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ON THE OBSERVABILITY OF POLYNOMIAL SYSTEMS, I: FINITE-TIME PROBLEMS*

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Abstract. Different notions of observability are compared for systems defined by polynomial difference equations. The main result states that, for systems having the standard property of (multiple-experiment initial-state) observability, the response to a generic input sequence is sufficient for final-state determination. Some remarks are made on results for nonpolynomial and/or continuous-time systems. An identifiability result is derived from the above.

Introduction. This paper deals with observability problems for (deterministic) control systems defined by simultaneous polynomial difference equations, and for other related classes of systems. These problems are natural from a (mathematical) system-theoretic viewpoint, and a strong motivation for their study is also provided by the goal of obtaining explicit solutions to filtering and regulation problems for rather general, yet tractable, classes of nonlinear systems.

Roughly, questions of observability deal with determining the internal state of a (known) dynamical system on the basis of available input/output data. "Observability" is a fundamental system property, due, among others, to the following reasons:

(a) The modern "state-variable" approach to regulator construction is based upon the possibility of feeding back a function of (good estimates of) the state, which must be obtained via "observers" operating on input/output data (in the linear case, "Luenberger observers").

(b) In the stochastic version of the above, the only known effective solution of the optimal nonlinear filtering problem, the Kalman filter, consists precisely of an effective observer construction (for a deterministic system), with parameters optimized on the basis of the available statistical data. This view of Kalman filtering as "deterministic system theory plus elementary theory of Gaussian processes" strongly suggests that a solution in the nonlinear case may be conditional upon a better understanding of nonlinear observers. Moreover, for the known cases, estimation is feasible (in the sense that the error covariance can be made small) only when the system has suitable observability characteristics, as is known for finite-dimensional linear systems (see, e.g., Kwakernaak and Sivan (1972, § 4.4)) and as recently found for infinite-dimensional linear systems (Vinter (1977)).

(c) Observability is one of the main concepts in realization theory, where it appears, under various technical variants, as a characterizing property of canonical systems.

(d) Even in problems not explicitly involving outputs, observability may appear as an important question. To insure the stability of the optimal state regulator, the unstable states must be "observed" by the performance index, as explained intuitively—and proved rigorously in the linear case—in Anderson and Moore (1971, § 3.2).

(e) Problems of identification, i.e., the possibility of determining the input/output behavior of an unknown system on the basis of a limited number of experiments, are closely related to observability questions, as further discussed below.

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The above rough description of observability as a specific property of systems is highly ambiguous, even at an intuitive, nontechnical, level. This ambiguity arises mainly in the following senses: it is not clear whether the state to be determined is that which existed *before* or *after* experimentation, nor whether *simple* or *multiple* experimentation is allowed, nor whether the steps in the experiments can be modified according to partial information (*open-* vs. *closed-loop* observation). Finally, other, rather different, interpretations are possible; for instance, state determination may be only "asymptotic" in that an infinite procedure permits obtaining progressively better estimates of the internal state, as opposed to the above "finite-time" interpretation, where states are precisely determined after experimentation.

As an example of the different possibilities, the canonical realization of any given input/output behavior is multiple-experiment initial-state observable, while an observer is a device solving a single-experiment final-state problem. Thus, for instance, regulator synthesis via the design philosophy "obtain a canonical realization/build an observer/feed-back 'observed' variables" presupposes a positive answer to the question: "is a canonical realization necessarily final-state observable ("reconstructible")?"

Possible observability notions. The main variations on the notions of observability to be studied and compared are, at an intuitive level:

(a) *Observability*: this terminology is reserved for the standard multiple-experiment initial-state notion. A system is observable when any two states can be distinguished by some input/output experiment. Since the experiment (i.e., the input to be applied) depends on the pair of states to be distinguished, practical determination of an initial state assumes the possibility of somehow resetting the system to this (unknown) state after experimentation, or alternatively having a number of copies of the original system, all in the same initial state. This notion of observability appears naturally in realization theory, since "canonical" or "minimal" realizations usually exhibit technical variants of this property (e.g., "algebraic observability," when each coordinate of the initial state can be obtained by *algebraic* manipulations—additions and multiplications—of input data; this property characterizes "canonical" polynomial systems, as discussed in Sontag and Rouchaleau (1975), Sontag (1976a)).

(b) *Single-experiment observability*: there exists a single input (over some finite time interval) which by itself permits the determination, through measurement of ensuing outputs, of the initial state. Clearly this is a much more desirable property than (a); it turns out to be, however, rather restrictive for *discrete*-time systems. (This is not surprising; already Moore (1956) showed that (a) and (b) are far from equivalent, at least for finite automata. For *linear* systems (a) is equivalent to (b), and in fact any long-enough input distinguishes any pair of states, as discussed for instance in Kalman (1968) or Wonham (1974).)

(c) *Final-state determinability*: there is an input sequence w which permits determination of the state of the system resulting *after* w is applied. (In other words, if two states produce the same output sequence under input w, then these two states are necessarily sent into the same state under the action of w.) This property is of interest from a control viewpoint, since control actions can be taken after the state of the system is determined, independently of the state before experimentation. Of course, (b) implies (c). What is not clear is what are the relations, if any, between (a) and (c), since in the former case multiple experimentation is required. It is known that, for finite automata, (a) (called in automata theory a "diagnosing" problem) implies (c) ("homing" problem). This was proved by Moore (1956); expositions are given by Gill

(1962) and Conway (1971); applications to regulation are given by Gatto and Guardabassi (1976). The same result holds for certain types of finite-dimensional systems—e.g., Theorem 4.8 below—; proofs are in fact totally analogous to the finite automata case, with a new type of finiteness (algebraic, linear, or analytic) replacing a set-theoretic finiteness.

(d) *Generic final-state determinability*: while (c) concerns the *existence* of an input such that final states can be determined by testing the system with this input, (d) concerns the much more desirable case when no "experimentation" is needed, but, strictly speaking, "observation" of the input/output behavior is enough. The extreme case of (d) would correspond to that case in which *any* (long-enough) input permits final-state determination. This extreme case is easily seen to be too restrictive, but it may be weakened to only requiring that "almost any" (i.e., a "generic") long-enough input permits this determination. (The rigorous definition of "generic" is a purely technical question, to be discussed later.) In other words, real-time observation of a system, not influencing it in any way (or even, observation of data from past behavior) should be enough for final-state determination. This property is totally different from (c), except in the very special case of linear systems, where (c) = (d). In the automata-theoretic case, "genericity" cannot be even *defined* in a satisfactory way, so this is a genuinely *new* system-theoretic concept.

The main result of this paper states that (a) implies (d) for polynomial systems. Thus, for instance, final states can be determined for canonical realizations of polynomial systems, just observing the "generic" input/output behavior. The proof of the main result uses some elementary notions from algebraic geometry. Since all results remain true when system parameters are not necessarily real or complex but belong to an arbitrary field, everything is stated for arbitrary infinite fields (the finite field case belongs properly to finite automata theory; infinite fields permit identifying polynomials and polynomial functions). Some technical variants of the above observability properties are also discussed and relations between all such notions are clarified.

The last section deals with (i) the particular case of state-affine systems, (ii) generalizations to related classes of systems, in particular state-analytic and continuous-time analytic, and (iii) a restatement of the main result as a system identification problem.

This paper does not treat questions of closed-loop and/or asymptotic observability (closely related to problems of stability), nor the effective construction of "observers." Another interesting set of problems left open is that of finding numerical values for smallest lengths of observability experiments; except for the state-affine case, only qualitative results are given (even for the case of finite automata many of these problems are still unresolved; see Conway (1971)).

The results of this paper strongly suggest that the proper definition of "observer" in the nonlinear context may be that of a dynamical system which determines the state of the "observed" system on the basis of a *generic* set of data.

1. Definitions and characterizations. Let k be an arbitrary but fixed infinite field, and m, n, p arbitrary positive integers. Recall that an *algebraic subset* S of the affine space k^q, $q \geqq 0$, is a set defined by polynomial equations $S = \{Q_i(x_1, \cdots, x_q) = 0\}$. An *irreducible* algebraic set is one which cannot be expressed as the union of two proper algebraic subsets. In this context, a subset R of an irreducible algebraic set S is *generic* when its complement is contained in a proper algebraic subset of S. (These definitions are justified by the fact that for $k = \mathbb{R}$ or $\mathbb{C}$, a proper algebraic set is "thin" in most possible senses, including Baire category and measure-theoretic.)

DEFINITION 1.1. A (discrete-time) *polynomial system* Σ is given by a set of equations

$$x(t+1)=P(x(t),u(t)), \qquad y(t)=h(x(t)), \qquad t=0,1,2,\cdots,$$

where *inputs* $u(t)$, *states* $x(t)$ and *outputs* $y(t)$ belong to algebraic subsets U of k^m, X of k^n, and Y of k^p respectively, U is irreducible, and $P: X\times U\to X$ and $h: X\to Y$ are polynomial maps.

Allowing proper algebraic subsets, rather than insisting on finite dimensional spaces, for U, X, Y, permits increasing the generality of the results to include input or state constraints of a polynomial type. The irreducibility assumption on U is made purely for technical convenience. For instance, the unit real circle $U=\{x^2+y^2-1=0\}$, as well as any space k^m, are admissible input sets. Nonpolynomial systems will be considered later.

Some extra notation will be useful. The extension of P to input sequences is also denoted by $P: X\times U^*\to X$ (for the empty sequence e, $P(x,e)=x$). Applying an input sequence $w=u_1\cdots u_r$ to a system in state x produces an output sequence

$$H^w(x)=(h(x),h(P(x,u_1)),\cdots,h(P(x,w)))$$

in Y^{r+1}.

In what follows, Σ is a fixed polynomial system. The input sequence w *distinguishes* between the states x and z iff $H^w(x)\neq H^w(z)$. The following are several possible definitions of "observability":

(A) *Single-experiment observability*: there exists an input sequence w which distinguishes every pair of states.

(B) *Single-experiment observability with a generic input*: there are a positive integer r and a generic subset R of U^r such that any w in R distinguishes every pair of states.

(C) *Observability*: each pair of states can be distinguished by some input sequence.

(D) *Finite observability*: there are a positive integer r and input sequences $w_1,\cdots,w_s$ of length r such that each pair of states x, z is distinguished by some w_i.

(E) *(Finite) observability with generic inputs*: there are integers r, s and a proper generic subset R of U^{rs} such that (D) holds for any set $w_1,\cdots,w_s$ of inputs of length r for which $(w_1,\cdots,w_s)$ is in R.

(F) *Algebraic observability*: for each polynomial function $\hat{q}: X\to k$ there are input sequences $w_1,\cdots,w_s$ and a polynomial function $q: Y^s\to k$ such that $\hat{q}(x)=q(h(P(x,w_1)),\cdots,h(P(x,w_s)))$ for all x in X.

(G) *Final-state determinability*: there is an input sequence w such that for each pair of states x, z either $H^w(x)\neq H^w(z)$ or $P(x,w)=P(z,w)$.

(H) *Final-state determinability with generic inputs*: there are a positive integer r and a generic subset R of U^r such that (G) holds for all w in R.

The characterizations below are useful in checking observability. They are stated in terms of the polynomial functions h_{ij} defined as follows by induction on j. First, an (infinite) basis B is chosen for the vector space of all polynomial functions on U. (If $U=k^m$, the natural choice is the set of all m-variable monomials; if U is a proper algebraic set one may choose a linearly independent subset of such monomials.) The polynomial map $h: X\to Y\subseteq k^p$ gives rise to p polynomial functions

$$h_{01},\cdots,h_{op}$$

by composing with the coordinate projections. If the h_{ij} have been defined for some i

and $j = 1, \cdots, q_i$, one may express

$$h_{ir}(P(x, u)) = \sum_s a_{rs}(x) g_s(u), \qquad r = 1, \cdots, q_i \tag{1.2}$$

for some finite subset $g_1, \cdots$ of B. The $h_{i+1,j}$ are then given by the a_{rs}, $r = 1, \cdots, q_i$, all s, listed in any order except that an a_{rs} is dropped if it is redundant, i.e., if a_{rs} is in the algebra generated by the previous h_{ij}'s.

LEMMA 1.3. (a) *Σ is observable iff the map*

$$x \to (h_{11}(x), h_{12}(x), \cdots, h_{21}(x), \cdots) \tag{1.4}$$

is one-to-one.

(b) *Σ is algebraically observable iff each coordinate function $x_i : X \to k$, $i = 1, \cdots, n$, is a polynomial combination of the $h_{ij}(x)$.*

Proof. Observability clearly implies that (1.4) is one-to-one, since the functions $x \mapsto h(P(x, w))$ are combinations of the h_{ij}. Conversely, from Sontag (1976a, "Main lemma" (10.7)), the $h_{ij}(\cdot)$ are linear combinations of the functions $h(P(\cdot, w))$; it follows that if x, z are indistinguishable then $h_{ij}(x) = h_{ij}(z)$ for all i, j. The proof of (b) is similar.

The above result permits checking observability without having to consider, for each pair of states, if there is an input sequence separating them. The result can be tightened considerably, in that it is theoretically possible to specify an integer s (which depends only on the degrees of the polynomials defining Σ) such that it is enough to check, in order to determine (algebraic) observability, if the map

$$X \to Y^{sr_s} : x \to (h_{11}(x), \cdots, h_{sr_s}(x)) \tag{1.5}$$

is one-to-one (or if each coordinate function is a combination of the h_{ij}'s); this follows from the decidability theory in commutative algebra, as remarked in Sontag and Rouchaleau (1975). The problem of checking if (1.5), or a general polynomial map, is one-to-one is very difficult, and it appears also in trying to determine if a system is observable with respect to a *fixed* input $w = u_1 \cdots u_r$, since one must then check

$$x \mapsto (h(x), h(P(x, u_1)), \cdots, h(P(x, w)));$$

in that context, sufficient conditions for one-to-oneness (with k = reals) were surveyed by Fitts (1972).

As a very simple illustration of Lemma 1.3, take the polynomial system Σ_1 with equations

$$x_1(t+1) = x_2(t), \qquad x_2(t+1) = x_1(t), \qquad x_3(t+1) = x_3(t),$$

$$x_4(t+1) = x_1(t)u_1^2(t) + x_2(t)u_2^2(t) + x_3(t),$$

$$y(t) = x_4(t),$$

where $U = k^2$, $X = k^4$, $Y = k$. Then h_{01} = the coordinate function x_4. From the fourth equation, and noting that u_1^2, u_2^2, 1 are linearly independent functions, one has x_1, x_2, x_3 for the h_{1j}. Thus Σ is algebraically observable, and in particular observable.

If, instead, now U is the circle $u_1^2 + u_2^2 = 1$, then $u_2^2 = 1 - u_1^2$ as functions on U, so

$$x_1 u_1^2 + x_2 u_2^2 + x_3 = (x_1 - x_2) u_1^2 + (x_2 + x_3),$$

so $h_{11} = x_2 + x_3$, $h_{12} = x_1 - x_2$. Now, in obtaining the h_{2j}, $x_1 - x_2$ yields $x_2 - x_1$ (from the first two equations), which is $-(x_1 - x_2)$ and hence belongs to the algebra generated by previous h_{ij}'s. On the other hand, $x_2 + x_3$ yields $x_1 + x_3$, which is equal to $(x_1 - x_2) +$

(x_2+x_3), hence in the algebra generated by previous h_{ij}'s. Thus no h_{ij} are added for $i=2,3,\cdots$. The system is therefore *not* observable, since

$$(x_1, x_2, x_3, x_4) \mapsto (x_4, x_1-x_2, x_2+x_3)$$

is not one-to-one. In fact, the indistinguishable pairs of states are those in the lines parallel to $\{x_4=0, x_1-x_2=0, x_2+x_3=0\}$.

When k = reals or complexes, observability can be checked using only inputs of arbitrarily small amplitude; this is easily derived from the above characterization using Sontag (1976a, Lemma (2.11)).

2. Implications among observability notions.

THEOREM 2.1. *With the notations in the previous section, the only implications are those indicated by the following diagram*:

$$\begin{array}{c} \mathrm{F} \\ \downarrow \\ \mathrm{B} \to \mathrm{A} \to \mathrm{C} = \mathrm{D} = \mathrm{E} \to \mathrm{H} \to \mathrm{G}. \end{array} \tag{2.2}$$

Proof. The following implications are immediate from the definitions: $E \to D \to C$, $B \to A \to C$, $H \to G$, and $F \to C$. That $C \to D$ is proved in Sontag and Rouchaleau (1975, Prop. 7.2). Proofs are given below for $D \to E$ (2.4) and $C \to H$ (Theorem 3.5). To complete the proof of 2.1, counterexamples must be given to $B \to F$, $A \to B$, $F \to A$, $G \to H$ and $H \to C$. For the latter it is sufficient to consider the trivial system with both transition and output maps equal to zero: after one step, the state is known (zero), no matter which input was "applied", but the initial state cannot be determined. The remaining counterexamples are given by:

$B \to F$: let $k=\mathbb{R}$, $X=Y=k$, U = arbitrary, $P(x,u)=0$ for all x, u, and $h(x)=x^3$.

$A \to B$: let $U=Y=k$, $X=k^2$, and Σ_2 given by

$$x_1(t+1)=0, \qquad x_2(t+1)=x_1(t)+x_1^2(t)u(t), \qquad y(t)=x_2(t).$$

An input $w=u_1\cdots u_r$ distinguishes initial states if and only if $u_1=0$. But the set of all such inputs is not generic in U^r, for any r.

$F \to A$: let $U=Y=k$, $X=k^2$, and Σ_3 given by

$$x_1(t+1)=0, \qquad x_2(t+1)=x_1(t)u(t)-x_1^2(t), \qquad y(t)=x_2(t).$$

Algebraic observability follows from criterion 1.3: recursively, one generates x_2 and then x_1 (and x_1^2, which is redundant). But no single sequence w serves to distinguish every pair of states: let $w=uw'$, with u in U; if $u=0$ then $(1\ 0)'$ and $(-1\ 0)'$ are not distinguished by w, while if $u \neq 0$ then $(u\ 0)'$ is indistinguishable from $(0\ 0)'$.

$G \to H$: let $U=X=k$ and

$$x(t+1)=x(t)u(t), \qquad u(t)=0.$$

Then $w=u_1\cdots u_r$ determines the final state if and only if some $u_i=0$. The set of all such w is not generic.

It will be now proved that finite observability (D) implies, for polynomial systems, generic finite observability (E). This is somewhat surprising because the corresponding implication for single-experiment observability ($A \to B$) is false. (Σ_3 above is, however, generically finitely observable: any two length-one inputs u, v permit observing $x_1+x_1^2u$, $x_1+x_1^2v$, hence also

$$x_1=[(x_1+x_1^2u)v-(x_1+x_1^2v)u](v-u)^{-1}$$

is known. Thus the generic set R of all (u,v) in $U^2=k^2$ with $u-v\neq 0$ satisfies definition E.)

The following algebraic result is needed; its proof is essentially the same as that in Sontag (1976a, Lemma (10.6)):

LEMMA 2.3. *Let V, W be algebraic sets, W irreducible, and $f: V \times W \to k$ a polynomial function. There exists then an integer s and a nonzero polynomial function $d: W^s \to k$ such that, for each $w, w_1, \cdots, w_s$ in W there are $a_1, \cdots, a_s$ in k with*

$$d(w_1, \cdots, w_s)f(v, w) = \sum_i a_i f(v, w_i).$$

One can now give the

(2.4) *Proof of $D \to E$.* Assume that Σ is finitely observable, and let $\bar{w}_1, \cdots, \bar{w}_t$ be such that $x \neq z$ implies $h(P(x, \bar{w}_i)) \neq h(P(z, \bar{w}_i))$ for some i. For each i, let $f_i = h \circ P: x \times U^{r_i} \to k$. Applying 2.3 with $V = X$, $W = U^{r_i}$, $f = f_i$, a $d_i: U^{s_i r_i} \to k$ is obtained for each i. Let q:=largest of the s_i. In the definition of generic observability, take r:=largest of the r_i and s:=$t.q$. An element of U^{rs} can be written as

$$(w_{11}, \cdots, w_{t1}, w_{12}, \cdots, w_{t2}, \cdots, w_{tq}),$$

with each w_{ij} in U. Define the proper algebraic subset F of U^{rs} by the equations

$$d_i(w_{i1}, \cdots, w_{is_i}) = 0, \qquad i = 1, \cdots, t.$$

Then generic observability holds with R = complement of F.

3. Proof of the main result.

LEMMA 3.1. *For any polynomial system Σ there exists an integer $r \geqq 0$ and a proper algebraic subset F of U^r such that, for every $w = (u_1, \cdots, u_r)$ not in F, and for any x, z in X,*

$$H^w(x) = H^w(z)$$

implies that

$$P(x, w) \text{ is indistinguishable from } P(z, w).$$

Proof. Since $Y \subseteq k^p$ for some integer p and since a union of proper algebraic subsets of U^r is again a proper algebraic subset, it is sufficient to prove the lemma with $Y = k$. The general case can be reduced to this by considering the p projections $Y \to k$.

Let $s \geqq 0$ be such that any pair of distinguishable states is already distinguished by inputs of length $\leqq s$ (Sontag and Rouchaleau (1975, Cor. 7.3)).

For any algebraic set Z, let $A(Z)$ denote the algebra of polynomial functions on Z. Irreducibility of U means that $A(U^t)$ is an integral domain for all t. Let D be the direct limit of the sequence of k-algebras

$$A(U) \to \cdots \to A(U^t) \to A(U^{t+1}) \to \cdots,$$

where

$$A(U^t) \to A(U^{t+1}) = A(U^t) \otimes A(U): f \to f \otimes 1.$$

Let K be the quotient field of D (which is an integral domain, being a direct limit of integral domains); K contains all $A(U^t)$.

Since $Y = k$, a polynomial map $X \times X \times U^t \to Y$ is an element of $A(X \times X) \oplus A(U^t)$; in particular the functions h_t defined by

$$h_t(x, z, u_1, \cdots, u_t) := h(P(x, u_1, \cdots, u_t)) - h(P(z, u_1, \cdots, u_t))$$

are in $A(X \times X) \otimes K$. The latter is a finitely generated algebra over the field K, hence Noetherian. Thus there is some integer r such that all h_t are in the ideal of $A(X \times X) \otimes K$ generated by $h_0, \cdots, h_r$. In particular, there are therefore equations

$$\text{(3.2)} \qquad ch_{r+j} = \sum_{t=0}^{r} a_{jt} h_t, \qquad j = 1, \cdots s,$$

with all a_{jt} in $A(X\times X)\otimes D$ and c a nonzero element of D. Since D is the union of the $A(U^t)$, there is some integer q such that all a_{jt} are in $A(X\times X)\otimes A(U^q)$ and c is in $A(U^q)$. Without loss of generality, we shall assume that $q \geqq r+s$.

Define the proper algebraic set

$$F := \{(u_1, \cdots, u_r) \text{ in } U^r \text{ such that } c(u_1, \cdots, u_r, \cdots, u_q) = 0 \text{ for all } (u_{r+1}, \cdots, u_q)\}.$$

Claim: F satisfies the requirements of the lemma. Indeed, assume that $\underline{w} = (u_1, \cdots, u_r)$ is not in F. Take x, z in X such that $h(P(x, u_1, \cdots, u_t)) = h(P(z, u_1, \cdots, u_t))$ for all $t = 0, \cdots, r$, i.e.,

$$h_t(x, z, u_1, \cdots, u_t) = 0, \qquad t = 0, \cdots, r. \tag{3.3}$$

Denote $\underline{x} := P(x, \underline{w})$, $\underline{z} := P(z, \underline{w})$. It must be proved that $\underline{x}$, $\underline{z}$ are indistinguishable.

Assume that $\underline{x}$, $\underline{z}$ are distinguished by an input sequence v, which can be taken of length j, $0 \leqq j \leqq s$, by definition of s. let

$$F_1 := \{w \text{ in } U^j \text{ such that } h_{r+j}(x, z, \underline{w}, w) = 0\};$$

this is an algebraic set, proper because v is not in F_1. Let

$$F_2 := \{w \text{ in } U^j \text{ such that } c(\underline{w}, w, w') = 0 \text{ for all } w' \text{ in } U^{q-r-j}\};$$

this is also an algebraic set, and it is proper because $\underline{w}$ was taken not in F.

It follows that $F_1 \cup F_2$ is also a proper algebraic set. Let then w be in neither F_1 nor F_2. Then $c(\underline{w}, w, w') \neq 0$ for some w', so

$$c(\underline{w}, w, w')h_{r+j}(x, z, \underline{w}, w) \neq 0. \tag{3.4}$$

But (3.2), (3.3) and (3.4) taken together are contradictory.

THEOREM 3.5. *Observability implies, for polynomial systems, final-state determinability with generic inputs.*

Proof. Immediate from the lemma.

Remark 3.6. As shown in Sontag (1976), canonical realizations Σ_f of polynomial response maps are not, in general, polynomial systems. So the Theorem above is not applicable directly ($A(X_f)$ is not Noetherian). However, if f admits a polynomial realization Σ, then the reachable states of Σ_f form a set which is a quotient of the reachable set of Σ. Then Lemma 3.1 can be applied to Σ, implying that the reachable part of Σ_f does satisfy Theorem 3.5. Another generalization regards the case in which X is a nonaffine variety: taking an affine cover of X, equations as in (3.2) result on each piece of the corresponding decomposition of $X\times X$, and Lemma 3.1 is again true. This generalization is of interest in identifiability questions, with nonaffine parameter spaces.

4. Particular cases, applications, generalizations.

(Polynomial) State-affine systems. For this class of systems, whose realization theory was studied in Sontag (1976b), most of the implications among observability properties are easy generalizations of the linear case.

DEFINITION 4.1. A polynomial system Σ is *state-affine* iff $X = k^n$, $U = k^m$, P is affine (linear + translation) in states, and h is linear.

Fixing a basis in X, the equations for a state-affine system have the form

$$x(t+1) = F(u(t))x(t) + G(u(t)),$$
$$y(t) = Hx(t),$$

where $F(\cdot)$ is a (polynomial) matrix function of u, $G(\cdot)$ is a vector function of u, and H

is a constant matrix. A particular case is that of internally-bilinear systems (see, e.g., Brockett (1972), D'Alessandro, Isidori and Ruberti (1974), Fliess (1973)), when F and G are themselves linear or affine in u.

For state-affine systems the table of implications given in § 2 collapses to

$$A = B \to C = D = E = F \to H \to G.$$

It must be proved that C → F and A → B. That C → F is clear from (1.3), since the h_{ij} are linear functions of x, observability thus meaning that the coordinates x_i are *linear* combinations of the h_{ij}. (An explicit matrix criterion for observability is described in Sontag (1976b, Lemma 1.32).) That A → B follows from the following characterization, which can be also generalized to the case U = proper algebraic set by considering a basis of functions $U \to k$ instead of all monomials u^{α_i}:

PROPOSITION 4.2. *The state-affine system* Σ *is single-experiment observable iff*

$$\text{rank}\begin{bmatrix} H & & \\ HF(U_1) & & \\ \vdots & & \\ HF(U_{n-1}) & \cdots & F(U_1) \end{bmatrix} = n \tag{4.3}$$

over the field $K = k(U_1, \cdots, U_{n-1})$ *of rational functions in* $m(n-1)$ *variables. Moreover, if* (4.3) *holds, then any* $w = u_1 \cdots u_{n-1}$ *such that the rank in* (4.3) *remains n after specializing* $U_1 = u_1, \cdots, U_{n-1} = u_n$ *solves the single-experiment observation problem.* (The set of all such w is generic.)

For example, consider the three-dimensional state-affine system Σ_4:

$$x_1(t+1) = x_1(t)u_1(t) + x_2(t)u_2(t) + x_3(t)u_3(t) \quad x_2(t+1) = 0, \quad x_3(t+1) = 0,$$

$$y(t) = x_1(t).$$

This system is observable, but (with $U_1 = [U_{11}, U_{21}, U_{31}]$, $U_2 = [U_{12}, U_{22}, U_{32}]$) the matrix in (b) is

$$\begin{bmatrix} 1 & 0 & 0 \\ U_{11} & U_{21} & U_{31} \\ U_{12}U_{11} & U_{12}U_{21} & U_{12}U_{31} \end{bmatrix},$$

which has rank two, so the system is not single-experiment observable.

Proof of Proposition 4.2. Single-experiment observability with an input $w = u_1 \cdots u_t$ is equivalent to the map $x \to H^w(x)$ being one-to-one. Since

$$H^w(x) = (Hx, \cdots, HF(u_t) \cdots F(u_1)x) + \text{translation},$$

H^w is one-to-one if and only if the rank of

$$\begin{bmatrix} H & & \\ HF(u_1) & & \\ \vdots & & \\ HF(u_t) & \cdots & F(u_1) \end{bmatrix}$$

is n. Being full rank means that some $n \times n$ minor is nonzero, so the same minor is nonzero as a polynomial in $u_1 = U_1, \cdots, u_t = U_t$ (variables) over K. Thus the rank of this matrix is also n. Consider the chain of subspaces V_r of K^n defined by

$$V_r := \text{span over } K \text{ of } F'(U_1) \cdots F'(U_i)H_j, \qquad i < r, \ j = 1, \cdots, p,$$

where H_j is the jth column of H. It is easy to see that if $V_r = V_{r-1}$ for some r then $V_r = V_{r+1} = \cdots$. Thus $V_n = V_{n+1} = \cdots = V_t$. This proves that the rank in (4.3) is n. The rest of the statement is clear from the proof.

The proof of Lemma 3.1 can be rederived for the state-affine case, using only linear-algebraic methods (over rational function fields). A constructive proof is thus obtained, with the precise value $r = n$.

Parametric identification. The result in § 3 can be applied to the following identification problem: a family of polynomial systems is given, parametricized by polynomial functions. It follows that, if the output is known for a generic input, then the future input/output behavior of the system is completely determined. Specifically, considering a family (or "structure"—see Bellman and Aström (1970)):

$$\Sigma_\lambda : \begin{cases} x(t+1) = P(\lambda, x(t), u(t)) = P_\lambda(x(t), u(t)), \\ y(t) = h(\lambda, x(t)) = h_\lambda(x(t)), \end{cases} \tag{4.4}$$

where $P: \Lambda \times X \times U \to x$ and $h: \Lambda \times X \to y$ are polynomial maps and Λ, X, U, Y are algebraic subsets of k^q, k^n, k^m, k^p respectively, U irreducible. The input/output map of Σ_λ for initial state x is

$$f_{\lambda,x}: w \mapsto H_\lambda^w(x).$$

THEOREM 4.5. *There is a positive integer r and a generic subset R of U^r such that, for each input sequence $w = u_1 \cdots u_r$ in R,*

$$f_{\lambda,x}(w) = f_{\mu,z}(w)$$

implies that

$$f_{\lambda,x}(wv) = f_{\mu,z}(wv) \quad \textit{for all input sequences } v.$$

Proof. Let $\hat{\Sigma}$ be the polynomial system with $\hat{X} := \Lambda \times X$; $\hat{U} = U$, $\hat{Y} = Y$ and equations

$$\lambda(t+1) = \lambda(t), \qquad x(t+1) = P(\lambda(t), x(t), u(t)),$$
$$y(t) = h(\lambda(t), x(t)).$$

Then Lemma 3.1 applied to $\hat{\Sigma}$ gives an r and an R such that $\hat{H}^w(\lambda, x) = f_{\lambda,x}(w)$ determines the final state $(\lambda, x(r))$ up to indistinguishability, i.e., all future outputs coincide.

For instance, the future input/output behavior of the system Σ_5 (with $U = Y = k$, $X = k^3$):

$$x_1(t+1) = x_3(t), \quad x_2(t+1) = \lambda x_1(t) + x_2(t), \quad x_3(t+1) = x_2(t)u(t) + x_2(t),$$
$$y(t) = x_3(t)$$

is uniquely determined once that the output corresponding to a $w = u_1u_2u_3$, $u_i \neq -1$ is known, since $x_3(0)$, $x_2(0)$, $\lambda x_1(0)$, and $\lambda x_3(0)$ are successively obtained. (Note that the parameter λ itself is in general not determinable, for instance if $x_3(0)$ is zero; an additional "parameter-identifiability" condition is needed on the given family in order to determine λ.)

The above definition of family of systems includes the case in which the identification is desired of a system of which one only has a bound on dimension and a bound on the degree of the polynomials in its defining equations: it is then obviously enough to add one parameter for each unknown coefficient and one for each coordinate of the (unknown) initial state. (Such a parametrization is of course highly redundant; realization theory may give lower order ones; see Remark 3.6.).

Nonpolynomial systems. Many of the remarks and results of previous sections apply to more general finite-dimensional systems than polynomial systems, i.e., systems

$$(4.6) \qquad x(t+1) = P(x(t), u(t)), \qquad y(t) = h(x(t))$$

where, say, U, X, Y are subsets of $\mathbb{R}^m$, $\mathbb{R}^n$, $\mathbb{R}^p$ (or corresponding complex spaces) and P, h are analytic, infinitely differentiable, or just continuous, either in both x and u, or only in x. The "generic" conditions, defined in terms of algebraic sets, should of course, be redefined according to the category to be worked on (analytic sets, nowhere dense sets, etc.). We conjecture, but have not yet proved, that Theorem 3.5 is true in the analytic case. (In certain cases this is trivially true, e.g. for "analytic state-affine systems," when P, G are analytic in u and linear in x.) The weaker result C→G: observability implies final-state determinability, holds for the following kind of system (analogous definitions for the complex case):

DEFINITION 4.7. A *state-analytic system* Σ has equations (4.6) with X an open subset of $\mathbb{R}^n$, Y a subset of $\mathbb{R}^p$, and both P and h analytic in x.

(U, and the dependence of P on inputs u, are completely arbitrary.)

THEOREM 4.8. *Let the state-analytic system Σ be observable. Let K be any compact subset of X. Then there exists an input sequence w such that, for each pair of states x, z in K, either $H^w(x) \neq H^w(z)$ or $P(x, w) = P(x, z)$.*

Proof. For each input sequence w, let

$$K_w := \{(x, z) \text{ in } K \times K \mid H^w(x) = H^w(z)\}.$$

Each K_w is a subset of the compact set $K \times K$, defined by analytic equations in $X \times X$. It can be proved, using compactness and applying the generalized form of the Weierstrass preparation theorem given by Hervé [1963, Thm. 2.7, Cor. 3], that sets defined by analytic equations satisfy a descending chain condition on compact sets. Thus, there is a minimal K_w.

Then w satisfies the conclusion of the theorem. Indeed, assume that, on the contrary, there is a pair (x, z) in $K \times K$ with $H^w(x) = H^w(z)$ but $P(x, w) \neq P(z, w)$. By observability of Σ, there is an input sequence v such that

$$H^{wv}(x) = H^w(P(x, v)) \neq H^w(P(z, v)) = H^{wv}(z).$$

So K_{wv} is properly contained in K_w, contradicting minimality of the latter.

Except for our use of the result from analytic functions, the above is essentially the standard proof of C→G for automata (all sets finite, so there is again a minimal K_w) and for internally-bilinear systems (all sets are linear subspaces), in particular as given by Muchnik (1973) and independently (strictly speaking, for continuous-time) by Grasselli and Isidori (1977).

The compactness assumption cannot be dropped: the one-dimensional state-analytic system Σ_6 with equations

$$x(t+1) = \tfrac{1}{2}x(t), \qquad y(t) = \sin x(t)$$

is observable but is not final-state determinable with any (finite length) input. Similarly, infinite differentiability (instead of analyticity) will not be sufficient: consider the one-dimensional system Σ_7 with $X := (-1, 1)$ and

$$x(t+1) = a(x(t)), \qquad y(t) = b(x(t)),$$

where a, b are infinitely differentiable with $a(x) = 2x$ on $[-\frac{1}{4}, \frac{1}{4}]$ (arbitrary otherwise), and $b(x) = 0$ on $[-\frac{1}{4}, \frac{1}{4}]$ and bijective in the complement. Then Σ_7 is observable, but pairs of states in $K=[-\frac{1}{4},\frac{1}{4}]$ do not satisfy the conclusion of Theorem 4.8. (It is interesting to remark that in both these examples there is an "asymptotic" final-state determinability; infinite-time conditions are more appropriate for nonpolynomial systems.)

Continuous-time. Many of the previous results can be generalized to continuous-time finite-dimensional systems

$$\dot{x}(t)=P(x(t),u(t)),\qquad y(t)=h(x(t)), \tag{4.9}$$

where appropriate restrictions are placed on the state-space, input set, spaces of input functions, and P, h. The continuous case is simpler than the discrete one, due to the time-reversibility of (finite dimensional) differential equations. This implies that no information is lost when an experiment is performed on such a system, i.e., the maps

$$x\mapsto P(x,w) \tag{$*$}$$

are homeomorphisms for all w ($P(x,w)$ = solution of (4.9) at time T with $x(0)=x$ and input $w(\cdot)$ on $[0,T]$). It follows that final-state-determination is equivalent to single-experiment observability. If P is analytic in both x, u and h is analytic in x (so that the maps ($*$) are analytic), and under suitable technical assumptions insuring existence and uniqueness of solutions of (4.9) for admissible input functions w, it follows by essentially the same argument as in Theorem 4.8 that *observability implies single-experiment observability*. (The internally-bilinear case of this result was proved via linear-algebraic techniques by Grasselli and Isidori (1977).) When P, h are *polynomial* in x, the methods in Sontag and Rouchaleau (1975) can be applied to jets of outputs corresponding to smooth inputs, resulting in finiteness results for the continuous case.

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