NEW RESULTS ON POLE-SHIFTING FOR PARAMETRIZED FAMILIES OF SYSTEMS*

by

M.L.J. Hautus**
Department of Mathematics
Technical University Eindhoven
Eindhoven, Netherlands

Eduardo D. Sontag
Department of Mathematics
Rutgers University
New Brunswick, NJ 08903, USA

ABSTRACT

New results are given on the pole-shifting problem for commutative rings, and these are then applied to conclude that rings of continuous, smooth, or real-analytic functions on a manifold X are PA rings if and only if X is one-dimensional.

---

* **Research Supported in Part by US Air Force Grant AFOSR-85-0196E.**

** This work was completed while the author was a visitor at the Department of Mathematics, Rutgers, The State University of New Jersey.

1. Introduction

This paper establishes new results regarding control problems for parametrized families of pairs ("systems") {(A(x),B(x)), x∈**X**}, where A(x) is an n×n and B(x) is an n×m real matrix for each x (with n,m fixed integers), and the parameter x belongs to a manifold **X**. To be found is a new parametrized family {K(x), x∈**X**} such that a given design criterion is satisfied by the *closed-loop* matrix A(x)+B(x)K(x) for all x, and the K(x) depend in a suitably 'nice' form on the parameter. The design criterion we shall be concerned with is that of *pole assignment*, and consists of obtaining arbitrary characteristic polynomials for the closed loop matrix. (Pole assignment problems are a central issue in the more "classical" case studied in control theory, that of single systems; see e.g. [KA].) 'Nice' will mean continuous, smooth (C^∞), or (real-)analytic. See [S2] for an introduction to the topic of control of families of systems, including several motivating examples as well as a survey of other problems, different from pole-assignment, that are also of interest.

The results to be given, that the pole-assignment problem is solvable in general if and only if **X** has dimension 1, will be obtained as consequences of new results on *systems over rings*, which extend those in [S1], [MO], [BY], [BSSV], [TA1], [TA2], [SH], and other references. New necessary conditions are given, as well as sufficient conditions that apply for various clases of rings, including Dedekind domains. These results should be of independent interest. For an introduction to the general topic of systems over rings, see [S1], [HA], [KA], [BBV], and the references there and in [S2].

2. Preliminaries

Let R be a reduced (no nilpotents) commutative ring. "(Sub)module" will always mean *finitely generated* (sub)module, unless otherwise stated, and "projective" will always mean (finitely generated) projective of constant rank. All maps are R-linear, unless otherwise stated. When R is in particular an integral domain with quotient field F, the *rank* of the module M is the dimension of M⊗F, and if f: M→N, *rank*[f] is the rank of f⊗F: M⊗F→N⊗F. For any R and any matrix G over R, (equivalently, any map between free modules,) *rank*[G] is the determinantal rank of G, the largest integer r such that there is a nonzero r-minor of G. For integral domains, this coincides with the above definition.

Let Spec(R) denote the prime spectrum of R. If p∈Spec(R) and h is in R, we write h(p) for the residue of h in R/p. Thus, h is in p iff h(p)=0. This notation will also be used for R-modules and maps, so that M(p) denotes the R/p-module M⊗(R/p) and, if f: M→N, then f(p) is map f⊗(R/p): M(p)→N(p). If G is a vector or even a matrix over R, G(p) denotes the vector or matrix obtained by componentwise reduction; this is consistent with the notations for abstract maps and modules just given, when applied to free modules.

A *nondegenerate submodule G* of an R-module M will be one for which the following property holds: if ι: *G*→M is the inclusion map, and p is in Spec(R), then ι(p): *G*(p)→M(p) is a nonzero map. This is equivalent to the requirement that rank[(M/*G*)(p)] < rank[M(p)] for all p∈Spec(R). If M is a direct summand of the module N, then *G*⊆M is nondegenerate as a submodule of M iff it is nondegenerate as a submodule of N. When M=R^n is free and G is an n×m matrix over R whose columns span *G*, the nondegeneracy condition can be expressed simply as: c(G) = R, where c(G) is the *content* of the matrix G, the ideal generated by its entries.

We shall be interested below in determining conditions under which a submodule *G* of the projective module M contains a rank 1 (projective) summand of M. As before, if M is a summand of N then *G*⊆M has such a summand with respect to M iff the same is true with respect to N. If M is free and G is a matrix as above, this is equivalent to the existence of a matrix V such that the columns of GV span a rank 1 summand of M. This latter condition is in turn equivalent to the requirement that the rank of GV be constant, equal to 1, when reducing modulo all primes in Spec(R) (by [BO], II.5, exercise 4, applied to coker(GV); recall the assumption that R is reduced). In other words, c(GV) = R and GV has rank ≤ (hence,=) 1.

Let Ω = Ω(R) ⊆ Spec(R) be the set of maximal ideals of R. A *principal closed set* (relative to the Zariski topology) of Ω is a set of the form

$$V(h) := \{p \in \Omega \text{ s.t. } h(p) = 0\},$$

with h in R. The set of p∈Ω for which h(p)≠0 will be denoted supp(h), and we use the same notation for vectors and matrices. For any matrix G, c(G)=R is equivalent to supp(G)=Ω. The *constructible* subsets of Ω are those in the Boolean algebra generated by the sets of the form V(h). A constructible C has *codimension at least one* if C is contained in some V(h) with h≠0. Consider the following two properties for rings R:

(*) Every constructible of codimension at least one is a principal closed set.

(2) If *G* is a nondegenerate (finitely generated) submodule of the projective R-module M, there is a rank 1 summand of M contained in *G*.

Lemma 1: Property (*) implies property (2)

Proof: Embedding M as a summand of a free module, we can by the above remarks restrict attention to the case where M is free. Let G be an n×m matrix whose columns generate *G*. Thus c(G) = R, (equivalently, supp(G)=Ω,) and we seek a V such that c(GV) again equals R (i.e. supp(GV)=Ω,) but such that rank[GV]=1. If G has rank 1, we can choose V = identity. So assume that G has a 2×2 minor Δ≠0. Let g_1, g_2 be the columns of G involved in Δ. We shall find a linear combination $g = \alpha g_1+\beta g_2$ with the property that supp(g) = supp(g_1,g_2); the result will then follow by induction on the number of columns m. Let

$C_o := \{p\in\Omega \text{ s.t. } \Delta(p)\neq 0\}$,
$C_1 := \{p\in\Omega \text{ s.t. } \Delta(p)=0 \text{ and } g_1(p)\neq 0\}$,
$C_2 := \{p\in\Omega \text{ s.t. } \Delta(p)=g_1(p)=0 \text{ and } g_2(p)\neq 0\}$.

Thus, supp(g_1,g_2) is the union of the disjoint sets C_1, C_2, and C_o. Further, the sets C_1 and C_2 are constructibles of codimension at least one. By property (*), $C_1 = V(\beta)$ and $C_2 = V(\alpha)$ for some α, β in R. Let $g := \alpha g_1 + \beta g_2$, and pick any p in Ω. If p is in C_o then $\alpha(p)$ and $\beta(p)$ are both nonzero, and $\Delta \neq 0$ implies that $g(p) \neq 0$. If p is in C_1 then $g(p) = \alpha(p) g_1(p)$ is nonzero, since $\alpha(p)$ and $g_1(p)$ are both nonzero (here we use that C_1 and C_2 are disjoint). Finally let p be in C_2. Then $g(p) = \beta(p) g_2(p)$ is again nonzero. It follows that $\mathrm{supp}(g_1,g_2) = \mathrm{supp}(g)$, as desired.n

An example of rings for which the above discussion applies is provided by the following remark.

Corollary 2: Let R be a ring for which V(h) is finite whenever $h \neq 0$, and for which the following property holds: for each $p \in \Omega$ there is an $h \in R$ such that $V(h) = \{p\}$. Then R satisfies (2).

Proof: Let C be a constructible of codimension at least one. Thus C is contained in a finite V(h) and is itself finite, say $C = \{p_1,\cdots,p_r\}$. Now let $\{h_1,\cdots,h_r\}$ be such that $V(h_i) = \{p_i\}$. Then $C = V(h_1 \cdots h_r)$, and (2) holds.n

In particular, a Dedekind domain whose ideal class group is torsion satisfies (*), and hence also (2): in that case each V(h), h nonzero, is indeed finite. If the ideal class group is torsion, i.e. R is a "QR" domain in the terminology in ([GI], Theorem 40.3,) then for any $p \in \Omega$ there is an integer k such that $p^k = hR$ = principal, so $V(h) = \{p\}$. However, it is possible to prove property (2) directly for *every* Dedekind domain. We are grateful to Wolmer Vasconcelos for directing us to the invariant factor theorem for Dedekind domains:

Lemma 3: A Dedekind domain satisfies property (2).

Proof: Let M be a projective R-module, and *G* a nondegenerate submodule of M. Without loss, we again assume that M is free. By the invariant factor theorem for Dedekind domains, in the form given in ([CR], exercise 22.6, we may take *G* to have the form $E_1 \oplus \cdots \oplus E_k$, with each E_i a submodule of the i-th factor of $M = R \oplus \cdots \oplus R$, $k \leq n$. Further, the invariant factors E_i are ideals of R with $E_{i+1} \subseteq E_i$, $i=1,\cdots,k-1$. Thus, *G* is included in $E_1 M$. Since *G* is nondegenerate, $E_1 = R$, and it follows that this is a rank 1 summand of M included in *G*.n

An algorithm for explicitely carrying out the computation of invariant factors will be all that is be needed to make effective our theorem on pole shifting in the case of Dedekind domains.

Note that it is sufficient that property (2) be satisfied for enough finitely generated subrings of R. More precisely, if G is a matrix as above, then there is a linear combination $\Sigma r_{ij} g_{ij} = 1$, for suitable $\{r_{ij}\}$. It is enough that there be a subring S of R containing the $\{g_{ij}\}$ and $\{r_{ij}\}$ and for which the property is true. A V over S such that GV has rank 1 and has content S will be in particular a good V over R. In that sense, (2) behaves very similarly to the QR property (see [GI], IV.27, exercise 10). Other classes of rings satisfying (*) have been pointed out to us by Wolmer Vasconcelos. These include for instance one-dimensional affine algebras over **Z**; see [WI].

3. Reachable Systems

Definition 4: A *system* is a pair (A,*B*), where A: M→M is an endomorphism of a projective module M and *B* is a (finitely generated) submodule of M. The *state-space* of the system is M, and its *rank* is the rank of M. A *free* system is one for which M = R^n is free.

Usually one specifies *B* in terms of generators, giving a linear B: R^m→M whose image is *B*. We shall feel free to do so when convenient. Whenever refering to a specific system, n will denote its rank, and ι: *B*→M the natural inclusion map. A *reachable* (or "*controllable*") system is one for which the submodule

$$\text{Reach}(A,\mathcal{B}) := \mathcal{B} + A\mathcal{B} + \cdots + A^k\mathcal{B} + \cdots \qquad (1)$$

(the smallest A-invariant submodule containing *B*) equals M. If M has rank n then, by the Cayley-Hamilton Theorem, Reach(A,*B*) is the sum of the $A^i\mathcal{B}$ with i at most n-1. From the definition it is clear that, for reachable systems, M = *B* + AM; or equivalently:

Lemma 5: If (A,*B*) is reachable, then the map M⊕*B* → M: (x,b) → Ax+b is onto.n

Note that if (A,*B*) is reachable then (A+λI,*B*) is again reachable, for any λ in R (I = identity map). Thus the maps (x,b) → λx+Ax-b are all onto. These are all particular consequences of a general criterion for reachability, which is easier to state in terms of a presentation *B* = im(B), B: R^m→M. The criterion is: (A,*B*) is reachable if and only if the R[z]-map M[z]⊕R^m[z] → M[z] with block form [zI-A,B] is onto; see [KS] for details.

If (A,*B*) is a system and p is in Spec(R), we denote by (A,*B*)(p) the system (A(p),ι(p)*B*(p)) over the ring R/p, with state space M(p). Assume that B: R^m→*B* is onto, so that ι(p)*B*(p) = im(B(p)). Consider the map γ: R^{nm} → M, $\gamma(u_o,\cdots,u_{n-1}) := \sum A^i B u_i$. Then (A,B) is reachable iff γ is onto. By the local-global criterion for surjectivity (see e.g. [BO], II.3, Proposition 11), this is equivalent to γ(p) being onto for each maximal ideal p, or equivalently for all p in Spec(R). This implies that (A,B) is reachable iff all the residue systems (A,*B*)(p) are, in which case i(p)*B*(p) cannot be zero. Thus:

Lemma 6: If (A,*B*) is reachable, then *B* is a nondegenerate submodule of M.

The following observation will play a central role.

Lemma 7: If (A,*B*) is reachable, then M/*B* ≃ M/A^{-1}*B*. In particular, rank[(M/*B*)(p)] = rank[(M/A^{-1}*B*)(p)] for all p in Spec(R), and A^{-1}*B* is a nondegenenerate submodule of M.

Proof: Consider the following commutative diagram:

$$
\begin{array}{ccccccc}
 & & B & \xrightarrow{\iota} & \mathbf{M} & & \\
 & & \downarrow j & & \downarrow 1 & & \\
\mathbf{K} & \xrightarrow{u} & \mathbf{M}\oplus B & \xrightarrow{v} & \mathbf{M} & \rightarrow & \mathbf{0} \\
\downarrow a & & \downarrow \pi & & \downarrow c & & \\
\mathbf{M} & \xrightarrow{1} & \mathbf{M} & \rightarrow & \mathbf{0} & & \\
\downarrow \xi & & \downarrow & & & & \\
\mathbf{M/A^{-1}}B & \rightarrow & \mathbf{0} & & & &
\end{array}
$$

Here we took j(b):= (0,b), π(x,b):= x, v(x,b):= Ax+b, (K,u):= the kernel of v with its natural inclusion, ξ:= natural quotient map, and 1:= identity on M. The map a is defined as the composition of u with π. By construction, the two middle rows are exact, since v is onto by lemma 5. Note that since c=0, the last column is exact, so that ker(c) = M, and similarly ker(π) = *B* (up to a natural isomorphism). We claim that the first column is exact too, i.e. that im(a) = $A^{-1}B$. But x is in im(a) iff it is in π(ker(v)), i.e., there exists a b∈ *B* such that Ax+b=0, which means that x is in $A^{-1}B$ as required.

We are thus in the situation of the usual "snake lemma" (notations of maps are in fact almost as in the proof given in [BO], I.4, Proposition 2). There exists then a map d: M→ $M/A^{-1}B$ such that the sequence

$$B \xrightarrow{i} \mathbf{M} \xrightarrow{d} \mathbf{M/A^{-1}}B \rightarrow \mathbf{0} \qquad \mathbf{(2)}$$

is exact, so $M/B \simeq M/A^{-1}B$, as desired. Since tensoring is right-exact, for each p∈Spec(R) it holds that rank[i(p)] + rank[$(M/A^{-1}B)$(p)] = n = rank[i(p)] + rank[(M/*B*)(p)], and the rank condition is obtained. The last conlusion now follows from lemma 6. Note also that, by Schanuel's Lemma, applied to the above sequence and to the natural sequence *B* → M → M/*B*, we know also that $M\oplus B \simeq M\oplus A^{-1}B$, but this does not seem to play any role in what follows.n

Remark. W.Vasconcelos and A.Roy have independently pointed out to us that the above proof can be simplified, and the result strengthened, by noticing that A itself induces an isomorphism $M/B \simeq M/A^{-1}B$.

4. Pole assignment

For projective modules there is a theory of determinants that generalizes the standard one for free modules. See for instance ([BO], II.5, exercise 9), ([SW], page 148), or ([MD], Chapter V). Briefly, one considers for an A: M→M the associated wedge mapping detA := $\Lambda^n A$: $\Lambda^n M \rightarrow \Lambda^n M$ (n=rank M), and observes that $End(\Lambda^n M) \simeq R$, so that detA is canonically identified to an element of R. If M is free, this is the usual determinant. DetA behaves well under tensoring, and this permits checking all the standard properties by reducing to the free case (localize at all primes). The characteristic polynomial χ_A is defined as the determinant of zI-A, seen as an endomorphism of the projective M[z]. This is a monic polynomial in R[z] of degree n, and the Cayley Hamilton Theorem is valid just as in the free case.

The following situation will occur below. Assume that π: M$\to$N is onto and that (A,*B*) [resp., (A',*B*')] has state-space M [resp., N]. Further, assume that

$$\pi(B) = B' \text{ and } A'\circ\pi = \pi\circ A. \qquad (3)$$

Then, π(Reach(A,*B*)) = Reach(A',*B*'); in particular, the second system is reachable if the first one is. Further, since N is a projective of well-defined rank, Q:= ker(π) is also of this type (and is A-invariant). Since π splits, it follows by a localization argument that χ_A is the product of the characteristic polynomials of A' and of the restriction of A to Q.

Definition 8: An F: M$\to$M is *congruent to* A *(mod B)*, denoted F~A (mod *B*), iff the image of F-A is contained in *B*.

If B: $R^m \to$M has image *B*, and F is as above, there is by projectivity of M a linear "feedback" map K: M$\to R^m$ such that F = A+BK; thus this definition is equivalent to the one usually given. Note that if (A,*B*) is reachable and F~A (mod *B*) then (F,*B*) is again reachable.

Definition 9: χ is a *(characteristic) polynomial for* A *(mod B)* iff there exists an F with $\chi_F = \chi$ and F~A (mod *B*).

A *splitting* polynomial is a monic polynomial χ in R[z] whose roots are in R: $\chi = (z-\alpha_1)\cdots(z-\alpha_n)$, all $\alpha_i \in R$.

Definition 10: The system (A,*B*) is *pole-assignable* iff every splitting polynomial of degree n is a polynomial for A (mod *B*). The *ring* R is *pole-assignable (PA)* iff every reachable system over R is pole-assignable. A *PAF* ring is one for which every reachable *free* system is pole-assignable.

As known for free systems, (and with the same proof,) a pole-assignable system is necessarily reachable (see for instance [BSSV]); this motivates the definition of PA ring.

Previous work has always defined pole-assignability only with respect to *free* systems, i.e. what we called "PAF" rings above. However previous results dealt with projective-free rings, for which both notions coincide. In general, it is more natural to work with the definition of PA given here, because as we shall see the main induction step needed to prove that some classes of rings are PAF rings is most naturally presented in terms of nonfree systems. In any case, the positive results to be presented in the next section will establish that certain rings are PA, so in particular are PAF rings, while the negative results will show that other rings are not even PAF.

5. General Pole-Shifting Results

All counterexamples to the PA property studied in the past have been based on the observation that for pole assignable systems over projective-free rings, B (or its preimage $A^{-1}B$) must contain an unimodular element of M. The next result shows that this type of property has more to do with *reachability* itself, (and a weak property of A,) and is only *a fortriori* a consequence of pole assignability. The proof is conceptually quite different from that based on the stronger pole assignability assumption.

Proposition 11: Let (A,B) be reachable. Assume that there is a rank n-1 summand of M which is A-invariant. Then B contains a rank 1 summand of M. In particular, this happens if χ_A has the form $(z-\alpha_1)\chi_2$, where $((z-\alpha_1),\chi_2)$ (ideal generated over R[z]) = R[z].

Proof: Let $M = P\oplus Q$, with P of rank 1 and $AQ\subseteq Q$, and let $\pi: M\to P$ be the canonical projection. We claim that $\pi|B$ is onto. If this is established then, since P is projective, there will exist a linear $j: P\to M$ with $\pi^\circ j = 1_P$ and $im(j)\subseteq B$, so that the image of j will be the desired summand. The last conclusion will then follow: since $z-\alpha_1$ and χ_2 are relatively prime over R[z], M as an R[z] module (with z.x:= Ax) splits into two submodules, one of which is $Q:= \ker(\chi_2(A))$ (a complement is $im(\chi_2(A))$, and Q is A-invariant. Tensoring with all possible $p\in Spec(R)$ preserves the above characteristic polynomial, so Q indeed has rank n-1.

So we must prove that $\pi^\circ\iota: B\to P$ is onto, where ι is the inclusion map. Equivalently, since P has rank 1 we must establish that this composition remains nonzero when tensoring by all R/p, p maximal. Pick any such p, and assume that the composition is zero, i.e. $\pi(p)^\circ\iota(p)=0$. Since Q induces a summand Q(p) which is A(p)-invariant, we are reduced to the case where R is a field and there is a proper A-invariant subspace Q of R^n which contains B. This contradicts the fact that the system $(A,B)(p)$ is reachable, and the proof is complete.n

Corollary 12: If the system (A,B) is pole-assignable, B contains a rank 1 summand of M. If R is a PA ring, all projective modules over R split into rank 1 summands.

Proof: As remarked earlier, (A,B) is reachable. By pole assignability, there is an F~A (mod B) such that $\chi_F = z(z-1)^{n-1}$, and (F,B) is again reachable. Thus proposition 11 applies to this new system. If M is any given projective module and R is a PA ring, we may apply the argument to (0,M), which is a reachable system and hence is pole assignable. Thus M has a rank 1 summand and the last conclusion follows by induction on the rank of M.n

The following theorem generalizes the result in [BSSV], Corollary 3.6. The proof is the obvious generalization of the projective-free case treated there.

Theorem A. R is a PA ring iff $A^{-1}B$ contains a rank 1 summand of M whenever (A,B) is reachable.

Proof: Assume that F~A (mod *B*) has characteristic polynomial $z(z-1)^{n-1}$. Since z and $(z-1)^{n-1}$ are relatively prime over R[z], M splits into two submodules one of which, P:= ker(F), is of rank 1. Since F~A (mod *B*), AP⊆*B*, as desired. Alternatively, we may argue using the previous lemma: first find F~A (mod *B*) with characteristic polynomial $(z+1)^n$; now F is invertible so $F^{-1}B \simeq B$ contains a unimodular, and the congruence implies that $A^{-1}B = F^{-1}B$.

Conversely, assume that (A,*B*) is reachable. If n=1 then *B* = M, so A~F for *every* F: M→M, and the result follows from the identification End(M) ≃ R. Assume inductively that all systems of rank n-1 are pole-assignable. Pick any α_1 in R, and let J be the map $A-\alpha_1 I$. Since (J,*B*) is again reachable, by hypothesis there is a rank 1 summand Q of M contained in $J^{-1}B$, i.e. $(A-\alpha_1 I)Q \subseteq B$. Writing M = Q⊕P, there exists a C: M→M that satisfies $Cq = \alpha_1 q$ for all q∈Q and Cx = Ax for x∈P. Thus C~A (mod *B*) and CQ⊆Q.

Since Q is C-invariant, the latter induces an endomorphism D of N:= M/Q (≃P). Let *B*' be the image of *B* on N. Then, (D,*B*') is again a reachable system, since this is the situation in equation 3. By induction, there is an E: N→N such that D~E (mod *B*') and its characteristic polynomial is $\chi_2 := (z-\alpha_2)\cdots(z-\alpha_n)$. Let L':= E-D: N→N. Since im(L')⊆*B*' and N is projective, there is a lifting L": N→M which satisfies im(L")⊆*B*. Let L: M→M be the composition of the quotient map M→N with L", so that im(L)⊆*B*, L(Q)=0, and L induces L' on N. Define F:= C+L, so that F~A (mod *B*). By construction, F coincides with C on Q, and induces E on N, so $\chi_F = (z-\alpha_1)\chi_2$ as desired.n

Corollary 13: If property (2) holds, in particular if R is as in Corollary 2 or is a Dedekind domain, R is a PA ring.

Proof: Follows from theorem A and lemma 7.n

We now state some, mainly negative, consequences of proposition 11. These explain the various counterexamples in the literature regarding PAF rings (see [BSSV], [TA1], [TA2], [SH]). The constructions in the next two lemmas are minor modifications of those given independently by [BHL] and [SH], where they are given (in slightly different form) in the case when R is Noetherian (or a certain submodule is finitely generated). They are in turn the generalizations to n>2 of those given in [TA1].

Lemma 14: Let I be an ideal of R, and let π: R→S, S:= R/I, be the canonical map. Assume that (F,*G*) is a reachable free system over S. Then there is a reachable free system (A,*B*) over R such that *B*⊗S ≃ *G* and such that, if $\chi = (z-\alpha_1)\cdots(z-\alpha_n)$ is a polynomial for A (mod *B*) then $\pi(\chi) := (z-\pi(\alpha_1))\cdots(z-\pi(\alpha_n))$ is a polynomial for F (mod *G*).

Proof: It is more convenient to work with matrices, so let $F \in S^{n\times n}$ and $G \in S^{m\times n}$ be so that (F,im(G)) is the given system when interpreted in the standard basis of S^n. Let $A \in R^{n\times n}$ be any matrix lifting F, and let $B' \in R^{m\times m}$ lift G. Consider the matrix $T := (B', AB', \cdots, A^{n-1}B')$. Let T(I) be the reduction of T modulo I. Reachability of (F,im(G)) means that the columns of T(I) span S^n. Consider the elements $\pi(e_i)$, $i=1,\cdots,n$, where e_i is the i-th element of the standard basis of R^n. Then $\pi(e_i)$ is in the column space of T(I), so there exists for each i an element v_i in the column space of T such that $\pi(e_i-v_i)=0$, i.e. for some $w_i \in I^n$, $e_i = v_i+w_i$. Now let B be the matrix obtained by adjoining to B' the columns w_i. By construction, (A,im(B)) is reachable. Further, since the last n columns of B reduce to zero modulo I, im(B)⊗S ≃ im(G), as wanted. Finally, if A+BK has characteristic polynomial χ then, reducing modulo I results in F+GL having characteristic polynomial $\pi(\chi)$, where L is obtained by dropping the last n rows of K and reducing the resulting matrix modulo I.n

Corollary 15: If R is a PAF ring then R/I is a PAF ring for each ideal I of R.n

Lemma 16: Assume that P is a projective such that P^t (direct sum with itself) is free for some t. Then there is a reachable free system (A,*B*) with *B* ≃ P.

Proof: Let ϕ: P⊕···⊕P → R^n be the isomorphism. Pick *B*:= ϕ(first factor), and define A: $R^n \to R^n$ as the conjugate under ϕ of the cyclic permutation of the P's. This is as desired.n

Corollary 17: Let R be a PAF ring. If P is as in lemma 16 then P has a rank 1 summand.

Proof: Clear from lemma 16 and the first part of corollary 12.n

Recall that a *stably free* module is one for which $P \oplus R^s$ is free for some k, and that a *Hermite* ring is one for which all stably free modules are free. A theorem of Gabel and Lam ([MD], Theorem IV.44,) which can be proved as a simple consequence of Bass' theorem ([LAM], Theorem 7.3), shows that if P is a stably free module then P^k is free for some k.

Theorem B. Let R be a PAF ring for which rank 1 projectives are free. Let I be any ideal of R, and S:= R/I. Then, if P is any projective S-module such that P^t is free for some t, then P is itself free. In particular, every quotient of R is a Hermite ring.

Proof: Let P be of that form. We shall proceed by induction on the rank   of P, the case   =0 being trivial. By lemma 16, there is a free reachable system (F,*G*) with *G* ≃ P. By lemma 14, there is then a free reachable system (A,*B*) over R with *B*⊗S ≃ P. Since R is a PAF ring, *B* must by corollary 12 contain a rank 1, hence free, summand. Tensoring with S, we conclude that P has a free summand S. Write P = S⊕Q. Then, $Q^t \oplus S^t$ is free, so that Q^t is stably free and so, by the remark before the theorem, itself satisfies the hypothesis on P. Since rank(Q) =   -1, Q is by induction free, and hence so is P.n

For example, one may apply the theorem in order to conclude that, for many topological spaces X, the ring C^0(X) cannot be a PAF ring:

Corollary 18: If X is a normal topological space having a subspace W which is homeomorphic to a closed disk in $\Re^2$, then R = C^0(X) is not a PAF ring.

Proof: Since W is compact, it is closed, so by the Tietze extension theorem every function on W extends to a function on X. Thus C^0(W) is isomorphic to R/I, where I is the ideal of R consisting of those functions which vanish on W. It follows from corollary 15 that, if R were a PAF ring, C^0(W) would also be. So assume now that X = a closed disk in $\Re^2$. Without loss, we may assume that X contains $\mathbf{S}^1$, so $C^0(\mathbf{S}^1)$ is a quotient of C^0(X). The module P of sections of the Moebius band, seen as a line bundle over $\mathbf{S}^1$, satisfies P⊕P = free, but is not free. On the other hand, X being contractible implies that all projectives over R are free. The conclusion then follows from the theorem.n

The following positive result will not be used in the next section, but relates very naturally to the material presented here.

Proposition 19: If (A,*B*) is a free reachable system and *B* is a rank 1 summand then *every* monic polynomial of degree n is a polynomial of A (mod *B*).

Proof: In terms of matrices, we are assuming that *B* = im(B), where B is a nonzero m×n matrix of rank 1. We want to show that the mapping ϕ: $R^{mn} \rightarrow R^n$: K $\rightarrow$ coefficients of the characteristic polynomial of A+BK, is onto. We claim that this map is *affine* (linear+translation), and tensoring modulo any maximal ideal p of R results in the analogous map for (A,*B*)(p). Since the corresponding result is true over fields (the classical pole-shifting theorem), the local-global principle will give the desired conclusion.

We shall prove that det(zI-A-BK) = χ_A - trace{cof(zI-A)BK}, where "cof" denotes the transpose matrix of cofactors. This will establish that ϕ is indeed affine and behaves as claimed under tensoring. First note that, for any matrix G of determinantal rank ≤1, det(I-G) = 1-trace(G). This is because for any G the characteristic polynomial of G is of the form z^n - trace(G)z^{n-1} + ···, where the coefficients not displayed are all linear combinations of r×r minors of G, with r≥2, and hence all vanish in this case. Substituting z=1 gives det(I-G) = 1-trace(G). Now apply this observation to the matrix G:= $(zI-A)^{-1}$BK, over the fraction ring T^{-1}R[z], where T = set of monic polynomials. Then G has rank at most 1 (Cauchy-Binet), so det(zI-A-BK) = χ_A.det(I-G) = χ_A(1-trace{$(zI-A)^{-1}$BK}) is as claimed.n

Since R is reduced, the proof could have been simplified somewhat by arguing modulo all primes and hence basically dealing with the case of fields, but as given it applies to arbitrary commutative rings. Note that even if property (2) holds, one may *not* reduce pole assignment to this case, since replacing *B* by a submodule of it will in general destroy reachability. In fact, it is known that the conclusion is in general false for reachable systems over most rings satisfying (2), including principal ideal domains like **Z** and $\Re$[z] (see the material on "coefficient assignment" in [BSSV]).

Added note: We have recently noticed the reference [EI], which provides an ingenious construction over principal ideal domains. As the author notes, some facts are valid over more general rings. It is clear that the construction in that paper could serve as the basis for an alternative version of theorem A, stated in terms of *B* instead of A^{-1}*B*.

6. Families of Systems

We now study the case of rings of real-valued functions $C^k(\mathbf{X})$, k = 0, ∞, or ω, where **X** is a topological, smooth, or real-analytic manifold respectively. By "manifold" we mean Hausdorff, second countable, connected. "Diffeomorphism" means smooth or analytic, depending on the context. The ring structure is that of pointwise operations. Systems over this type of ring correspond to "families of systems" (A(x),B(x)) obtained by pointwise evaluations at the x in **X**. See [S2] for an introduction to the topic of control of families of systems. Since an element of $C^k(\mathbf{X})$ is a unit iff it is nowhere vanishing, reachability corresponds to the simultaneous reachability of all systems in the family. The result to be proved is as follows.

Theorem C. $C^k(\mathbf{X})$ is a PA ring iff **X** has dimension 1.

The proof will actually show that in the case of dimension>1, $C^k(\mathbf{X})$ is not even a PAF ring. It is easy to prove that the manifold **X** has dimension 1 iff **X** is homeomorphic (diffeomorphic for k=∞,ω) to $\Re$ or to $\mathbf{S}^1$; see for instance [DD], 3.16.2, problem 6 (which deals with the smooth case C^∞, but the same proof applies in general). Thus, we need to prove the positive result for these two manifolds, and the negative result for manifolds of dimension d≥2.

6.1. Negative Results

Corollary 18 already shows that $C^0(\mathbf{X})$ is not a PAF ring for **X** with d≥2. A similar proof could be used in the smooth case, but not in the analytic case. For those two cases, we may argue however as follows.

By Whitney's embedding theorem (see e.g. [NA], p.149), we may, and shall, assume without loss that **X** is an embedded submanifold of $\Re^q$, for some large enough q. Further, we may assume that $0 \in \mathbf{X}$ and that the tangent space to **X** at 0 (as a subspace of the tangent space to $\Re^q$ at 0, identified to $\Re^q$ itself) is the subspace $\Re^d$ defined by the equations $x_{d+1}=\cdots=x_q=0$. Consider the orthogonal projection $\pi: \Re^q \to \Re^d$, and the composition $\theta := \pi \circ i$, where $i: \mathbf{X} \to \Re^q$ is the inclusion mapping. Thus the differential of θ at 0 is the identity. By the inverse function theorem, there are open neighborhoods U of 0 in **X** and V of 0 in $\Re^d$ which are diffeomorphic under θ. Applying if needed a linear transformation on $\Re^q$, we may assume that V is an open subset of $\Re^d$ containing the unit sphere of $\Re^d$. Let σ: V→U be a diffeomorphism such that $\pi \circ \sigma = 1_V$.

Assume that (F,*G*) is a free reachable system over $C^k(\Re^q)$, G a matrix whose columns span *G*, with the property that all entries of F and G are invariant under π, i.e. that all $f_{ij}(x) = f_{ij}(\pi(x))$ and similarly for G. Restricting all entries of F and G, (F,*G*) induces a reachable system (A,*B*) over $C^k(\mathbf{X})$, with B = restriction of G. Assume that (A,*B*) would be a PAF ring, and consider any splitting polynomial χ over $C^k(\Re^q)$ which is also invariant as above. Its restriction to **X** is then a polynomial of A (mod *B*). Thus there is a matrix K' over $C^k(\mathbf{X})$ such that A+BK' has characteristic polynomial χ. On V define K(x):= K'(σ(x)). It follows from π-invariance that the restriction of A+BK to V has characteristic polynomial χ. If we give then an example of an (F,*G*) and χ as above for which the restrictions to V are such that χ is not a polynomial of F (mod *G*), we'll have a contradiction, and $C^k(\mathbf{X})$ will not be a PAF ring.

In [BSSV] an instance is given of a system in $\Re^2$ with the property that its restriction to any open set containing the unit circle does not admit e.g. z^2 as a characteristic polynomial under feedback. Embedd $\Re^2$ in the first two coordinates of $\Re^d$ (recall that we are dealing with the case d≥2), and let (F,*G*) be the same system seen as a family over $\Re^q$, that is, the system that extends the one in [BSSV] and is invariant under the orthogonal projection on $\Re^2$. This provides the required example.

6.2. Analytic Cases, dimension 1

The case $C^\omega(\Re)$ was already treated in [BSSV]: R is an elementary divisor domain, so corollary 13 applies. (Since elementary divisor rings are projective-free, there is no difference in that case between the PA and the PAF properties.) For $C^\omega(\mathbf{S}^1)$ that argument will not apply, since this ring is not (even) Bezout. However, we shall still use corollary 13. In fact, most results in this paper were derived in the process of proving that $C^\omega(\mathbf{S}^1)$, and those in the next section, are PA rings.

Proposition 20: R=$C^\omega(\mathbf{S}^1)$ satisfies the hypothesis of corollary 2.

Proof: Note first that the maximal ideals of R are precisely the ideals p_ξ:= {f∈R s.t. f(ξ)=0}, for ξ∈$\mathbf{S}^1$. This is a standard compactness argument: if I is an ideal with no common zeroes then for each ξ there is an f∈I with f(ξ)≠0 in a neighborhood of ξ. Choosing a finite cover by such neighborhoods the sum of the squares of the corresponding f's is an element of I which is nowhere zero and hence a unit. Thus every proper ideal is contained in some p_ξ. If h≠0 it can have only finitely many zeroes in $\mathbf{S}^1$, by compactness and analiticity. Thus V(h) is indeed finite. Finally, each p_ξ is of the form V(h_ξ) for some h_ξ: identifying $C^\omega(\mathbf{S}^1)$ with the ring of real analytic functions of period π, pick $h_\xi(x) := \sin^2(x-\xi)$.n

Actually, $C^\omega(\mathbf{S}^1)$ is Dedekind, with torsion ideal class group: each p_ξ is generated by the element given above and an element that has a zero of order 1 at ξ and another zero of order 1 at a different point. The essential valuations of R are the v_ξ(h):= order of h at ξ, and $p_\xi^2 = (h_\xi)$.

6.3. Continuous and Smooth Cases, dimension 1

We again use corollary 13. As in the proof of lemma 1, it will suffice to establish that, for R = $C^k(\mathbf{X})$, k = 0,∞, **X** = $\Re$, $\mathbf{S}^1$, and for any n×m matrix G over R with c(G)=R, nd rank>1 there is a matrix V such that GV has rank=1 at all x in **X** and c(GV)=R. The condition c(G)=R is equivalent here to: G(x)≠0 for all x in **X**. We'll prove somewhat more than explicitely needed: for any such G, even just continuous, there is a smooth vector v(x) such that G(x)v(x) is nonzero for all x.

Lemma 21: (a) Let G be an n×m matrix of real-valued continuous functions defined on the half-closed interval [0,1), and assume that $G(x)\neq 0$ for all x in U. Suppose that $G(0)\nu_o\neq 0$. Then there exists an m-vector v of real-valued functions, smooth on [0,1), and an $\varepsilon>0$, such that

(i) $v(x) = \nu_o$ (constant) for all $0\leq x\leq\varepsilon$, and
(ii) $G(x)v(x) \neq 0$ for all $0\leq x<1$.

(b) Let G be an $n\times m$ matrix of real-valued continuous functions defined on the closed interval [0,1], and assume that $G(x)\neq 0$ for all x in U. Suppose that $G(0)\nu_o\neq 0$ and $G(1)\nu_f\neq 0$. If rank[G(1)]>1 then there exists an m-vector v of real-valued functions, smooth on [0,1], and an $\varepsilon>0$, such that

(i) $v(x) = \nu_o$ (constant) for all $0\leq x\leq\varepsilon$,
(ii) $v(x) = \nu_f$ (constant) for all $1-\varepsilon<x\leq 1$, and
(iii) $G(x)v(x) \neq 0$ for all $0\leq x\leq 1$.

Proof: (a) For each $\xi\in$ [0,1) there is a ν such that $G(\xi)\nu\neq 0$, and hence such that $G(x)\nu\neq 0$ for all x in an open interval (relative to [0,1)) around ξ. Obtain a locally finite countable subcover of [0,1) consisting of such intervals. We may assume the subcovering to be a union of sets of the form (a_i,b_i), i = 1,2,···, and $[a_o,b_0)$, $a_o=0$; further we may assume that $a_o < a_1 < b_o < a_2 < b_1 < \cdots$. Associated to these are vectors v_i with $G(x)v_i\neq 0$ in the corresponding interval, and $v_o=\nu_o$. We shall say that v_i and v_{i+1} are *compatible* at x iff $G(x)v_{i+1}$ is *not* a negative multiple of $G(x)v_i$. For i=0,1,···, we modify the intervals (a_i,b_i) and the v_i as follows: If v_i and v_{i+1} are incompatible for every x in (a_{i+1},b_i) then we replace v_{i+1} by its negative, $-v_{i+1}$. Otherwise, there is a subinterval (α,β) of (a_{i+1},b_i) on which they are compatible for all x; in that case, we replace a_{i+1} by α and b_i by β. Thus we achieve that v_i and v_{i+1} are compatible on each interval (a_{i+1},b_i), the overlap between successive intervals in the above cover. Next, choose C^∞ functions ϕ_i: $[0,1)\to[0,1]$, i=0,···, such that ϕ_i vanishes for $x\leq a_{i+1}$ and $\phi_i(x)=1$ for $x\geq b_i$. Finally, define $v(\cdot)$ on the first interval of the covering, and on successive intervals of the form $[b_{i-1},b_i]$ as follows:

$v(x):= \nu_o$ for $a_o \leq x \leq a_1$,
$v(x):= v_i$ for $b_{i-1} \leq x \leq a_{i+1}$, and
$v(x):= (1-\phi_i)v_i + \phi_i v_{i+1}$ for $a_{i+1} \leq x \leq b_i$.

This satisfies the requirements.

(b) The construction of $v(\cdot)$ is similar to (a). The only difference is that we have here a *finite* cover, and in the last interval, say $(a_r,b_r=1]$, we choose $v_r=\nu_f$. In addition, since G(1) has rank at least 2, we also have that G(x) has rank at least 2 for x in some interval $(1-\varepsilon,1]$, $\varepsilon>0$. We choose x_o larger than $1-\varepsilon$ and b_{r-1} and u such that $R(x_o)u$ and $R(x_o)\nu_f$ are independent. Let a_{r+1} be a point between b_{r-1} and x_o, and redefine b_r to be any point in the interval $(x_o,1)$, such that G(x)u and $G(x)\nu_f$ are independent in that interval. Now let b_{r+1}:=1, and redefine v_r:= u and $v_{r+1}:= \nu_f$. Now proceed as before; by construction, v_r and v_{r+1} are compatible in some interval, so we do not need to replace ν_f.n

Corollary 22: $C^k(\Re)$ satisfies property (2) for k = 0, ∞.

Proof: Pick any ν_o such that $G(0)\nu_o\neq 0$. We apply part (a) of the lemma twice, identifying first [0,1) with $[0,+\infty)$ and then with $(-\infty,0]$. The functions in both intervals match well to give a smooth function, since they are constant in a neighborhood of zero.n

Corollary 23: $C^k(\mathbf{S}^1)$ satisfies property (2) for k = 0, ∞.

Proof: We represent $\mathbf{S}^1$ as [0,1] with endpoints indentified, and assume without loss that G(0) = G(1) has rank at least 2. Choose ν_o such that $G(0)\nu_o = G(1)\nu_o$ is nonzero, and apply part (b) of the lemma with $\nu_o = \nu_f$. The function obtained has v(0) = v(1) and is constant in a neighborhood of 0, so defines a smooth function on $\mathbf{S}^1$.n

7. References

[BK] Bourbaki, N., *Elements of Mathematics, Commutative Algebra*, Addison-Wesley, Reading, 1972.

[BBV] Brewer,J.W., John W.Bunce, and F.S.Van Vleck, *Linear Systems over Commutative Rings*, to appear.

[BHL] Brewer,J., W.Heinzer, and D.Lantz, "The pole assignability property in polynomial rings over GCD domains," *J.Algebra*, to appear.

[BSSV] Bumby,R., E.D.Sontag, H.Sussmann, and W.Vasconcelos, "Remarks on the pole-shifting problem over rings," *J. Pure Appl. Algebra* **20**(1981):113-127

[BY] Byrnes,C.I., "On the control of certain infinite-dimensional systems by algebro-geometric techniques," *American J.Math.* **100**(1978): 1333-1381.

[CR] Curtis,C.W., and I.Reiner, *Representation Theory of Finite Groups and Associative Algebras*, Wiley, New York, 1962.

[DD] Dieudonné,J., *Elements d'analyse*, Gauthier-Villars, Paris, 1970.

[EI] Eising,R., "Pole assignment for systems over rings," *Systems & Control Letters* **2**(1982): 225-229.

[GI] Gilmer,R., *Multiplicative Ideal Theory*, Dekker, N.Y., 1972.

[HA] Hazewinkel,M., "A partial survey of the uses of algebraic geometry in systems and control theory," in *Proc. Severi Centennial Conference*, Rome, 1979.

[KA] Kailath,T., *Linear Systems*, Prentice Hall, Englewood Cliffs, 1980.

[KA] Kamen,E.W., "Lectures on algebraic system theory: linear systems over rings," NASA Contractor Report #3016.

[KS] Khargonekar,P.P. and E.D.Sontag, "On the relation between stable matrix fraction decompositions and regulable realizations of systems over rings," *IEEE Trans.Autom. Control* **27**(1982):627-638.

[LA] Lam, T.Y., *Serre's Conjecture*, Springer, Berlin, 1978.

[MD] McDonald, Bernard R., *Linear Algebra over Commutative Rings*, Dekker, New York, 1984.

[MO] Morse,A.S., "Ring models for delay differential systems," *Automatica* **12**(1976): 529-531.

[NA] Narasimham,R., *Analysis on Real and Complex Manifolds*, North-Holland, Amsterdam, 1968.

[SH] Sharma,P.K., "On pole assignment problem in polynomial rings," *Systems and Control Letters* **5**(1985): to appear.

[S1] Sontag,E.D., "Linear systems over commutative rings: A survey," *Ricerche di Automatica* **7**(1976):1-34.

[S2] Sontag,E.D., "An introduction to the stabilization problem for parametrized families of linear systems," in *Linear Algebra and its Role in Systems Theory*, (B.N.Datta, ed.), AMS Publications, 1985.

[SW] Swan,R.G., *Lecture Notes in Math. #76: Algebraic K-Theory*, Springer, Berlin, 1968.

[TA1] Tannenbaum,A., "On pole assignability over polynomial rings," *Systems and Control Letters* **2**(1982):13-16.

[TA2] Tannenbaum,A., "Polynomial rings over arbitrary fields in two or more variables are not pole assignable," *Systems and Control Letters* **2**(1982): 222-224.

[WI] Wiegand,R., "The prime spectrum of a two-dimensional affine domain," preprint, 1984.

Table of Contents