UNIVERSAL NONSINGULAR CONTROLS[1]

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ABSTRACT

In this note we prove that, for analytic systems satisfying the strong accessibility rank condition, generic inputs produce trajectories along which the linearized system is controllable. Applications to the steering of systems without drift are briefly mentioned.

Keywords: nonlinear control, linearization, numerical control

1 Introduction and Statement of Result

The systems considered here will be of the type

$$\dot{x}(t) \;=\; f(x(t), u(t)) \tag{1}$$

(arguments "t" will be deleted below when clear from the context) where $x(t) \in \mathcal{X}$, $u(t) \in U$, and:

- $\mathcal{X} \subseteq \mathbb{R}^n$ is open and connected, for some $n \geq 1$;
- $\mathcal{U} \subseteq \mathbb{R}^m$ is open and connected, for some $m \geq 1$;
- $f : \mathcal{X} \times \mathcal{U} \to \mathbb{R}^n$ is real-analytic.

This defines a time-invariant control system (for basic system-theoretic definitions, consult [11]).

A *control* is a measurable essentially bounded map $\omega : [0,T] \to \mathcal{U}$; it is said to be *smooth* (respectively, *analytic*) if it is infinitely differentiable (respectively, real-analytic) as a function of $t \in [0,T]$. We denote by $\phi(t,x,\omega)$ the solution of (1) at time t with initial condition x and control ω. This is defined for all small $t < \tau(x,\omega) > 0$; when we write $\phi(\cdot,x,\omega)$, we mean the solution as defined on the largest interval $[0,\tau)$ of existence.

Recall that the system (1) is said to be *strongly accessible* if for each $x \in \mathcal{X}$ there is some $T > 0$ so that

$$\operatorname{int} \mathcal{R}^T(x) \;\neq\; \emptyset \,,$$

where as usual $\mathcal{R}^T(x)$ denotes the reachable set from x in time exactly T. Equivalently, the system must satisfy the *strong accessibility rank condition*: $\dim \mathcal{L}_0(x) = n$ for all x, where $\mathcal{L}_0$ is the ideal generated by all the vector fields of the type $\{f(\cdot,u) - f(\cdot,v), u,v \in \mathcal{U}\}$ in the Lie algebra $\mathcal{L}$ generated by all the vector fields of the type $\{f(\cdot,u), u \in \mathcal{U}\}$; see [14]. For systems affine in controls:

$$\dot{x} \;=\; f(x) + \sum_{i=1}^{m} u_i g_i(x) \tag{2}$$

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the algebra $\mathcal{L}_0$ is the Lie algebra generated by all vector fields $\mathrm{ad}_f^k(g_i)$, $k \geq 0$, $i = 1, \ldots, m$.

Given a state x, a control ω defined on $[0, T]$, and a positive $T_0 \leq T$ so that $\xi(t) = \phi(t, x, \omega)$ is defined for all $t \in [0, T_0]$, we may consider the *linearization along the trajectory* (ξ, ω):

$$\dot{z}(t) \;=\; A(t)z(t) + B(t)u(t) \tag{3}$$

where $A(t) := \frac{\partial f}{\partial x}(\xi(t), \omega(t))$ and $B(t) := \frac{\partial f}{\partial u}(\xi(t), \omega(t))$ for each t. A control ω will be said to be *nonsingular for* x if the linear time-varying system (3) is controllable on the interval $[0, T_0]$, for some $T_0 > 0$. When u is analytic, this property is independent of the particular T_0 chosen, and it is equivalent to a Kalman-like rank condition (see e.g. [11], Corollary 3.5.17). Nonsingularity is equivalent to the Fréchet derivative of $\phi(T_0, x, \cdot)$ being full rank at ω.

If ω is nonsingular for $x \in \mathcal{X}$, and T_0 is as above, then $\mathcal{R}^{T_0}(x)$ has a nonempty interior. This is a trivial consequence of the Implicit Function Theorem (see for instance [11], Theorem 6). Thus, if for each state x there is some control which is nonsingular for x, then (1) is strongly accessible. The converse of this fact is also true, that is, if a system is strongly accessible then for each state x there is some control which is nonsingular for x. This converse fact was proved in [10] (the result in that reference is stated under a controllability assumption, which is not needed in the proof of this particular fact; in any case, we review below the proof). The main purpose of this note is to point out that ω can be chosen *independently* of the particular x, and moreover, that in a certain sense a "random" ω has this property. We now give a precise statement of these facts.

A control $\omega : [0, T] \to \mathcal{U}$ will be said to be a *universal nonsingular control* for the system (1) if it is nonsingular for every $x \in \mathcal{X}$.

Theorem 1 *If (1) is strongly accessible, there is an analytic universal nonsingular control.*

Let $\mathcal{C}^\infty([0, T], \mathcal{U})$ denote the set of smooth controls $\omega : [0, T] \to \mathcal{U}$, endowed with the $\mathcal{C}^\infty$ topology (uniform convergence of all derivatives). A *generic* subset of $\mathcal{C}^\infty([0, T], \mathcal{U})$ is one that contains a countable intersection of open dense sets.

Theorem 2 *If (1) is strongly accessible, the set of smooth universal nonsingular controls is generic in $\mathcal{C}^\infty([0, T], \mathcal{U})$, for any fixed $T > 0$.* ∎

The proof will be heavily based on the paper [13]. This in turn generalized a weaker result in [7], which would have given only Theorem 1 for compact $\mathcal{X}$; see also this special case in [15], Lemma 4.10).

In the last section we make some remarks about applications to the control of systems without drift.

2 Proof of Result

We first recall the fact, mentioned above, that for each x there is a control nonsingular for x. This can be proved as follows. Pick x, and assume that the system (1) is strongly accessible. Let y be in the interior of $\mathcal{R}^T(x)$, for some $T > 0$. It follows from [8], Lemma 2.2 and Proposition

2.3, that there exists some real number $\delta > 0$ and some positive integer k so that y is in the interior of the image of

$$F : \mathcal{U}^k \to \mathcal{X}\,,\ (u_1, \ldots, u_k) \mapsto \exp(\delta f_{u_1}) \ldots \exp(\delta f_{u_k})(x)\ ,$$

where we are using the notation $\exp(\delta f_u)(z) = \phi(\delta, z, \omega)$ for the control $\omega \equiv u$ on $[0, \delta]$. This map F is smooth, so by Sard's Theorem it must have full-rank Jacobian at some point $(u_1^0, \ldots, u_k^0)$. This implies that the piecewise-constant control ω, defined on $[0, k\delta]$ and equal to the values u_i^0 on consecutive intervals of length δ, is nonsingular for the given state x, as desired.

What we need next is basically a restatement of the main results in [13]:

Proposition 2.1 Consider the system (1) and assume that $h : \mathcal{X} \to \mathbb{R}$ is a real-analytic function. Let G be the set of states x for which there is for some control $\omega = \omega(x)$ so that $h(\phi(\cdot, x, \omega))$ is not identically zero. Then, there exists an analytic control ω^* —independent of x— so that, for every $x \in G$, $h(\phi(\cdot, x, \omega^*))$ is not identically zero; moreover, for each $T > 0$, the set of smooth such controls is generic in $\mathcal{C}^\infty([0, T], \mathcal{U})$.

Proof. We consider the extended system (with state space $\mathcal{X} \times \mathbb{R}$):

$$\begin{aligned} \dot{x} &= f(x, u) \\ \dot{z} &= 0 \\ y &= zh(x)\ , \end{aligned}$$

which is an analytic system with outputs. Consider two states of the form $(x, 0)$ and $(x, 1)$, with $x \in \mathcal{X}$. A control ω distinguishes these states if and only if $h(\phi(\cdot, x, \omega))$ is not identically zero.

Let ω^* be a control for the extended system which is universal with respect to observability. There are analytic such controls, and the desired genericity holds, by Theorems 2.1 and 2.2 in [13]. Now pick any x in the set G. Then $(x, 0)$ and $(x, 1)$ are distinguishable, and hence ω^* distinguishes among them. This means that $h(\phi(\cdot, x, \omega^*))$ is not identically zero, as desired. ∎

We now prove Theorems 1 and 2. Let (1) be given, and take the following new system:

$$\begin{aligned} \dot{x} &= f(x, u) \\ \dot{Q} &= AQ + QA + BB' \end{aligned}$$

with output $h(x, Q) = \det Q$, where $A = A(x, u) = \frac{\partial f}{\partial x}(x, u)$ and $B = B(x, u) = \frac{\partial f}{\partial u}(x, u)$. This is seen as a system with state space $\mathcal{X} \times \mathbb{R}^{n \times n}$. For an initial state of the form $z = (x, 0)$, and a control ω, the solution $\hat{\phi}$ of the larger system at time t, if defined, is so that

$$h(\hat{\phi}(t, (x, 0), \omega)) \;=\; \det\left(\int_0^t \Phi(t, s) B(s) B'(s) \Phi(t, s)'\, ds\right)$$

(where Φ denotes the fundamental solution of $\dot{x} = A(x, u)x$), so ω is nonsingular for x precisely when $h(\hat{\phi}(t, (x, 0), \omega))$ is not identically zero.

By the remarks at the begining of this section, strong accessibility guarantees that every state of the form $(x, 0)$ is in the set G defined in Proposition 2.1 (for the enlarged system); thus our Theorems follow from the Proposition. ∎

Remark 2.2 Even though the results are stated merely in terms of genericity, rather than probabilistically, the theorems in [13], and hence also Theorem 2, hold in a stronger sense. The universal controls include those whose jets at $t = 0$ do not lie in a countable union of analytic subsets; see the proofs in [13] for details. □

Remark 2.3 More generally, all results will remain true if $\mathcal{X}$ is a real-analytic —second countable, connected,— manifold and $f : \mathcal{X} \times \mathcal{U} \to T\mathcal{X}$ is real-analytic and satisfies $f(x, u) \in T_x\mathcal{X}$ for all (x, u), but this requires care in defining linearizations in a coordinate-free fashion. Also, provided enough transversality is assumed (e.g. use of the observability rank condition in the proof of the result in [13]), one may be able to generalize the proof to the smooth case. □

3 Systems with no Drift

Consider now the case of analytic systems of the type (2) *without drift*, that is, with $f \equiv 0$. For these, Chow's theorem guarantees that the strong accessibility property is equivalent to complete controllability. Such systems have been the focus of much activity lately; see e.g. [4], [5]. It is in general of interest to give algorithms for steering a given state ξ into another given state, which without loss of generality can be taken to be the zero state. The paper [12] explains how to use the result given here for that purpose; next we sketch the main ideas.

The basic principle is to use an iterative technique, either steepest descent or Newton, for finding controls that steer ξ closer and closer to the origin. For this, one starts by obtaining a "nonsingular loop": Generate a generic control ω on some small interval $[0, T]$, and let ξ be the solution obtained by using the control ν on $[0, 2T]$ that equals ω followed by its antisymmetric extension $\omega(2T - t) = -\omega(t)$, $t \in [T, 2T]$. Thus, unless the solution ceases to be defined, it follows that $\xi(2T) = x$. Furthermore, with "probability one" the linearization along this trajectory is nonsingular. Thus a linear least-squares (pseudo-inverse) method can be used to obtain a perturbation ν' on $[0, 2T]$ so that the state that results when using ν' instead of ν is closer to zero than x was (basically Newton's method); alternatively, the adjoint of the differential of ϕ can be used instead of the pseudo-inverse (gradient descent). In either case, the perturbation is easily computed numerically, solving a differential equation. This procedure can now be iterated in the same fashion, starting from the new state that had been reached. If one uses always the same control ν, there is guaranteed convergence in finite time to any arbitrary neighborhood of the origin, for small enough stepsize (see [12]). One may also combine this approach with line searches, or even conjugate gradient algorithms. Of course, such techniques are always used in nonlinear control; see for instance [1], [3]. What appears to be new is the observation that for analytic systems without drift, generic loops indeed provide nonsingularity. This is all also related to the material in [9], which relied on pole-shifting along nonsingular trajectories. The paper [12] presents details and experimental results with this method.

Finally, we mention also a connection with the recent work on time-varying feedback laws for systems without drift (see especially [2] and [6]). In [2], Coron proves for smooth systems with no drift that there is a smooth feedback law $u = k(t, x)$, peridic on t and with $k(t, 0) \equiv 0$, such that, for each initial state x and each t, the control that results by aplying the feedback law, over a period, is nonsingular. The basic step is to prove that there is a smooth choice of controls $u(x)$ so that each u is of fixed length T and nonsingular for x. We can rederive this result —at least in the analytic case— using Theorem 1, as follows. Let ω be any smooth universal nonsingular control, which for convenience we take to be defined on some interval of the form $[T, 2T]$. Patching smoothly, we may extend ω to the interval $[0, 5T]$ in such a manner that it satisfies the antisymmetry condition $\omega(5T - t) = -\omega(t)$ and also $\omega(0) = 0$. Furthermore, by openness of the domain of definition of solutions, there is a scalar smooth real function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ such that the control $\nu(x) = \rho(\|x\|^2)\omega(\cdot)$ is so that the solution $\phi(t, x, \nu(x))$ is defined on all of $[0, 5T]$, and one can also make ρ vanish at zero. The control $\nu(x)$ is as needed in the constructions in [2]; note that it is universal because its restriction to $[T, 2T]$ is. Our result and constructions are closely related to those obtained by Coron, and in a sense connect his work with more classical results in systems theory.

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