REAL ADDITION AND THE POLYNOMIAL HIERARCHY

Eduardo D. SONTAG *

Department of Mathematics, Rutgers University, Hill Center, Busch Campus, New Brunswick, NJ 08903, U.S.A.

Communicated by David Gries
Received 19 July 1984
Revised 15 September 1984

The $k$th alternation level of the theory of real numbers under addition and order is log-complete for the $k$th level of the polynomial hierarchy.

Keywords: Computational complexity, non-numerical mathematics

1. Introduction

There has been some interest in the literature on decision methods for $\text{Th}(\mathbb{R}, +, <)$, the theory of real numbers with addition (and order). See, for instance, [6], [3], and [2], as well as [4] and [1] in the more general context of reals with multiplication. This theory is of interest also from the point of view of piecewise linear algebra and its potential applications in control system theory [9,10]. If $n$ is the length of a formula in the language $L$ associated to this theory, it is known that the decision problem has deterministic time complexity at least exponential and at most doubly exponential. Here we remark that the $k$th level of this problem (i.e., deciding the truth of a sentence given in prenex form with $k-1$ quantifier alternations) is complete for $\Sigma^p_k$. In particular, the satisfiability problem ($k=1$) is NP-complete, and for each fixed level $k$ we have a deterministic single exponential time bound.

The proof proceeds roughly as follows. In the next section we establish some facts on polyhedra, in particular, that a nonempty polyhedron defined by equations with 'small' coefficients must contain 'small' points, and, more importantly, that every projection of such a polyhedron can also be defined by equations with small coefficients. This gives rise to a theoretical procedure for quantifier elimination in the last section. The procedure results in exponentially large formulas, but the lengths of the numbers appearing in these formulas remain small, so an alternating algorithm can be deduced which only checks rational points of size polynomial in the size of the input data.

2. Some polyhedral geometry

We need a few facts regarding the geometry of polyhedra. We use [8] and [7] as our main references, but much of the terminology is standard. A flat in $\mathbb{R}^d$ is an affine submanifold (= translate of a subspace) of $\mathbb{R}^d$. A polyhedron $P \subseteq \mathbb{R}^d$ is by definition an intersection of finitely many hyperplanes and closed half spaces,

$$a_1x_1 + \cdots + a_dx_d \leq a_0.$$  \hspace{1cm} (2.1)

(It is of course redundant to include hyperplanes explicitly.) The affine hull of $P$ is $\text{aff}(P) =$ smallest flat containing $P$. The relative interior $\text{ri}(P)$ is the interior of $P$ relative to $\text{aff}(P)$; its dimension is the dimension of $\text{aff}(P)$. A line-free polyhedron is one that does not contain any lines (= one-dimensional flats). If $P$ is the intersection of the hyperplanes
$H_1, \ldots, H_r$ and the half spaces $K_1, \ldots, K_s$, and if $H_{r+1}, \ldots, H_{r+s}$ are the bounding hyperplanes of the $K_i$ (obtained by replacing $\leq$ by $=$ in (2.1)), we call $\Gamma = \{H_1, \ldots, H_{r+s}\}$ a 

**supporting family** (of hyperplanes) for $P$. Let $(K_1, \ldots, K_u)$, $u \leq s$, be the set of those half spaces among the above for which $P$ is in fact included in the corresponding $H_{r+i}$. If we replace $K_i$ by the corresponding $H_{r+i}$, for $i = 1, \ldots, u$, we obtain a new representation as an intersection, giving rise to the 

**same supporting family**, but with the property that $r_i(P)$ and aff($P$) can be represented as follows:

\[
\text{aff}(P) = \text{intersection of the } \{H_i, i = 1, \ldots, r + u\},
\]

(2.2)

\[
\text{ri}(P) = \text{aff}(P) \text{ intersected with the open halfspaces int}(K_i), i = u + 1, \ldots, s.
\]

(2.3)

A vertex (or extreme point) of the polyhedron $P$ is a point not in the relative interior of any segment contained in $P$; equivalently [7, result 2.6.3], a zero-dimensional face of $P$. If $\Gamma$ as above is a supporting family for $P$ and $v$ is a vertex of $P$, then $v$ is the intersection of those $H_i$ containing it; thus, by a trivial dimensionality argument, $v$ is the intersection of precisely $d$ such hyperplanes. The Minkovski-Weil theorem [8, Theorem 2.12] says that $P$ is a polyhedron iff it is generated by finitely many points and rays. More precisely, there exist two sets of points $V = \{v_1, \ldots, v_r\}$ and $D = \{y_1, \ldots, y_s\}$ such that

\[
P = \text{conv}(V; D) = \{\sum \lambda_i v_i + \sum \mu_j y_j, \sum \lambda_i = 1, \text{ all } \lambda_i \geq 0, \mu_j \geq 0\}.
\]

(2.4)

We call $V \cup D$ an **extreme family** for $P$. With such a representation, $\text{ri}(P)$ is the set of points as in (2.4) but such that all $\lambda_i$ and $\mu_j$ are strictly positive. Appropriate sets $V$ and $D$ can be obtained as follows. First write $P$ as the union of finitely many line-free polyhedra $P_i$; this can be done, for instance, through intersection with the various orthants of $\mathbb{R}^d$. Each such $P_i = \text{conv}(V_i; D_i)$, where $V_i$ is the set of vertices of $P_i$ and $D_i$ is a set of representatives for the directions of recession of $P_i$. A direction of recession (or vertex at infinity) is a ray $\mathbb{R}_+ y$, where $y$ is a vector in $\mathbb{R}^d$ such that, for some $v_0 \in P$, the line $v_0 + \mathbb{R}_+ y = \{v_0 + \mu y, \mu \geq 0\}$ is a one-dimensional face of $P$.

If $y$ is of this form and $\Gamma_i$ is a supporting family for $P_i$, then there are $d - 1$ hyperplanes in $\Gamma_i$, having equations $\sum q_{ij} x_j = b_i, i = 1, \ldots, d - 1$, which intersect precisely at a line containing $v_0 + \mathbb{R}_+ y$. Thus $\sum q_{ij} y_j = 0$ for each $i$. These $d - 1$ equations determine $\mathbb{R}_+ y$ uniquely except for sign. The sets $V, D$ can be obtained now as the union of the $V_i$ and $D_i$ respectively. Note that one can always obtain rational $y_i$, and that the $x_i$ are necessarily rational, if $P$ admits a supporting family with rational-coefficient equations. Rationality will be implicitly assumed in all that follows.

Denote by $\#(q)$ the length of the rational $q$: if $q = a/b$ with $a, b$ relatively prime, then $\#(q)$ is the sum of the number of bits in the binary representations of $a$ and $b$ (disregarding signs). If $v \in \mathbb{R}^d$, then $\#(v)$ denotes the sum of the lengths of the entries of $v$. If $H$ is a hyperplane (definable by rational equations), consider the possible equations

\[
q_1 x_1 + \cdots + q_d x_d = q_0
\]

(2.5)

for $H$. For any such equation, we sum the lengths of the rationals $q_i$. We denote the smallest such possible sum, over all representations of $H$, by $\#(H)$. If $\Gamma$ is a finite family $(H_i, i = 1, \ldots, k)$ of hyperplanes, we let

\[
\#(\Gamma) = \max\{\#(H_i), i = 1, \ldots, k\}.
\]

If $V \cup D$ is an extreme family for $P$, we write

\[
\#(V \cup D) = \max\{\#(v_i), \#(y_j), i = 1, \ldots, r, j = 1, \ldots, s\}.
\]

The main point of the following series of lemmas is that these dual kinds of representations (extreme family, supporting family) are equivalent in a polynomial-space sense.

**Lemma 2.1.** There is a polynomial $p_1$ such that for every polyhedron $P \subseteq \mathbb{R}^d$ and each supporting family $\Gamma$ for $P$, there is an extreme family $V \cup D$ for $P$ such that

\[
\#(V \cup D) \leq p_1(d) \#(\Gamma).
\]
Proof. Without loss (see above construction) we may assume that $P$ is line-free; notice that $\#(\Gamma)$ does not increase when adding constraints for intersections with orthants $(x_1 \geq 0, x_2 \leq 0, \ldots)$. Let $v$ be any vertex of $P$. As remarked above, $v$ is the intersection of $d$ elements of $\Gamma$, say $H_1, \ldots, H_d$. Let $A$ and $b$ be the following matrix and vector respectively:

$$
A = \begin{pmatrix}
q_{11} & \cdots & q_{1d} \\
\vdots & & \vdots \\
q_{d1} & \cdots & q_{dd}
\end{pmatrix}, \quad b = \begin{pmatrix}
q_{10} \\
\vdots \\
q_{d0}
\end{pmatrix}
$$

(2.6)

obtained from a representation of the hyperplanes in $\Gamma$ that achieves $\#(\Gamma)$, with $H_i$ having equations

$$q_{ii} x_1 + \cdots + q_{id} x_d = q_{i0}.$$

Thus $v$ is the solution of the system $Ax = b$. In general, let $\#(C)$ be the sum of the $\#(c_{ij})$, for any matrix $C$; here, the composite matrix $(A, b)$ has $\#(A, b) \leq (d^2 + d)\#(\Gamma)$. For any $d \times d$ matrix $C$, $\#(\det(C))$ is bounded by a polynomial in $d$ and $\#(C)$: this follows by Hadamard's inequality; in fact, the following very nice bound is given in [5]:

$$\#(\det(C)) \leq (d^2 + 2)^d \#(C) + \frac{1}{2} (d \log d). \quad (2.7)$$

By Cramer's rule, the coordinates $v_i$ are quotients of determinants of submatrices of $(A, b)$. So $\#(v) \leq 5d^4 \#(\Gamma)$. To obtain directions of recession the argument is similar: we must find small vectors $y$ in any one-dimensional intersection L of $d - 1$ hyperplanes $\Sigma q_{ij} x_j = 0$, $i = 1, \ldots, d - 1$ (see discussion preceding the statement of the lemma). But this can be reduced to finding the unique solution of a nonsingular system of $d$ equations, as follows. Introduce $A$, $b$ as in (2.6) (but with $d - 1$ rows). Since rank $A = d - 1$, some row vector of the form $(0, \ldots, 0, 1, 0, \ldots, 0)$ can be adjoined to $A$ in order to obtain a matrix $A'$ of full rank. Adjoining the entry $q_{d0} = 0$ to $b$, there results a nonsingular system $A'x = b'$. Let $z$ be the unique solution of this system. Then, $z$ has a small size (same argument as above), and either $y = z$ or $-z$ has the property that $R_+ y = L$ and can hence be used to represent the direction of recession $L$. □

**Lemma 2.2.** There is a polynomial $p_2$ such that for every nonempty polyhedron $P \subseteq \mathbb{R}^d$ and each extreme family $\Gamma \subseteq \mathbb{R}^d$ for $P$ there is a point $z$ in $\text{ri}(P)$ such that $\#(z) \leq p_2(d)\#(\Gamma)$. 

**Proof.** Assume that $P$ has dimension $e \leq d$. Thus there are $e + 1$ affinely independent elements in $V \cup D$, say $v_1, \ldots, v_e, y_{r+1}, \ldots, y_{e+1}$. Let $\sigma$ be the (generalized) simplex

$$\sigma = \text{conv}(\{v_1, \ldots, v_e\} \cup \{y_{r+1}, \ldots, y_{e+1}\}).$$

Thus $\dim \sigma = e$, so $\text{ri}(\sigma) \subseteq \text{ri}(P)$. Pick now the point

$$z = (1/r)(v_1 + \cdots + v_e) + y_{r+1} + \cdots + y_{e+1}.$$

Then $z \in \text{ri}(v)$, and $\#(z) \leq 6d^3\#(V \cup D)$. □

**Corollary 2.3.** There is a polynomial $p_4$ such that for every nonempty polyhedron $P \subseteq \mathbb{R}^d$ and each supporting family $F$ for $P$ there is a point $z$ in $\text{ri}(P)$ such that

$$\#(z) \leq p_4(d)\#(F).$$

**Proof.** The proof clearly follows from the two previous lemmas, taking $p_3 = p_1p_2$. □

**Lemma 2.4.** There is a polynomial $p_4$ such that for every polyhedron $P \subseteq \mathbb{R}^d$ and each extreme family $\Gamma \subseteq \mathbb{R}^d$ for $P$ there is a supporting family $F$ for $P$ such that

$$\#(F) \leq p_4(d)\#(V \cup D).$$

**Proof.** We claim first that it is sufficient to consider the case in which $P$ contains the origin 0. Indeed, assume that case is established, and pick an arbitrary $P$. If $P$ is empty, the problem is trivial, so assume $P$ is nonempty. Thus, by Lemma 2.2 there is a $z$ with small size in $P$. Translating coordinates $x \mapsto x - z$, $P$ is transformed into a polyhedron $P_0$ containing 0, and $P_0 = \text{conv}(V_0; D)$, with $V_0$ corresponding to translates by $z$. So $V_0 \cup D$ increases at most polynomially from $V \cup D$. If the family $\Gamma_0$ obtained for $P_0$ is small, a new translation of coordinates $x \mapsto x + z$ results in a family $\Gamma$ for $P$ with $\#(\Gamma)$ again small. Thus we may assume that $P$ contains 0.

Let $P^0 \subseteq \mathbb{R}^d$ be the polar of $P$. This is the polyhedron consisting of all the vectors $q =$
(q₁, ..., qₖ)' such that
qᵢ'vᵢ ≤ 1 and qᵢ'yⱼ ≤ 0
for all vᵢ in V and all yⱼ in D (prime indicates transpose). Thus P° admits the supporting family Γ° consisting of all hyperplanes vᵢx = 1 and yⱼx = 0; this family has #(Γ°) ≤ #(V ° ∪ D °) + 1. By Lemma 2.1 applied to P° there is an extreme family V ° ∪ D ° for P° with #(V ° ∪ D °) small. Since P is a closed convex set and contains the origin, it is reflexive in the sense that (P°)° = P [8]. Again applying the above argument, the extreme family V ° ∪ D ° gives rise to a small length support family F = (F°) °, as desired. □

Corollary 2.5. There is a polynomial p₅ such that for every polyhedron P ⊆ Rᵈ, each projection π(x₁, ..., xₙ) = (x₁, ..., xₑ) ∈ P, and each supporting family F for P, P₁ := π(P) is a polyhedron admitting a supporting family F₁ such that
#(F₁) ≤ p₅(d) #(F).
Proof. By Lemma 2.1 there exists an extreme family V ∪ D with #(V ∪ D) ≤ p₁(d) #(Γ). Let
V₁ = π(V), D₁ = π(D). Thus P₁ = conv(V₁ ∪ D₁) and #(V₁ ∪ D₁) ≤ #(V ∪ D). By Lemma 2.4 applied to P₁ with extreme family S₁ ∪ D₁ there is a supporting family Γ₁ as desired, with p₁ = p₁P₄. □

3. Complexity results

We consider prenex-form formulas
Φ = Qᵢ₁xᵢ₁ ... Qᵢₖxᵢₖ F(x₁, ..., xₙ) (3.1)
formed by propositional combinations of terms τ of the form
qᵢ₁xᵢ₁ + ... + qᵢₖxᵦ ρ q₀, (3.2)
where ρ is one of the relational symbols <, ≤, =, and #:; the qᵢ are rational constants, represented as quotients of integers written in binary, and the variables xᵢ are those in the list x₁, x₂, ..., with binary subscripts. Let L be the set of closed such formulas for which there exist real values for the variables in Φ such that Φ is a true sentence, when the rational and relational symbols, and +, are interpreted over the reals.

Let x₁, ..., xₙ be all the variables appearing in F. Without loss of generality we assume that these are the first d variables in the language. We shall assume that all terms τ in F are as in (3.2) but with r = d, adding zero coefficients qᵢ if necessary; padding equations of F with zeroes at most squares its length.

The kth level Lₖ of L is formed from all formulas Φ in L beginning with an existential quantifier and having k − 1 alternations (thus, L₀ corresponds to the quantifier-free formulas); more precisely, a formula Φ of L is in Lₖ if there is grouping of its variables x₁, ..., xₙ into subsets
z₁ = {x₁, ..., xₙ₁}, z₂ = {xₙ₁+1, ..., xₙ₂}, ...

such that Φ has the form
∃z₁ ∀z₂ ... Qₖzₖ F, (3.3)
where ∃z₁ stands for the group of quantifications ∃x₁ ... ∃xₙ₁, etc., and Qₖ is existential if k is odd and universal if k is even. Let Lₖ⁺ denote the corresponding π-hierarchy, obtained from formulas that start with universal quantification; we phrase results in terms of Lₖ, but the analogous ones apply to Lₖ⁺.

We denote quantification over finite ranges by ∃ₐz and ∀zᵦ, meaning that, for each variable in the group z = {x₁, ..., xₙ}, only (rational) values with #(xᵦ) < a are considered in checking truth over R. Let Σₖ denote the kth level set of the polynomial hierarchy [11]. Thus, for any fixed k, Lₖ is in Σₖ if and only if there exists a polynomial p(n) such that a formula Φ as in (3.3) is in Lₖ iff
∃p(|Φ|)z₁ ... Qₖp(|Φ|)zₖ F
is true. Here, |Φ| denotes the length of Φ. The main result is as follows.

Theorem 3.1. Lₖ is in Σₖ⁺.

Before proving the theorem, we establish a couple of technical results. To each term (3.2) in formula Φ (with r = d = number of variables) we associate the hyperplane H in Rⁿ obtained when ρ is replaced by equality, if this is a nontrivial hyper-
plane. Let \( \Gamma = \Gamma(\Phi) \) be the set of hyperplanes of \( \Phi \). We denote \( \#(\Phi) = \#(\Gamma) \). Of course, \( \#(\Phi) < \#(\Gamma) \). If \( \Phi = \Phi(z) \) is a formula as above, with free variables \( z = \{x_1, \ldots, x_d\} \), we let

\[ S(\Phi) = \{ z \in \mathbb{R}^d \text{ s.t. } \Phi(z) \text{ is true} \} . \]

We first establish the case \( k = 0 \) of the theorem, i.e., that \( L_0 \) is in NP.

**Lemma 3.2.** There exists a polynomial \( q_1 \) such that, if \( F \) is quantifier-free, with variables \( z = \{x_1, \ldots, x_d\} \), then \( F \) is satisfiable, i.e., \( \Phi = \exists z F(z) \) holds, if and only if \( \exists q_1(d) \#(\Phi) \leq \#(F(z)) \) holds.

**Proof.** Let \( q_1 \) be the polynomial \( p_3 \) obtained in Corollary 2.3. We consider the extended form \( F^* \) of \( F \) obtained by the following procedure. First bring \( F \) to disjunctive normal form. Then replace each negation by the corresponding positive term, e.g.,

\[ -(s < c) \rightarrow (-s \leq -c) , \]

\[ -(s \neq c) \rightarrow (s = c) . \]

Next replace the \( \leq \) and \( \neq \) signs:

\[ (s \leq c) \rightarrow (s < c) \lor (s = c) , \]

\[ (s \neq c) \rightarrow (s < c) \lor (-s < -c) , \]

and bring the expression again into DNF; this is \( F^* \). Thus, \( F^* \) is a disjunction of formulas \( G \) of type

\[ (a_1 = a_1) \land \cdots \land (a_s = a_s) \land (\beta_1 < b_1) \land \cdots \land (\beta_t < b_t) . \]

The above construction is such that \( \mathcal{S}(F) = \mathcal{S}(F^*) \). Further, this set is the union of the sets \( \mathcal{S}(G) \) corresponding to the formulas \( G \) appearing in (3.4) above. If \( F \) is satisfiable, then at least one such \( G \) is satisfiable. Assume then that \( Q = \mathcal{S}(G) \) is nonempty, for some \( G \). Consider the polyhedron \( P = \mathcal{S}(G') \) corresponding to the non-strict formula \( G' \):

\[ (a_1 = a_1) \land \cdots \land (a_s = a_s) \land (\beta_1 \leq b_1) \land \cdots \land (\beta_t \leq b_t) . \]

Since \( Q \) is nonempty, \( Q = \text{ri}(P) \). Further, and this is the critical observation, the family \( \Gamma \) consisting of all the hyperplanes \((a_i = a_i)\) and \((\beta_j = b_j)\) is a supporting family for \( P \) and

\[ \#(\Gamma) \leq \#(\Phi) . \]

This last inequality is due to the fact that all these hyperplanes arise from terms in the original formula \( F \), i.e., \( \Gamma \subseteq \Gamma(\Phi) \). By Corollary 2.3, \( \mathcal{Q} \) — and hence \( \mathcal{S}(F) \) — contains a point \( z \) as desired.

**Lemma 3.3.** There exists a polynomial \( q_2 \) such that, for any \( \Phi = \exists u F(z, u) \) with free variables \( z = \{x_1, \ldots, x_e\} \) and with bound variables \( u = \{x_{e+1}, \ldots, x_d\} \), there is a quantifier-free formula \( \Psi(z) \) such that \( \mathcal{S}(\Phi) = \mathcal{S}(\Psi) \) and \( \#(\Psi) \leq q_2(d) \#(\Phi) \).

**Proof.** Let \( q_2 \) be the polynomial \( p_5 \) from Corollary 2.5. Consider again the expanded form \( F^* \) of \( F \). Note that, with the notations of Corollary 2.5, \( \mathcal{S}(F) = \pi(\mathcal{S}(F^*)) \). Since \( \mathcal{S}(F^*) \) is the union of the \( \mathcal{S}(G) \) in the proof of Lemma 3.2, it is enough to consider the projections of such sets: if quantifier-free formulas \( \Psi_G \) are found with \( \mathcal{S}(Q_G) = \mathcal{S}(G) \) for each of these such that \( \#(\Psi_G) \leq q_2(d) \#(G) \leq q_2(d) \#(\Phi) \), the disjunction \( \Psi \) of such \( \Psi_G \) will be as desired. So consider any \( G \). Without loss of generality, \( Q = \mathcal{S}(G) \) is nonempty. Again with the notations of the previous proof, consider \( G' \) and \( P = \mathcal{S}(G') \), so that \( Q = \text{ri}(P) \). Let \( P_1 \) be the projection \( \pi(P) \) and \( Q_1 = \pi(Q) \); thus \( Q_1 = \text{ri}(P_1) \). Let \( \Gamma = \Gamma(G') \) be the set of hyperplanes of formula \( G' \). By Corollary 2.5, applied to \( P \) and \( \Gamma \), \( P_1 \) can be written as an intersection of halfspaces and hyperplanes giving rise to a supporting family \( \Gamma_1 \) with \( \#(\Gamma_1) \leq q_2(d) \#(\Gamma) \). Thus, \( P_1 = \mathcal{S}(\Psi_1) \), where \( \Psi_1 \) is some formula as in (3.5) whose hyperplanes are in \( \Gamma' \). We need then a formula \( \Psi \) such that \( \mathcal{S}(\Psi) = \text{ri}(P_1) \). By (2.3), \( \Psi \) can be obtained from \( \Psi_1 \) through changing some of the inequalities \( \leq \) to equalities and others to strict inequalities. This preserves the hyperplanes, and hence \( \#(\Psi) = \#(\Gamma_1) \).

**Proof of Theorem 3.1.** We may now prove the theorem, by induction on the number \( k \) of alternations. Specifically, we shall prove by induction that all formulas \( \Phi \) in \( L_k \) (and \( L_k^* \)) can be decided
by checking points whose coordinates have length less than \( p_k(d) \#(\Phi) \), where

\[
p_k(d) = (8q_1(d)q_2(d)k^{-1})^k.
\]

Note that the case \( k = 1 \) follows from Lemma 3.2. Since \( \#(\Phi) < n \Rightarrow |\Phi| \), it will then follow that \( p(n) = np_k(n) \) satisfies the requirements.

Let then \( \Phi \) be as in (3.3) (the case of \( \pi \)-formulas is analogous). Using Lemma 3.3 repeatedly, we eliminate \( k - 1 \) quantifiers, starting with \( Q_k \). Thus, there exists a quantifier-free formula \( \Psi(z_1) \) such that

\[
S(\forall z_2 \ldots Q_k z_k F(z_1, \ldots, z_k)) = S(\Psi),
\]

and \( \#(\Psi) \leq q_2(d)^{k-1} \#(\Phi) \). It follows from Lemma 3.2 that \( \Phi \) is true iff \( \exists \lambda z_1 \Psi(z_1) \) holds, where \( \lambda = q_1(d)q_2(d)^{k-1} \#(\Phi) \). For any fixed \( \xi \) with \( \#(\xi) \leq \lambda \), consider the formula

\[
\Phi_\xi := \forall z_2 \ldots Q_k z_k F(\xi, z_2, \ldots, z_k).
\]

Since only a substitution is performed, \( \#(\Phi_\xi) \leq 8q_1(d)q_2(d)^{k-1} \#(\Phi) \). We conclude that \( \Phi \) is true if and only if one of the formulas \( \Phi_\xi \) holds, for some \( \xi \) as above. By the induction hypothesis, each such \( \Phi_\xi \) can be decided by checking at points \( (\xi, z_2, \ldots, z_k) \) such that the components of the \( z_i \) have lengths less than \( p_{k-1}(d-1) \#(\Phi_\xi) \). Thus these lengths are less than \( p_k(d-1) \#(\Phi) \). Since the same bound holds for \( \xi \), the induction step is completed. \( \square \)

Remark. As a decision method for (all of) \( L_n \), we obtain an alternating space complexity of the order of

\[
2^{cn^2 \log n},
\]

which is considerably worse than that in [6], namely \( 2^{\omega n} \). This strongly suggests that the actual constructions given in this paper could be improved considerably.

Let \( B_k \) be, as in [11], the \( k^{th} \) level quantified Boolean satisfiability problem. Given a formula in \( B_k \), with variables \( x_1, \ldots, x_d \), replace each occurrence of \( x_i \) by the term \( (x_i = 1) \) and each occurrence of its negation \( \neg x_i \) by the term \( (x_i = 0) \), for each \( i = 1, \ldots, d \). Adding the conjuncts \( (x_i = 0) \lor (x_i = 1) \) for all \( i \), such a formula is log-space reduced to a formula in \( L_k \), preserving truth. From this, the above theorem, and the completeness of \( B_k \cap 3\text{DNF} \) or \( B_k \cap 3\text{CNF} \) (see, e.g., [11]) we have the following.

**Corollary 3.4.** \( L_k \) is log-complete in \( \Sigma_k^p \).

Acknowledgment

We would like to thank Martin Dowd for useful discussions and for pointing out to us reference [5].

References