in \Delta_i$ such that $H \cap G_i = \text{Alt}(\Delta_i \setminus \{ x_i \})$.

If $G = \bigcup_{i \in \mathbb{N}} G_i$ is the strictly diagonal limit of the finite alternating groups $G_i = \text{Alt}(\Delta_i)$, then we will refer to $G \acts \Delta$ as the *canonical minimal action*. We are now in a position to state the main result of this paper.

**Theorem 1.2.** If $G$ is an $L(\text{Alt})$-group and $X \subseteq \text{Sub}_G$ is a nontrivial URS, then $G$ can be expressed as a strictly diagonal limit of finite alternating groups and $X$ is the stabilizer URS of the corresponding canonical minimal action $G \acts \Delta$.

The proof of Theorem 1.2 makes use of an observation that is potentially useful in the setting of arbitrary countable amenable groups; namely, that if $G$ is a countable amenable group and $X \subseteq \text{Sub}_G$ is a URS, then there exists a $G$-invariant ergodic Borel probability measure $\nu$ on $\text{Sub}_G$ which concentrates on $X$. Consequently, measure-theoretic techniques (such as the Pointwise Ergodic Theorem for countable amenable groups [8]) can be employed in the study of the URSs of countable amenable groups.

A Borel probability measure $\nu$ on $\text{Sub}_G$ which is invariant under the conjugation action of $G$ on $\text{Sub}_G$ is called an *invariant random subgroup* or IRS. Since the introduction of this notion by Abert-Glasner-Virag [1], a number of authors have studied the IRSs of specific countable groups. For example, see Bowen [3], Bowen-Grigorchuk-Kravchenko [4], Vershik [11] and Benli-Grigorchuk-Nagnibeda [2]. In particular, Thomas-Tucker-Drob [10] have recently classified the ergodic IRSs of the $L(\text{Alt})$-groups; and, as we will see, Theorem 1.2 is a straightforward consequence of this classification. In this paper, an IRS $\nu$ will be said to be *trivial* if $\nu$ is the Dirac measure $\delta_N$ corresponding to some normal subgroup $N \trianglelefteq G$.

The remainder of this paper is organized as follows. In Section 2, we will present a brief account of the classification of the ergodic IRSs of the $L(\text{Alt})$-groups. Then, in Section 3, we will present the proof of Theorem 1.2.

### 2. The Ergodic IRSs of the $L(\text{Alt})$-Groups

In this section, we will present a brief account of the classification [10] of the ergodic IRSs of the $L(\text{Alt})$-groups. First we need to introduce some notation. Throughout this section, suppose that $G = \bigcup_{i \in \mathbb{N}} G_i$ is the union of the strictly increasing chain of finite alternating groups $G_i = \text{Alt}(\Delta_i)$.

**Definition 2.1.** For each $i \in \mathbb{N}$, let

- $n_i = |\Delta_i|$;
- $s_{i+1}$ be the number of natural $G_i$-orbits on $\Delta_{i+1}$;
- $f_{i+1}$ be the number of trivial $G_i$-orbits on $\Delta_{i+1}$;
Remark 2.2. The embedding $G_i \hookrightarrow G_{i+1}$ is said to be diagonal if $e_{i+1} = 0$; and $G = \bigcup_{i \in \mathbb{N}} G_i$ is said to be a diagonal limit if every embedding $G_i \hookrightarrow G_{i+1}$ is diagonal.

Definition 2.3. For each $i < j$, let $s_{ij} = s_{i+1}s_{i+2} \cdots s_j$.

Remark 2.4. If $G = \bigcup_{i \in \mathbb{N}} G_i$ is a diagonal limit, then $s_{ij}$ is the number of natural $G_i$-orbits on $\Delta_j$. However, it is easy to construct examples of increasing chains in which some $G_i$ have strictly more than $s_{ij}$ natural orbits on some $\Delta_j$.

Definition 2.5. $G = \bigcup_{i \in \mathbb{N}} G_i$ is an almost diagonal limit if $s_{i+1} > 0$ for all $i \in \mathbb{N}$ and $\sum_{i=1}^{\infty} e_i/s_{i0} < \infty$.

We are now ready to state the first of the classification theorems of Thomas-Tucker-Drob [10].

Theorem 2.6. If $G$ is an $L(\text{Alt})$-group, then $G$ has a nontrivial ergodic IRS if and only if $G$ can be expressed as an almost diagonal limit of finite alternating groups.

For the remainder of this section, suppose that $G = \bigcup_{i \in \mathbb{N}} G_i$ is an almost diagonal limit of finite alternating groups $G_i = \text{Alt}(\Delta_i)$. The classification of the ergodic IRSs of such groups involves a fundamental dichotomy which was introduced by Leinen-Puglisi [6, 7] in the more restrictive setting of diagonal limits of alternating groups, i.e. the linear vs sublinear natural orbit growth condition. For the sake of completeness, we include the proof of the following easy but fundamental lemma. (Note that Lemma 2.7 does not require that $G = \bigcup_{i \in \mathbb{N}} G_i$ is an almost diagonal limit.)

Lemma 2.7. For each $i \in \mathbb{N}$, the limit $a_i = \lim_{j \to \infty} s_{ij}/n_j$ exists.

Proof. If $i < j < k$, then $s_{ik} = s_{ij} s_{jk}$ and clearly $n_j s_{jk} \leq n_k$. Hence

$$\frac{s_{ik}}{n_k} = \frac{s_{ij}}{n_j} \cdot \frac{n_j s_{jk}}{n_k} \leq \frac{s_{ij}}{n_j},$$

and the sequence $(s_{ij}/n_j \mid i < j \in \mathbb{N})$ converges to $\inf_{j>1} s_{ij}/n_j$. \hfill \Box

Definition 2.8. An almost diagonal limit $G = \bigcup_{i \in \mathbb{N}} G_i$ has linear natural orbit growth if $a_i > 0$ for all $i \in \mathbb{N}$. Otherwise, $G = \bigcup_{i \in \mathbb{N}} G_i$ has sublinear natural orbit growth.

Remark 2.9. Note that $a_i = s_{i+1} a_{i+1}$. Thus an almost diagonal limit $G = \bigcup_{i \in \mathbb{N}} G_i$ has linear natural orbit growth if and only if $a_i > 0$ for some $i \in \mathbb{N}$.

An explicit classification of the ergodic IRSs of the almost diagonal limits $G = \bigcup_{i \in \mathbb{N}} G_i$ can be found in Thomas-Tucker-Drob [10]. In this paper, we will only state some structural consequences of this classification which will be sufficient to prove Theorem 1.2. It turns out that the infinite alternating group $\text{Alt}(\mathbb{N})$ has a far richer collection of ergodic IRSs than the other almost diagonal limits; and consequently, $\text{Alt}(\mathbb{N})$ needs to be considered separately. The following result is enough to prove that $\text{Alt}(\mathbb{N})$ has no nontrivial URs.

Theorem 2.10. If $\nu$ is an ergodic IRS of $\text{Alt}(\mathbb{N})$ such that $\nu \neq \delta_1$, then for $\nu$-a.e. $H \in \text{Sub}_{\text{Alt}(\mathbb{N})}$, there exists an infinite subset $B \subseteq \mathbb{N}$ such that $\text{Alt}(B) \leq H$. 
Corollary 2.11. Alt(ℕ) has no nontrivial URSs.

Proof. If X ⊆ Sub_{Alt(ℕ)} is a nontrivial URS, then there exists a nontrivial ergodic IRS ν which concentrates on X. Hence, by Theorem 2.10, there exists H ⊆ X such that Alt(B) ⊆ H for some infinite subset B ⊆ ℕ. But now it is clear that Alt(ℕ) is in the closure of \{ g H g^{-1} | g ∈ Alt(ℕ) \}, which is a contradiction. □

The following result will be enough to deal with the almost diagonal limits \( G = \bigcup_{i∈ ℕ} G_i \) such that \( G \not\cong Alt(ℕ) \).

Theorem 2.12. Suppose that \( G = \bigcup_{i∈ ℕ} G_i \) is an almost diagonal limit of finite alternating groups \( G_i = Alt(Δ_i) \) and that \( G \not\cong Alt(ℕ) \). If ν is a nontrivial ergodic IRS of G, then for ν-a.e. \( H ∈ Sub_G \), for all but finitely many \( i \), there exists a nonempty subset \( Σ_i ⊆ Δ_i \) such that \( H ∩ G_i = Alt(Δ_i \setminus Σ_i) \). Furthermore:

(a) If \( G = \bigcup_{i∈ ℕ} G_i \) has linear natural orbit growth, then there exists an integer \( r ≥ 1 \) such that \( |Σ_i| = r \) for all but finitely many \( i \).

(b) If \( G = \bigcup_{i∈ ℕ} G_i \) has sublinear natural orbit growth, then \( \lim_{i→∞} |Σ_i| = ∞ \) and \( \lim_{i→∞} |Σ_i|/n_i = 0 \).

Finally it is necessary to present an explicit classification of the ergodic IRSs of the strictly diagonal limits. Let \( G = \bigcup_{i∈ ℕ} G_i \) be a strictly diagonal limit and let \( G ∩ Δ = \prod_{i∈ ℕ} s_i \) be the corresponding canonical minimal action. Let m be the product probability measure of the uniform probability measures on the \( s_i \). Then it is easily seen that m is the unique G-invariant ergodic probability measure on \( Δ \). Furthermore, by Thomas-Tucker-Drob [10, Theorem 2.4], the action \( G ∩ (Δ, m) \) is weakly mixing and it follows that the action of G on the product space \( (Δ^r, m^{⊗ r}) \) is also ergodic for each \( r ≥ 2 \). Fix some \( r ≥ 1 \) and let \( f : Δ^r → Sub_G \) be the G-equivariant map defined by

\[ \bar{x} = (x_1, \ldots, x_r) ↦ G_x = \{ g ∈ G | g(x_ℓ) = x_ℓ \ \text{for each} \ 1 ≤ ℓ ≤ r \} \].

Then the stabilizer distribution \( ν_r = f_* m^{⊗ r} \) is an ergodic IRS of G.

Theorem 2.13. If \( G = \bigcup_{i∈ ℕ} G_i \) is a strictly diagonal limit, then the ergodic IRSs of G are the \( \{ δ_1, δ_G \} \cup \{ ν_r | r ∈ ℕ^+ \} \).

3. The proof of Theorem 1.2

Suppose that G is an L(Alt)-group and that \( X ⊆ Sub_G \) is a nontrivial URS. Then there exists a nontrivial ergodic IRS ν of G which concentrates on X. Hence, by Theorem 2.6, we can express G as an almost diagonal limit \( \bigcup_{i∈ ℕ} G_i \) of finite alternating groups \( G_i = Alt(Δ_i) \). By Corollary 2.11, \( G \not\cong Alt(ℕ) \). Hence, by Theorem 2.12, there exists a subgroup \( H ∈ X \) such that for all but finitely many \( i \), there exists a nonempty subset \( Σ_i ⊆ Δ_i \) such that \( H ∩ G_i = Alt(Δ_i \setminus Σ_i) \) and such that:

(a) if \( G = \bigcup_{i∈ ℕ} G_i \) has linear natural orbit growth, then there exists an integer \( r ≥ 1 \) such that \( |Σ_i| = r \) for all but finitely many \( i \);

(b) if \( G = \bigcup_{i∈ ℕ} G_i \) has sublinear natural orbit growth, then \( \lim_{i→∞} |Σ_i| = ∞ \) and \( \lim_{i→∞} |Σ_i|/n_i = 0 \).

After deleting a finite initial segment from the sequence \( (G_i | i ∈ ℕ) \), we can suppose that such a subset \( Σ_i ⊆ Δ_i \) exists for all \( i ≥ 0 \).
Lemma 3.1. There exists an integer $n_0$ such that for all $i \geq n_0$, the embedding $G_i \hookrightarrow G_{i+1}$ is diagonal.

Proof. Suppose not. Then, by Praeger-Zalesskii [9, Theorem 1.7], for all $i \geq 0$, there exists $j > i$ such that $G_i$ has a regular orbit $\Phi$ on $\Delta_j$. Let $g \in G_j$ be an element such that $g(\Sigma_j) \cap \Phi \neq \emptyset$. Then $g H g^{-1} \in X$ and $g H g^{-1} \cap G_j = \operatorname{Alt}(\Delta_j \setminus g(\Sigma_j))$; and this implies that $g H g^{-1} \cap G_i = 1$. Since $i \geq 0$ was arbitrary, it follows that $1 \in X$, which is a contradiction. 

Hence, after deleting a finite initial segment from the sequence $(G_i \mid i \in \mathbb{N})$, we can suppose that $G = \bigcup_{i \in \mathbb{N}} G_i$ is a diagonal limit.

Lemma 3.2. There exists an integer $n_1$ such that for all $i \geq n_1$, the embedding $G_i \hookrightarrow G_{i+1}$ is strictly diagonal.

Proof. Suppose not. Let $i \geq 0$. Then for each $j > i$, the group $G_i$ has $s_{ij}$ natural orbits on $\Delta_j$ and fixes the remaining $n_j - s_{ij}n_i$ points. Let $\Phi_{ij} \subseteq \Delta_j$ be the union of the $s_{ij}$ natural $G_i$-orbits.

Claim 3.3. For each $i \geq 0$, there exists $j > i$ such that $|\Phi_{ij}| \leq |\Delta_j \setminus \Sigma_j|$.

Assuming that Claim 3.3 holds, let $g \in G_j$ be such that $\Phi_{ij} \subset g(\Delta_j \setminus \Sigma_j)$. Then $g H g^{-1} \in X$ and $g H g^{-1} \cap G_j = \operatorname{Alt}(g(\Delta_j \setminus \Sigma_j))$; and this implies that $g H g^{-1} \cap G_i = G_i$. Since $i \geq 0$ was arbitrary, it follows that $G \subseteq X$, which is a contradiction.

Thus it only remains to prove Claim 3.3. First suppose that $G = \bigcup_{i \in \mathbb{N}} G_i$ has linear natural orbit growth. Then there exists an integer $r \geq 1$ such that $|\Sigma_j| = r$ for all $j \geq 0$. Also, assuming Lemma 3.2 does not hold, it follows that for each $i \geq 0$, there exists $j > i$ such that $G_i$ fixes at least $r$ points on $\Delta_j$ and thus $|\Phi_{ij}| \leq |\Delta_j \setminus \Sigma_j|$. Next suppose that $G = \bigcup_{i \in \mathbb{N}} G_i$ has sublinear natural orbit growth. Then we have that

$$
\lim_{j \to \infty} \frac{|\Phi_{ij}|}{n_j} = \lim_{j \to \infty} \frac{s_{ij}}{n_j} = 0;
$$

and also that $\lim_{j \to \infty} |\Sigma_j|/n_j = 0$. The result follows easily. 

Thus, after deleting a finite initial segment from the sequence $(G_i \mid i \in \mathbb{N})$, we can suppose that $G = \bigcup_{i \in \mathbb{N}} G_i$ is the strictly diagonal limit of the finite alternating groups $G_i = \operatorname{Alt}(\Delta_i)$. Let $G \acts \Delta$ be the corresponding canonical minimal action and let $m$ be the unique $G$-invariant ergodic probability measure on $\Delta$. Then, by Theorem 2.13, there exists an integer $r \geq 1$ such that $\nu$ is the stabilizer distribution $\nu_r$ of the ergodic action $G \acts (\Delta^r, m^{\otimes r})$. Suppose that $r > 1$, so that $H$ is the pointwise stabilizer of $r$ elements $x_1, \ldots, x_r \in \Delta$. For each $1 \leq \ell \leq r$, let

$$x_\ell = (x_\ell(0), x_\ell(1), \ldots, x_\ell(i), \ldots);$$

and for each $1 \leq \ell \leq r$ and $i \geq 0$, let $x_\ell^i = (x_\ell(0), x_\ell(1), \ldots, x_\ell(i)) \in \Delta_i$ be the corresponding restriction. Then $\Sigma_i = \{ x_1^i, \ldots, x_r^i \}$. Fix some $i \geq 0$. Then if $j > i$ is sufficiently large, there exist distinct elements $y_1, \ldots, y_r \in \Delta_j$, all of which restrict to the same element $z \in \Delta_j$. Let $g \in G_j$ be such that $g(x_j^i) = y_j^i$ for all $1 \leq \ell \leq r$. Then $g H g^{-1} \in X$ and $g H g^{-1} \cap G_i = \operatorname{Alt}(\Delta_i \setminus \{ z \})$. Since $i \geq 0$ was arbitrary, it follows that $X$ also contains the stabilizer URS $X_\Delta$ of $G \acts \Delta$, which contradicts the minimality of $X$. Thus $r = 1$ and $X = X_\Delta$ is the stabilizer URS of $G \acts \Delta$. This completes the proof of Theorem 1.2.
References


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