A Descriptive View of Combinatorial Group Theory

Simon Thomas

Rutgers
The Soprano State University
"Jersey Roots, Global Reach"

28th July 2010

The Basic Theme:

Descriptive set theory provides a framework for explaining the inevitable non-uniformity of many classical constructions in mathematics.

Two Examples from Combinatorial Group Theory:

- *The word problem for finitely generated groups.*
- *The Higman-Neumann-Neumann Embedding Theorem.*

The word problem for finitely generated groups

For each $n \geq 1$, fix an effective enumeration $\{ w_k(x_1, \cdots, x_n) \mid k \in \mathbb{N} \}$ of the words in $x_1, \cdots, x_n, x_1^{-1}, \cdots, x_n^{-1}$.

Definition

If $G = \langle a_1, \cdots, a_n \rangle$ is a finitely generated group, then

$$\mathrm{Word}(G) = \{ k \in \mathbb{N} \mid w_k(a_1, \cdots, a_n) = 1 \}$$

Remark

The word problem for $G = \langle a_1, \cdots, a_n \rangle$ is the problem of deciding whether $k \in \mathrm{Word}(G)$.

Turing Reducibility

Convention

Throughout this talk, the powerset $\mathcal{P}(\mathbb{N})$ will be identified with $2^{\mathbb{N}}$ by identifying subsets of $\mathbb{N}$ with their characteristic functions.

Definition

If $A, B \in 2^{\mathbb{N}}$, then A is Turing reducible to B, written $A \leq_T B$, if there exists a B-oracle Turing machine which computes A.

Remark

In other words, there is an algorithm which computes *A* modulo an oracle which correctly answers questions of the form "Is $n \in B$?"

Turing Reducibility

Definition

If $A, B \in 2^{\mathbb{N}}$, then A is Turing equivalent to B, written $A \equiv_T B$, if both $A \leq_T B$ and $B \leq_T A$.

Definition

If $A \in 2^{\mathbb{N}}$, then the corresponding Turing degree is defined to be

$$\mathbf{a} = \{\, B \in 2^{\mathbb{N}} \mid B \equiv_T A \,\}.$$

Proposition

If $G = \langle\, a_1, \cdots, a_n \,\rangle = \langle\, b_1, \cdots, b_m \,\rangle$ is a finitely generated group, then

$$\{\, k \in \mathbb{N} \mid w_k(a_1, \cdots, a_n) = 1 \,\} \equiv_T \{\, \ell \in \mathbb{N} \mid w_\ell(b_1, \cdots, b_m) = 1 \,\}.$$

Prescribing the Turing degree of the word problem

Theorem (Folklore)

For each subset $A \subseteq \mathbb{N}$*, there exists a finitely generated group* G_A *such that* $\mathrm{Word}(G_A) \equiv_T A$*.*

Sketch Proof.

Let G_A be the group generated by the elements a, b subject to the following defining relations, where $c_n = [\, b, a^{-(n+1)} b\, a^{n+1}\,]$.
(Here $[\, c, d\,] = c^{-1}\, d^{-1}\, c\, d$.)

- $a\, c_n = c_n\, a$ for all $n \in \mathbb{N}$.
- $b\, c_n = c_n\, b$ for all $n \in \mathbb{N}$.
- $c_n^2 = 1$ for all $n \in \mathbb{N}$.
- $c_n = 1$ for all $n \in A$.

□

A natural question

Observation

The above construction of G_A is highly dependent on the specific subset $A \subseteq \mathbb{N}$, in the sense that if $A \neq B$ are subsets such that $A \equiv_T B$, then we "usually" have that $G_A \not\cong G_B$.

Question

Does there exist a more uniform construction $A \mapsto G_A$ with the property that the isomorphism type of G_A only depends upon the Turing degree of A?

The HNN Embedding Theorem

Theorem (Higman-Neumann-Neumann)

Every countable group G can be embedded into a 2-generator group.

Sketch Proof.

- Let $(\, g_n \mid n \in \mathbb{N}\,)$ be a sequence of generators with $g_0 = 1$.
- Let $\mathbb{F}$ be the free group on $\{\, a, b \,\}$ and let $G * \mathbb{F}$ be the free product.
- Then $\{\, b^{-n} a b^n \mid n \in \mathbb{N} \,\}$ and $\{\, g_n a^{-n} b a^n \mid n \in \mathbb{N} \,\}$ freely generate free subgroups of $G * \mathbb{F}$.
- Hence we can construct the *HNN* extension

$$G \hookrightarrow K_G = \langle\, G * \mathbb{F}, t \mid t^{-1} b^{-n} a b^n t = g_n a^{-n} b a^n \,\rangle$$

- Since $g_n \in \langle a, b, t \rangle$ and $t^{-1} a t = b$, it follows that $K_G = \langle a, t \rangle$.

□

Another natural question

Observation

It is reasonably clear that the isomorphism type of the 2-generator group K_G usually depends upon both the generating set of G and the particular enumeration that is used.

Question

Does there exist a more uniform construction with the property that the isomorphism type of K_G only depends upon the isomorphism type of G?

The answers ...

Notation

- $\mathcal{G}$ denotes the Polish space of countably infinite groups.
- $\mathcal{G}_{fg}$ denotes the Polish space of finitely generated groups.

Definition

If X, Y are Polish spaces, then the map $\varphi : X \to Y$ is Borel if graph(φ) *is a Borel subset of $X \times Y$.*

An Analog of Church's Thesis

EXPLICIT = BOREL

The answers ...

Theorem

- *Suppose that* $A \mapsto G_A$ *is* *any* *Borel map from* $2^{\mathbb{N}}$ *to* $\mathcal{G}_{fg}$ *such that* $\text{Word}(G_A) \equiv_T A$ *for all* $A \in 2^{\mathbb{N}}$.
- *Then there exists a Turing degree* $\mathbf{d}_0$ *such that for all* $\mathbf{d} \geq_T \mathbf{d}_0$, *there exists an infinite subset* $\{ A_n \mid n \in \mathbb{N} \} \subseteq \mathbf{d}$ *such that the groups* $\{ G_{A_n} \mid n \in \mathbb{N} \}$ *are pairwise incomparable with respect to embeddability.*

The answers ...

Theorem (*LC*)

- *Suppose that $G \mapsto K_G$ is any Borel map from $\mathcal{G}$ to $\mathcal{G}_{fg}$ such that $G \hookrightarrow K_G$ for all $G \in \mathcal{G}$.*
- *Then there exists an uncountable Borel family $\mathcal{F} \subseteq \mathcal{G}$ of pairwise isomorphic groups such that the groups $\{ K_G \mid G \in \mathcal{F} \}$ are pairwise incomparable with respect to relative constructibility; i.e., if $G \neq H \in \mathcal{F}$, then $K_G \notin L[K_H]$ and $K_H \notin L[K_G]$.*

Remarks

- (*LC*): There exists a Ramsey cardinal κ.
- In *ZFC*, we can find an uncountable Borel family $\mathcal{F}$ such that the groups $\{ K_G \mid G \in \mathcal{F} \}$ are pairwise incomparable with respect to embeddability.

How to prove such theorems?

The Word Problem Theorem

- *Reduce to a problem in Recursion Theory and then apply Martin's Theorem on the determinacy of Borel games.*
- *To be explained in the remainder of this talk ...*

The Embedding Theorem

- *Collapse the continuum $\mathbb{R}$ to a countable set and then apply the Shoenfield and Martin-Solovay Absoluteness Theorems.*
- *A little too technical for a general audience ...*

An obvious follow-up question ...

Notation

Let $\mathcal{G}_m$ be the Polish space of m-generator groups.

Question (Cherlin, Hrushovski, ...)

Does there exist a Borel map $G \mapsto K_G$ from $\mathcal{G}_3$ to $\mathcal{G}_2$ such that

- $G \hookrightarrow K_G$; *and*
- *if* $G \cong H$, *then* $K_G \cong K_H$?

The Friedman Embedding Theorem

There exists a Borel map $G \mapsto K_G$ from $\mathcal{G}_{fg}$ to $\mathcal{G}_2$ such that

- $G \hookrightarrow K_G$; *and*
- *if* $G \cong H$, *then* $K_G \cong K_H$.

Polish Spaces

- The Cantor space $2^{\mathbb{N}}$ is a complete separable metric space with respect to the metric

$$d(x,y) = \sum_{n=0}^{\infty} \frac{|x(n) - y(n)|}{2^{n+1}}.$$

- The corresponding topological space is a Polish space with basic open neighborhoods

$$U_s = \{\, x \in 2^{\mathbb{N}} \mid x \restriction n = s \,\}, \qquad \text{where } s \in 2^{<\mathbb{N}}.$$

The Polish space of countably infinite groups

- Let $\mathcal{G}$ be the set of groups with underlying set $\mathbb{N}$.
- We can identify each group

$$G \in \mathcal{G} \longleftrightarrow m_G \in 2^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}$$

  with the graph of its multiplication operation.
- Then $\mathcal{G}$ is a G_δ subset of the Cantor space $2^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}$ i.e. $\mathcal{G}$ is a countable intersection of open subsets.
- Hence $\mathcal{G}$ is a Polish subspace of the Cantor space $2^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}$.

The Polish space of finitely generated groups

- A marked group $(G, \bar{s})$ consists of a f.g. group with a distinguished sequence $\bar{s} = (s_1, \cdots, s_m)$ of generators.
- For each $m \geq 1$, let $\mathcal{G}_m$ be the set of isomorphism types of marked groups $(G, (s_1, \cdots, s_m))$ with m distinguished generators.
- Then there exists a canonical embedding $\mathcal{G}_m \hookrightarrow \mathcal{G}_{m+1}$ defined by

$$(G, (s_1, \cdots, s_m)) \mapsto (G, (s_1, \cdots, s_m, 1_G)).$$

- And $\mathcal{G}_{fg} = \bigcup \mathcal{G}_m$ is the space of f.g. groups.

The Polish space of finitely generated groups

- Let $(G, \bar{s}) \in \mathcal{G}_m$ and let d_S be the corresponding word metric. For each $\ell \geq 1$, let

$$B_\ell(G, \bar{s}) = \{g \in G \mid d_S(g, 1_G) \leq \ell\}.$$

- The basic open neighborhoods of $(G, \bar{s})$ in $\mathcal{G}_m$ are given by

$$U_{(G,\bar{s}),\ell} = \{\, (H, \bar{t}) \in \mathcal{G}_m \mid B_\ell(H, \bar{t}) \cong B_\ell(G, \bar{s}) \,\}, \qquad \ell \geq 1.$$

Example

For each $n \geq 1$, let $C_n = \langle g_n \rangle$ be cyclic of order n. Then:

$$\lim_{n \to \infty} (C_n, g_n) = (\mathbb{Z}, 1).$$

A slight digression ...

Some Isolated Points

- Finite groups
- Finitely presented simple groups

The Next Stage

- $SL_3(\mathbb{Z})$

Question (Grigorchuk)

What is the Cantor-Bendixson rank of $\mathcal{G}_{fg}$?

What are the advantages of working with $\mathcal{G}_{fg}$?

Vague Conjecture

There does not exist a "purely group-theoretic" construction $G \mapsto K_G$ from $\mathcal{G}_3$ to $\mathcal{G}_2$ such that

- $G \hookrightarrow K_G$; *and*
- *if* $G \cong H$, *then* $K_G \cong K_H$.

Remarks

- By this, I mean a construction which only involves purely group-theoretic notions such as wreath products, free products with amalgamation, *HNN*-extensions, etc.
- In each case that I have considered, such a construction induces a continuous map on the space $\mathcal{G}_{fg}$ of f.g. groups.
- So it is natural to conjecture that there does not exists a continuous such map from $\mathcal{G}_3$ to $\mathcal{G}_2$.

Countable Borel equivalence relations

Observation

The isomorphism relation $\cong$ is a countable Borel equivalence relation on the space $\mathcal{G}_{fg}$ of f.g. groups.

Definition

- *An equivalence relation E on a Polish space X is Borel if E is a Borel subset of $X \times X$.*
- *A Borel equivalence relation E is countable if every E-class is countable.*

Example

The Turing equivalence relation $\equiv_T$ is a countable Borel equivalence relation on $2^{\mathbb{N}}$.

Borel reductions

Definition

Let E, F be Borel equivalence relations on the Polish spaces X, Y respectively.

- $E \leq_B F$ *if there exists a Borel map* $f : X \to Y$ *such that*

$$x \, E \, y \iff f(x) \, F \, f(y).$$

  In this case, f is called a **Borel reduction** *from E to F.*
- $E \sim_B F$ *if both* $E \leq_B F$ *and* $F \leq_B E$.
- $E <_B F$ *if both* $E \leq_B F$ *and* $E \nsim_B F$.

Universal countable Borel equivalence relations

Definition

A countable Borel equivalence relation E is universal if $F \leq_B E$ for every countable Borel equivalence relation F.

Theorem (Thomas-Velickovic)

The isomorphism relation $\cong$ on $\mathcal{G}_{fg}$ is a universal countable Borel equivalence relation.

Remark

It is currently not known whether the Turing equivalence relation $\equiv_T$ is countable universal.

Universal countable Borel equivalence relations

Corollary

There exists a Borel reduction from $\equiv_T$ *to* $\cong$.

Main Theorem

- *There does not exist a Borel reduction* $A \mapsto G_A$ *from* $\equiv_T$ *to* $\cong$ *such that* $\mathrm{Word}(G_A) \equiv_T A$ *for all* $A \in 2^{\mathbb{N}}$.
- *"Equivalently", there does not exist a continuous reduction from* $\equiv_T$ *to* $\cong$.

Question (Kanovei)

Find natural examples of Borel equivalence relations E, F *such that* $E \leq_B F$ *but there is no continuous reduction from* E *to* F.

Why are such examples hard to find?

Theorem (Folklore)

If X, Y are Polish spaces and $\varphi : X \to Y$ is a Borel map, then there exists a comeager subset $C \subseteq X$ such that $\varphi \restriction C$ is continuous.

Theorem (Lusin)

Let X, Y be Polish spaces and let μ be any Borel probability measure on X. If $\varphi : X \to Y$ is a Borel map, then for every $\varepsilon > 0$, there exists a compact set $K \subseteq X$ with $\mu(K) > 1 - \varepsilon$ such that $\varphi \restriction K$ is continuous.

Another notion of largeness ...

Definition

For each $z \in 2^{\mathbb{N}}$, the corresponding cone is $\mathcal{C}_z = \{ x \in 2^{\mathbb{N}} \mid z \leq_T x \}$.

- Suppose $z_n = \{ a_{n,\ell} \mid \ell \in \mathbb{N} \} \in 2^{\mathbb{N}}$ for each $n \in \mathbb{N}$ and define

$$\oplus z_n = \{ p_n^{a_{n,\ell}} \mid n, \ell \in \mathbb{N} \} \in 2^{\mathbb{N}},$$

  where p_n is the nth prime.

- Then $z_m \leq_T \oplus z_n$ for each $m \in \mathbb{N}$ and so $\mathcal{C}_{\oplus z_n} \subseteq \bigcap_n \mathcal{C}_{z_n}$.

Remark

It is well-known that if $\mathcal{C} \subsetneq 2^{\mathbb{N}}$ is a proper cone, then $\mathcal{C}$ is both null and meager.

Continuous maps on the Cantor space

Theorem (Folklore)

If $\theta : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$, then the following are equivalent:

(a) *θ is continuous.*

(b) *There exists $C \in 2^{\mathbb{N}}$ and $e \in \mathbb{N}$ such that $\theta(A) = \varphi_e^{C \oplus A}$.*

Corollary

If $\theta : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is continuous, then there exists a cone $\mathcal{C}$ such that $\theta(A) \leq_T A$ for all $A \in \mathcal{C}$.

Corollary

If $G \mapsto K_G$ is a continuous map from $\mathcal{G}_{fg}$ to $\mathcal{G}_{fg}$, then there exists a cone $\mathcal{C}$ such that if $\mathrm{Word}(G) \in \mathcal{C}$, then $\mathrm{Word}(K_G) \leq_T \mathrm{Word}(G)$.

The "obvious" Turing reductions ...

Definition

If $A, B \in 2^{\mathbb{N}}$, then A is one-one reducible to B, written $A \leq_1 B$, if there exists an injective recursive function $f : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$,

$$n \in A \iff f(n) \in B.$$

Example

If $G, H \in \mathcal{G}_{fg}$ and $G \hookrightarrow H$, then $\text{Word}(G) \leq_1 \text{Word}(H)$.

Proof.

Suppose that $G = \langle a_1, \cdots, a_n \rangle$ and $H = \langle b_1, \cdots, b_m \rangle$. Let $\varphi : G \to H$ be an embedding and let $\varphi(a_i) = t_i(\bar{b})$. Then

$$w_k(a_1, \cdots, a_n) = 1 \iff w_k(t_1(\bar{b}), \cdots, t_n(\bar{b})) = 1.$$

□

Turing Equivalence vs. Recursive Isomorphism

Definition

The sets $A, B \in 2^{\mathbb{N}}$ are recursively isomorphic, written $A \equiv_1 B$, if both $A \leq_1 B$ and $B \leq_1 A$.

Theorem (Myhill)

If $A, B \in 2^{\mathbb{N}}$, then $A \equiv_1 B$ if and only if there exists a recursive permutation $\pi \in \mathrm{Sym}(\mathbb{N})$ such that $\pi[A] = B$.

Theorem (Folklore)

The map $A \mapsto A'$ is a Borel reduction from $\equiv_T$ to $\equiv_1$.

Observation

The Borel reduction $A \mapsto A'$ from $\equiv_T$ to $\equiv_1$ is certainly not continuous.

Turing Equivalence vs. Recursive Isomorphism

Definition

Let E, F be Borel equivalence relations on the Polish spaces X, Y. Then the Borel map $\varphi : X \to Y$ is a homomorphism from E to F if

$$x \, E \, y \Longrightarrow \varphi(x) \, F \, \varphi(y).$$

Main Lemma

If $\theta : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is a continuous homomorphism from $\equiv_T$ to $\equiv_1$, then there exists a cone $\mathcal{C}$ such that θ maps $\mathcal{C}$ into a single $\equiv_1$-class.

Corollary

There does not exist a continuous reduction from $\equiv_T$ to $\equiv_1$.

Turing Equivalence vs. Isomorphism on $\mathcal{G}_{fg}$

Corollary

There does not *exist a continuous reduction from $\equiv_T$ to $\cong$.*

Proof.

- Suppose $A \mapsto H_A$ is a continuous reduction from $\equiv_T$ to $\cong$.
- Note that $H \mapsto \text{Word}(H)$ is an injective continuous homomorphism from $\cong$ to $\equiv_1$.
- Thus $A \mapsto \text{Word}(H_A)$ is a countable-to-one continuous homomorphism from $\equiv_T$ to $\equiv_1$, which is a contradiction.

□

Determinacy

Definition

For each $X \subseteq 2^{\mathbb{N}}$, let $G(X)$ be the two player game

$$\begin{array}{lccccccccc} I & s(0) & & s(2) & & s(4) & & s(6) & \cdots & \\ II & & s(1) & & s(3) & & s(5) & & s(7) & \cdots \end{array}$$

where I wins if and only if $s = (\, s(0)\, s(1)\, s(2)\, s(3)\, \cdots) \in X$.

Definition

- *A strategy is a map $2^{<\mathbb{N}} \to 2$ which tells the relevant player which move to make in a given position.*
- *The game $G(X)$ is determined if one of the players has a winning strategy.*

Determinacy

Theorem (Borel Determinacy)

If $X \subseteq 2^{\mathbb{N}}$ is a Borel subset, then $G(X)$ is determined.

Definition

A subset $X \subseteq 2^{\mathbb{N}}$ is $\equiv_T$-invariant if it is a union of $\equiv_T$-classes.

Theorem (Martin)

If $X \subseteq 2^{\mathbb{N}}$ is a $\equiv_T$-invariant Borel subset, then either X or $2^{\mathbb{N}} \smallsetminus X$ contains a cone.

Cf. Kolmogorov's Zero-One Law ...

Proof of Martin's Theorem

- Suppose that $X \subseteq 2^{\mathbb{N}}$ is a $\equiv_T$-invariant Borel subset.
- Consider the two player game $G(X)$

$$s(0) \quad s(1) \quad s(2) \quad s(3) \quad \cdots$$

where I wins if and only if $s = (\, s(0)\, s(1)\, s(2)\, \cdots\,) \in X$.

- Then the Borel game $G(X)$ is determined. Suppose, for example, that $\sigma : 2^{<\mathbb{N}} \to 2$ is a winning strategy for I.
- Let $\sigma \leq_T t \in 2^{\mathbb{N}}$ and consider the run of $G(X)$ where
  - II plays $t = (\, s(1)\, s(3)\, s(5)\, \cdots\,)$
  - I uses the strategy σ and plays $(\, s(0)\, s(2)\, s(4)\, \cdots\,)$.
- Then $s \in X$ and $s \equiv_T t$. Hence $t \in X$ and so $\mathcal{C}_\sigma \subseteq X$.

Some easy consequences of Martin's Theorem

Theorem (Martin)

If $X \subseteq 2^{\mathbb{N}}$ is a $\equiv_T$-invariant Borel subset, then either X or $2^{\mathbb{N}} \setminus X$ contains a cone.

Corollary

If $X \subseteq 2^{\mathbb{N}}$ is a $\equiv_T$-invariant $\leq_T$-cofinal Borel subset, then X contains a cone.

Corollary

If $X \subseteq 2^{\mathbb{N}}$ is an arbitrary $\leq_T$-cofinal Borel subset, then X contains representatives of a cone.

Pointed Trees

Definition

- *A subset* $S \subseteq 2^{<\mathbb{N}}$ *is a tree if it is closed under taking initial segments.*
- *If* S *is a tree, then* $[S] \subseteq 2^{\mathbb{N}}$ *denotes the set of infinite branches through* S*.*
- *The tree* S *is perfect if for each* $s \in S$*, there exist incomparable* $a, b \in S$ *with* $s \lessdot a, b$*.*
- *The perfect tree* S *is pointed if* $S \leq_T y$ *for all* $y \in [S]$*.*

Theorem (Martin)

If $X \subseteq 2^{\mathbb{N}}$ *is a* $\leq_T$*-cofinal Borel subset, then there exists a pointed tree* $S \subseteq 2^{<\mathbb{N}}$ *such that* $[S] \subseteq X$*.*

Proof of the Main Lemma

Main Lemma

If $\theta : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is a continuous homomorphism from $\equiv_T$ to $\equiv_1$, then there exists a cone $\mathcal{C}$ such that θ maps $\mathcal{C}$ into a single $\equiv_1$-class.

- Let $\mathcal{D}$ be a cone $\mathcal{D}$ such that $\theta(A) \leq_T A$ for all $A \in \mathcal{D}$.
- Then there exists a cone $\mathcal{C} \subseteq \mathcal{D}$ such that either
  - (a) $\theta(A) <_T A$ for all $A \in \mathcal{C}$; or
  - (b) $\theta(A) \equiv_T A$ for all $A \in \mathcal{C}$.

Proof of the Main Lemma

Theorem (Slaman-Steel)

- *Suppose that $\mathcal{C}$ is a cone and that $\theta : \mathcal{C} \to 2^{\mathbb{N}}$ is a Borel homomorphism from $\equiv_T \restriction \mathcal{C}$ to $\equiv_T$ such that $\theta(A) <_T A$ for all $A \in \mathcal{C}$.*
- *Then there exists a cone $\mathcal{D} \subseteq \mathcal{C}$ such that θ maps $\mathcal{D}$ into a single $\equiv_T$-class.*

The Non-Selector Theorem

- *If $\mathcal{C}$ is a cone, then there does not exist a Borel homomorphism $\theta : \mathcal{C} \to \mathcal{C}$ from $\equiv_T \restriction \mathcal{C}$ to $\equiv_1 \restriction \mathcal{C}$ such that $\theta(A) \equiv_T A$ for all $A \in \mathcal{C}$.*
- *In other words, if $\mathcal{C}$ is a cone, then there does not exist a Borel map which selects an $\equiv_1$-class within each $\equiv_T$-class.*

Proof of the Non-Selector Theorem

- Suppose $\theta : \mathcal{C} \to \mathcal{C}$ selects a $\equiv_1$-class within each $\equiv_T$-class.
- Then $\theta[\mathcal{C}]$ is a $\leq_T$-cofinal Borel subset of $2^{\mathbb{N}}$.
- By Martin's Theorem, there exists a pointed tree $S \subseteq 2^{<\mathbb{N}}$ such that $[\,S\,] \subseteq \theta[\mathcal{C}]$.
- Note that if $x, y \in [\,S\,]$, then $x \equiv_T y$ iff $x \equiv_1 y$.
- We can suppose that $(\,\pi_n \mid n \in \mathbb{N}\,) \leq_T S$, where $\{\,\pi_n \mid n \in \mathbb{N}\,\}$ is the group of recursive permutations.
- Let $x \in [\,S\,]$ be the left-most branch, so that $x \equiv_T S$.
- Then we can construct a branch $y \leq_T S$ such that $\pi_n(y) \neq x$ for all $n \in \mathbb{N}$.
- But then $y \equiv_T x$ and $y \not\equiv_1 x$, which is a contradiction!

Proof of the Main Theorem

Main Theorem

There does not exist a Borel reduction $A \mapsto G_A$ from $\equiv_T$ to $\cong$ such that $\mathrm{Word}(G_A) \equiv_T A$ *for all* $A \in 2^{\mathbb{N}}$.

- Suppose that $A \mapsto G_A$ is a Borel reduction from $\equiv_T$ to $\cong$ such that $\mathrm{Word}(G_A) \equiv_T A$ for all $A \in 2^{\mathbb{N}}$.
- Consider the Borel map $\theta : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ defined by $A \mapsto \mathrm{Word}(G_A)$.
- If $A \equiv_T B$, then $G_A \cong G_B$ and so $\mathrm{Word}(G_A) \equiv_1 \mathrm{Word}(G_B)$.
- Thus $\theta : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is a Borel map which selects an $\equiv_1$-class within each $\equiv_T$-class, which is a contradiction!

Some References

Available at: http://www.math.rutgers.edu/ sthomas/papers.html

- S. Thomas, *Continuous versus Borel Reductions*, Arch. Math. Logic **48** (2009), 761–770.
- S. Thomas, *The Friedman Embedding Theorem*, preprint (2009).
- S. Thomas, *A Descriptive View of Combinatorial Group Theory*, preprint (2010).