Finite Gain Stabilization of Discrete-Time Linear Systems
Subject to Actuator Saturation

Xiangyu Bao† Zongli Lin† Eduardo D. Sontag‡

† Department of Electrical Engineering, University of Virginia
Charlottesville, VA 22903, U.S.A.
‡ Department of Mathematics, Rutgers University, New Brunswick, NJ 08903, U.S.A.

Abstract

It is shown that, for neutrally stable discrete-time linear systems subject to actuator saturation, finite
gain $l_p$ stabilization can be achieved by linear output feedback, for all $p \in (1, \infty]$. An explicit construction
of the corresponding feedback laws is given. The feedback laws constructed also result in a closed-loop
system that is globally asymptotically stable, and in an input-to-state estimate.

Key Words : input saturation, discrete-time linear systems, finite gain stability, Lyapunov functions.

†This paper was not presented at any IFAC meeting. Corresponding author: Professor Zongli Lin. Tel. +1(804)924-6342;
Fax +1(804)924-8818; Email zl5y@virginia.edu.
1 Introduction

In this paper, we consider the problem of global stabilization of a discrete-time linear system subject to actuator saturation:

\[ P : \begin{cases} \dot{x} = Ax + B\sigma(u + u_1), & x \in \mathbb{R}^n, \ u \in \mathbb{R}^m \\ y = Cx + u_2, & y \in \mathbb{R}^r \end{cases} \]  

(we use the notation \( x^+ \) to indicate a forward shift, that is, for a function \( x \) and an integer \( t \), \( x^+(t) = x(t+1) \)), where \( u_1 \in \mathbb{R}^m \) is the actuator disturbance, \( u_2 \in \mathbb{R}^r \) is the sensor noise, and \( \sigma : \mathbb{R}^m \to \mathbb{R}^m \) represents actuator saturation, i.e., \( \sigma(s) = [\sigma_1(s_1) \ \sigma_2(s_2) \ \cdots \ \sigma_m(s_m)] \) with \( \sigma_i(s_i) = \text{sign}(s_i) \min\{1,|s_i|\} \), and the pair \((A, B)\) is stabilizable. The problem of global asymptotic stabilization (internal stabilization) of this system has recently been solved using nonlinear state feedback laws and under the condition that all the eigenvalues of \( A \) are inside or on the unit circle (Yang et al., 1997), and for neutrally stable open-loop system using linear state feedback (Choi, 1999). Here, we are interested not only in closed-loop state space stability (internal stability), but also in stability with respect to both measurement and actuator noises. More specifically, we would like to construct a controller \( C \) so that the operator \((u_1, u_2) \mapsto (y_1, y_2)\) as defined by the following standard systems interconnection (see Fig. 1.)

\[ \begin{cases} y_1 = P(u_1 + y_2) \\ y_2 = C(u_2 + y_1) \end{cases} \]  

is well-defined and finite gain stable.

![Figure 1: Standard closed-loop connection](image-url)

We note that the disturbance \( u_1 \) we consider here is input additive and enters the system together with the control input \( u \) through the actuators. Simple examples show that the problem we consider does not always have a solution if the disturbance enters the system from outside the actuators.

The above problem was first studied for continuous-time systems. It was shown in (Liu et al., 1996) that, for neutrally stable open loop systems, linear feedback laws can be used to achieve finite gain stability, with respect to every \( L_p \)-norm. For a neutrally stable system, all open loop poles are located in the closed left-half plane, with those on the \( j\omega \) axis having Jordan blocks of size one. In the case that full state is available for feedback (i.e., \( y_1 = x \) and \( u_2 = 0 \)), it was shown in (Lin et al., 1995) that if the external input signal is uniformly bounded, then finite-gain \( L_p \)-stabilization and local asymptotic stabilization can always be achieved simultaneously by linear feedback, no matter where the poles of the open loop system are. The uniform boundedness condition of (Lin et al., 1995) was later removed by resorting to nonlinear feedback (Lin, 1997). Some other works related to the topic are Hou et al. (1997), Chitour et al. (1995), Nguyen and Jabbari (1997), Saberi et al. (1998), Suarez et al. (1997) and the references therein.

There are also several studies in the discrete-time setting, showing some of the continuous-time results carry over to discrete-time (for example, Hou et al., 1997 and Yang et al., 1997) and some do not (for
We use where

We will in fact actually obtain the following stronger ISS-like property (see, Sontag (1998) and Remark 1 references there):

An explicit construction of the corresponding feedback laws is given. The feedback laws constructed also result in a closed-loop system that is globally asymptotically stable, and provide an input-to-state estimate.

While many of the arguments used are conceptually similar to those used in the continuous-time case (Liu et al., 1996), there are technical aspects that are very different and not totally obvious. For example, unlike in (Liu et al., 1996), the feedback gain for the discrete-time case needs to be multiplied by a small factor, say \( \kappa \), which causes the solution of a certain Lyapunov equation, and the subsequent estimation of the solution, to be dependent on \( \kappa \) (see Lemma 2). As another example, the difficulties in evaluating the difference of the non-quadratic Lyapunov function along the trajectories of the closed-loop system entail a careful estimation by Taylor expansion.

The remainder of the paper is organized as follows. Section 2 states the main results. Section 3 contains the proof of the results that were stated in Section 2. A brief concluding remark is given in Section 4.

## 2 Preliminary and Problem Statement

We first recall some notation. For a vector \( X \in \mathbb{R}^l \), \(|X|\) denotes the Euclidean norm of \( X \), and for a matrix \( X \in \mathbb{R}^{m \times n} \), the induced operator norm. For any \( p \in [1, \infty) \), we write \( l^p \) for the set of all sequences \( \{x(t)\}_{t=0}^{\infty} \), where \( x \in \mathbb{R}^n \), such that \( \sum_{t=0}^{\infty} |x(t)|^p < \infty \), and the \( l^p \)-norm of \( x \in l^p \) is defined as \( \|x\|_p = (\sum_{t=0}^{\infty} |x(t)|^p)^{1/p} \).

We use \( l^\infty \) to denote the set of all sequences \( \{x(t)\}_{t=0}^{\infty} \), where \( x \in \mathbb{R}^n \), such that \( \sup_t |x(t)| < \infty \), and the \( l^\infty \)-norm of \( x \in l^\infty \) is defined as \( \|x\|_\infty = \sup_t |x(t)| \).

The objective of this paper is to show the following result concerning the global asymptotic stabilization as well as \( l_p \)-stabilization of system \( P \), as given by (1), using linear output feedback.

**Theorem 1** Consider a system (1). Let \( A \) be neutrally stable, i.e., all the eigenvalues of \( A \) are inside or on the unit circle, with those on the unit circle having all Jordan blocks of size one. Also assume that \((A, B)\) is stabilizable and \((A, C)\) is detectable. Then, there exits a linear observer-based output feedback law of the form

\[
\begin{align*}
\dot{x}^+ &= A\dot{x} + B\sigma(F\dot{x}) - L(y - C\dot{x}) \\
u &= F\dot{x}
\end{align*}
\]

which has the following properties:

1. It is finite gain \( l_p \)-stable for all \( p \in (1, \infty] \), i.e., there exists a \( \gamma_p > 0 \) such that

\[
\|x\|_p \leq \gamma_p \left[ \|u_1\|_p + \|u_2\|_p \right], \quad \forall u_1 \in l^p, \ u_2 \in l^p, \text{ and } x(0) = 0, \dot{x}(0) = 0.
\] (4)

2. In the absence of actuator and sensor noises \( u_1 \) and \( u_2 \), the equilibrium \((x, \dot{x}) = (0, 0)\) is globally asymptotically stable.

**Remark 1** We will in fact actually obtain the following stronger ISS-like property (see, Sontag (1998) and references there):

\[
\| (x, \ddot{x}) \|_p \leq \theta_p (|x(0)| + |\dot{x}(0)|) + \gamma_p \left[ \|u_1\|_p + \|u_2\|_p \right]
\] (5)
where $\theta_p$ is a class-$K$ function. Observe that the single estimate (5) encompasses both the gain estimate (4) and asymptotic stability. Obviously, (4) is the special case of (5) for zero initial states. On the other hand, when applied with arbitrary initial states but $u_1 = u_2 = 0$, there follows that $(x, \dot{x})$ is in $l_p$, which implies, in particular, that $(x(t), \dot{x}(t))$ must converge to zero as $t \to \infty$ (global attraction) and that $|(x(t), \dot{x}(t))|$ is bounded by $\theta_p(|x(0)| + |\dot{x}(0)|)$ (stability).

3 Proof of Theorem 1

The proof of Theorem 1 will follow readily from the following proposition, which we establish first.

**Proposition 1** Let $A$ be orthogonal (i.e., $A' A = I$), and suppose that the pair $(A, B)$ is controllable. Then, the system

$$x^+ = Ax + B \sigma(-\kappa B' Ax + u), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m$$

(6)

is finite gain $l_p$-stable, $p \in (1, \infty]$, for sufficiently small $\kappa > 0$. Moreover, for each $p \in (1, \infty]$ there exist a real $\gamma_p$, an $\kappa^* \in (0, 1]$, and a class-$K$ function $\theta_p$ such that, for all $\kappa \in (0, \kappa^*)$,

$$\|x\|_{l_p} \leq \gamma_p \|u\|_{l_p} + \theta_p(|x(0)|)$$

(7)

for all inputs $u \in l_p^m$ and all initial states $x(0)$.

To prove this proposition, we need to establish a few lemmas.

**Lemma 1** For any $p > l > 0$, there exist two scalars $M_1, M_2 > 0$ such that, for any two positive scalars $\xi$ and $\zeta$,

$$\xi^{p-l} \zeta^l \leq M_1 \xi^p + M_2 \zeta^p$$

(8)

and consequently, for any $n > 0$ and $\kappa > 0$,

$$\xi^{p-l} \zeta^l \leq M_1 \kappa^n \xi^p + \kappa^{\frac{n(1-p)}{l}} \zeta^p.$$  

(9)

**Proof of Lemma 1.** Let $h : \mathbb{R}^+ \to \mathbb{R}^+$ be defined as $h(x) = x^{\frac{l}{p-l}}$, which is continuous and strictly increasing with $h(0) = 0$ and $h(\infty) = \infty$, and $k(x) = x^{-\frac{l}{p-l}}$ be its pointwise inverse. Define

$$H(x) = \int_0^x h(v)dv = \frac{p-l}{p} x^{\frac{p}{p-l}}$$

and

$$K(x) = \int_0^x k(v)dv = \frac{l}{p} x^{\frac{l}{p}}.$$  

(10)

(11)

Letting $a = \xi^{p-l}$ and $b = \zeta^l$, it follows from Young’s inequality (Hardy el at., 1952), $ab \leq H(a) + K(b)$ for all $a, b \in \mathbb{R}^+$, that

$$\xi^{p-l} \zeta^l \leq \frac{p-l}{p} \xi^p + \frac{l}{p} \zeta^p = M_1 \xi^p + M_2 \zeta^p,$$  

(12)

which also trivially implies (9).
Lemma 2} Let $A$ and $B$ be as given in Proposition 1. Then, for any $\kappa > 0$ such that $\kappa B' B < 2I$, $\bar{A}(\kappa) = A - \kappa BB' A$ is asymptotically stable. Moreover, let $P(\kappa)$ be the unique positive definite solution to the Lyapunov equation,

$$\bar{A}(\kappa)' P \bar{A}(\kappa) - P = -I.$$  

(13)

Then, there exists a $\kappa^* > 0$ such that

$$\frac{\chi_1}{\kappa} I \leq P(\kappa) \leq \frac{\chi_2}{\kappa} I, \quad \forall \kappa \in (0, \kappa^*]$$

(14)

for some positive constants $\chi_1$ and $\chi_2$ independent of $\kappa$.

**Proof of Lemma 2:** The asymptotic stability of $\bar{A}$ follows from a simple Lyapunov/LaSalle argument (Choi, 1999). Let $\kappa_1^* > 0$ be such that $\kappa B' B < 2I$ for all $\kappa \in (0, \kappa_1^*]$. We recall that the solution to the Lyapunov equation (13) is given by

$$P(\kappa) = \sum_{k=0}^{\infty} (\bar{A}(\kappa))' \bar{A}(\kappa) = \sum_{k=0}^{\infty} [(A - \kappa BB' A)^i]' [(A - \kappa BB' A)^i].$$

(15)

Using the fact that $AA' = I$, we have

$$(A - \kappa BB' A)'(A - \kappa BB' A) = I - 2\kappa A' BB' A + \kappa^2 A' BB' BB' A$$

$$= I - \kappa A' B (2I - \kappa B' B) B' A.$$  

(16)

Using now the fact that $\kappa B' B < 2I$ for $\kappa \in (0, \kappa_1^*]$, we know that there exists $\kappa_2^* \in (0, \kappa_1^*]$ such that

$$\frac{1}{2} I \leq (A - \kappa BB' A)'(A - \kappa BB' A) \leq I, \quad \forall \kappa \in (0, \kappa_2^*].$$

(17)

Again using the fact that $A' A = I$, we verify in a straightforward way that

$$(A - \kappa BB' A)^n = A^n - \kappa C_{A,B} C_{A,B}' A^n + \kappa^2 M_1(\kappa)$$

(18)

where $M_1(k)$ is a polynomial matrix in $\kappa$ of order $n - 2$, $n$ being the order of the system (6), and

$$C_{A,B} = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$$

is the controllability matrix of the pair $(A, B)$ and is of full rank. It then follows that

$$((A - \kappa BB' A)^n)'(A - \kappa BB' A)^n$$

$$= (A^n)' - \kappa (A^n)' C_{A,B} C_{A,B}' + \kappa^2 M_1(\kappa))(A^n - \kappa C_{A,B} C_{A,B}' A^n + \kappa^2 M_1(\kappa))$$

$$= I - 2\kappa (A^n)' C_{A,B} C_{A,B}' A^n + \kappa^2 M_2(\kappa)$$

(19)

where $M_2(\kappa)$ is a symmetric polynomial matrix in $\kappa$ of order $2n - 2$. Since $C_{A,B}$ is of full rank, and because $A$ is nonsingular, there exists a $\kappa^* \in (0, \kappa_2^*]$ such that

$$0 \leq I - \kappa M_1 I \leq (A - \kappa BB' A)^n)'(A - \kappa BB' A)^n \leq I - \kappa M_2 I < I, \quad \forall \kappa \in (0, \kappa^*]$$

(20)

for some constants $M_1^0, M_2^0 > 0$ independent of $\kappa$.

Using (17), (20) and the fact that $A' A = I$ in (15), we have that for all $\kappa \in (0, \kappa^*],$

$$P(\kappa) \leq \sum_{i=0}^{n-1} [(A - \kappa BB' A)^i]'(A - \kappa BB' A)^i \sum_{k=0}^{\infty} (1 - \kappa M_2^0)^k I \leq n \frac{1}{\kappa M_2^0} I = \frac{\chi_2}{\kappa} I.$$  

(21)
and
\[ P(\kappa) \geq \sum_{i=0}^{n-1} [(A - \kappa BB' A)^i]^t (A - \kappa BB' A)^i \sum_{\kappa=0}^{\infty} (1 - \kappa M_1^2)^k I \geq \left( \frac{1}{2} \right)^{n-1} \frac{1}{\kappa M_1^2} I = \frac{\chi_1}{\kappa} I \] (22)
where \( \chi_1 = \frac{n}{2n-1} \) and \( \chi_2 = \frac{n}{M_2^2} \).

**Lemma 3** Let \( \bar{A}(\kappa) \) be as given in Proposition 1, \( P(\kappa) \) as defined in Lemma 2, then for any \( p \in (1, \infty) \), there exists a \( \kappa^* > 0 \) such that
\[ |x' \bar{A}(\kappa) P(\kappa) \bar{A}(\kappa)x|^{p/2} - |x' P(\kappa)x|^{p/2} \leq -\kappa^{\frac{p-2}{2}} \zeta |x|^p, \ \kappa \in (0, \kappa^*], \] (23)
where \( \zeta > 0 \) is some constant independent of \( \kappa \).

**Proof of Lemma 3.** Inequality (23) holds trivially for \( x = 0 \). Hence in what follows, we assume, without loss of generality, that \( x \neq 0 \).

For simplicity, we introduce from now the following notation:
\[ \mu = x' \bar{A}(\kappa) P(\kappa) \bar{A}(\kappa)x \] (24)
(where \( x \) and \( \kappa \) will be clear from the context). By the definition of \( P(\kappa) \), we have
\[ \mu - x' P(\kappa)x = -x' x. \] (25)
From Lemma 2, there exists a \( \kappa_1^* > 0 \) such that for all \( \kappa \in (0, \kappa_1^*] \)
\[ \left| \frac{x' x}{x' P(\kappa)x} \right| \leq \frac{4}{5}, \ \forall x \neq 0. \] (26)

With (25) and (26), we can continue the proof using Taylor expansion with remainder,
\[ [x' \bar{A}'(\kappa) P(\kappa) \bar{A}(\kappa)x]^{p/2} - [x' P(\kappa)x]^{p/2} \]
\[ = [x' P(\kappa)x - x' x]^{p/2} - [x' P(\kappa)x]^{p/2} \]
\[ \leq \left[ x' P(\kappa)x \right]^{p/2} \left[ 1 - \left( \frac{x' x}{x' P(\kappa)x} \right)^{p/2} \right] - [x' P(\kappa)x]^{p/2} \]
\[ = -\frac{p}{2} [x' P(\kappa)x]^{p-2} |x|^2 + \delta [x' P(\kappa)x]^{p-4} |x|^4, \ \kappa \in (0, \kappa_1^*] \] (27)
where \( \delta = \max_{|z| \leq 2} \left\{ \frac{p}{8} (p-2)(1+z)^{\frac{p}{2}-2} \right\} \) is a constant independent of \( \kappa \).

Again by Lemma 2, there exists a \( \kappa^* \in (0, \kappa_1^*] \) such that
\[ |\mu|^{p/2} - |x' P(\kappa)x|^{p/2} \leq -\kappa^{\frac{p-1}{2}} \zeta |x|^p, \ \kappa \in (0, \kappa^*] \] (28)
for some \( \zeta > 0 \) independent of \( \kappa \).

**Lemma 4** Let \( A \) and \( B \) be as given in Proposition 1. For any \( l \in [1, \infty) \) and any \( \kappa \in (0, 1] \),
\[ |\sigma(-\kappa B' Ax + u)|^l \leq 2^{l-1} \kappa^l |B|^l |x|^l + 2^{l-1} |u|^l. \] (29)
Proof of Lemma 4. Since $\sigma$ is a standard saturation function and $|A|=1$, for any $l \geq 1$, we have
\[ |\sigma(-\kappa B' Ax + u)|^l \leq (\kappa |B||x| + |u|)^l \leq 2^{l-1} \kappa |B||x| + 2^{l-1} |u|^l , \]
where the last inequality follows from Jensen’s inequality applied to the convex function $s^l$:
\[ (a + b)^l \leq \frac{1}{2} (2a)^l + \frac{1}{2} (2b)^l, \quad \forall a, b \geq 0. \]
\[
\]
Lemma 5 Let $A$ and $B$ be as given in Proposition 1. Pick any $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$, any number $\eta \geq 3$, and any nonnegative real number $l$. Denote $\tilde{x} = -\kappa B' Ax + u$. Then, provided $|x| > \eta |B\sigma(\tilde{x})|$, we have:
\[ |Ax + B\sigma(\tilde{x})|^l \leq |x|^l + l|x|^{l-2}x'A'B\sigma(\tilde{x}) + M|x|^{-2}|B\sigma(\tilde{x})|^2 , \tag{31} \]
for some constant $M > 0$ which is independent of $\kappa$.

Proof of Lemma 5. We first note that, since $|x| > \eta |B\sigma(\tilde{x})| \geq 3|B\sigma(\tilde{x})|$, we have:
\[ \frac{2x' A' B\sigma(\tilde{x}) + |B\sigma(\tilde{x})|^2}{|x|^2} \leq \frac{4}{5} . \tag{32} \]
Hence, using Taylor expansion with remainder, we have
\[ |Ax + B\sigma(\tilde{x})|^l = |x|^l \left( 1 + \frac{2x' A' B\sigma(\tilde{x}) + |B\sigma(\tilde{x})|^2}{|x|^2} \right)^{\frac{l}{2}} \]
\[ \leq |x|^l \left[ 1 + \frac{l}{2} \frac{2x' A' B\sigma(\tilde{x}) + |B\sigma(\tilde{x})|^2}{|x|^2} + \delta \left( \frac{2x' A' B\sigma(\tilde{x}) + |B\sigma(\tilde{x})|^2}{|x|^2} \right)^2 \right] \]
\[ \leq |x|^l + l|x|^{l-2}x'A'B\sigma(\tilde{x}) + \frac{l}{2} |x|^{l-4} |B\sigma(\tilde{x})|^2 + \delta |x|^{-4} ((2 + 1/\eta)|x||B\sigma(\tilde{x}))^2 \] \tag{33}
where $\delta = \max_{|x| \leq \eta} \left\{ \frac{l}{8} \left( (l - 2)(1 + z)^{\frac{l}{2}} \right) \right\}$ is a constant independent of $\kappa$.

So we can see that the inequality (31) holds for $M = \frac{l}{2} + \delta (2 + 1/\eta)^2$. \hfill \[\Box\]

We are now ready to prove Proposition 1.

Proof of Proposition 1: We separate the proof for $p \in (1, \infty)$ and for $p = \infty$.

For clarity, let us repeat here the system equation (6):
\[ x^+ = Ax + B\sigma(-\kappa B'Ax + u), \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m . \tag{34} \]
This may also be rewritten as:
\[ x^+ = \tilde{A}(\kappa)x + B(-\tilde{x} + \sigma(\tilde{x}) + u), \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m , \tag{35} \]
where $\tilde{A}(\kappa) = A - \kappa BB'A$, $\tilde{x} = -\kappa B'Ax + u$.

For this system, define the function $V_1$ as:
\[ V_1(x) = (x' P(\kappa)x)^{p/2} , \tag{36} \]
where $P(\kappa)$ is as given in Lemma 2. We next evaluate the increments $V(x^+(t)) - V(x(t))$, which we denote as “$\Delta V_1$” for short, along any given trajectory of (35). It is convenient to treat separately the cases $|x| > \eta|B\sigma(\tilde{x})|$ and $|x| \leq \eta|B\sigma(\tilde{x})|$. Here $\eta \geq 3$ is a number to be specified soon.

Case 1: $|x| > \eta|B\sigma(\tilde{x})|$.

Using the definition of $V_1$, we now give an upper bound on $\Delta V_1$ along the trajectories of the system (35). To simplify the equations, we introduce the following notation:

$$\nu = 2\kappa' A'P(\kappa)BB'Ax + 2x' A' P(\kappa)B\sigma(\tilde{x}) + \sigma'(\tilde{x})B' P(\kappa)B\sigma(\tilde{x}),$$

in addition to $\mu$ as defined in Equation (24). Thus:

$$\Delta V_1 = V_1^+ - V_1$$

$$= [(x^+)^t P(\kappa)x^+]^{p/2} - [x^t P(\kappa)x]^{p/2}$$

$$= [[\tilde{A}(\kappa)x + \kappa BB'Ax + B\sigma(\tilde{x})]' P(\kappa)[\tilde{A}(\kappa)x + \kappa BB'Ax + B\sigma(\tilde{x})]]^{p/2} - [x^t P(\kappa)x]^{p/2}$$

$$= [\mu + 2x' A' P(\kappa)B(\kappa' B'Ax + \sigma(\tilde{x})) - \kappa^2 x' A' BB'P(\kappa)BB'Ax$$

$$+ \sigma'(\tilde{x})B' P(\kappa)B\sigma(\tilde{x})]^{p/2} - [x^t P(\kappa)x]^{p/2}$$

$$\leq \left[ \mu + \nu \right]^{p/2} - [x^t P(\kappa)x]^{p/2}$$

$$= \left[ \mu \right]^{p/2} \left[ 1 + \left( \frac{\nu}{\mu} \right)^{p/2} \right] - [x^t P(\kappa)x]^{p/2} \quad (37)$$

By Lemma 2, there exist a $\kappa* > 0$ and $\eta \geq 3$ independent of $\kappa$, such that for all $|x| > \eta|B\sigma(\tilde{x})|$

$$\left| \frac{\nu}{\mu} \right| \leq \frac{4}{5}, \quad \kappa \in (0, \kappa_1^*]. \quad (38)$$

To see this, let $\kappa_0^* > 0$ be such that (14) of Lemma 2 and (17) in the proof of Lemma 2 both hold for all $\kappa \in (0, \kappa_0^*]$. Then, for all $\kappa \in (0, \kappa_0^*]$, we have

$$|\nu| \leq 2\chi_2|B|^2 + 2\chi_2 \kappa \eta + \chi_2 \kappa \eta^2 \quad (39)$$

and

$$|\mu| \geq \frac{\chi_1}{2\kappa} |x|^2 \quad (40)$$

from which it is clear that there exist $\kappa_1^*$ and $\eta > 3$ such that (38) holds.

Next, we may use a Taylor expansion with remainder to continue the bounding of $\Delta V_1$ as follows:

$$\Delta V_1 \leq \left[ \mu \right]^{p/2} \left[ 1 + \frac{p\nu}{2\mu} + \delta \left( \frac{\nu}{\mu} \right)^{p/2} \right] - [x^t P(\kappa)x]^{p/2} \quad (41)$$

where $\delta = \max_{||x|| \leq \eta} \left\{ \frac{p}{8} \left( p - 2 \right) \left( 1 + z \right)^{p-2} \right\}$ is a constant independent of $\kappa$.

By Lemma 3 and Lemma 2, there exists $\kappa_2^* \in (0, \kappa_1^*]$ such that for any $\kappa \in (0, \kappa_2^*],$ we have

$$\Delta V_1 \leq -\kappa \frac{2\nu}{\kappa} |x|^p \left[ 1 + \frac{p\nu}{2\mu} + \delta \left( \frac{\nu}{\mu} \right)^{p/2} \right] + \delta |\nu|^{p/2} |x|^2$$

$$\leq -\kappa \frac{2\nu}{\kappa} |x|^p + \psi_1 \kappa \frac{2\nu}{\kappa} |x|^{p-2} \left[ 2|x||P(\kappa)B||\tilde{x} - \sigma(\tilde{x})| + 2|x||P(\kappa)B||u| + |P(\kappa)| |B\sigma(\tilde{x})|^2 \right]$$

$$+ \psi_2 \kappa \frac{2\nu}{\kappa} |x|^p \left[ 2|x|^2 |P(\kappa)| |B|^2 + 2|P(\kappa)||x||B\sigma(\tilde{x})| + |P(\kappa)| |B\sigma(\tilde{x})|^2 \right] \quad (42)$$
where $\zeta > 0$ is as defined in Lemma 3, and $\psi_1, \psi_2 > 0$ are some constants independent of $\kappa$.

Before continuing, we digress to observe that

$$|\tilde{x} - \sigma(\tilde{x})| \leq \tilde{x}' \sigma(\tilde{x}).$$

Using (43), Lemma 1, Lemma 4, and the condition $|x| > \eta |B\sigma(\tilde{x})|$, we can show that there exists a $\kappa_3^* \in (0, \kappa_2^*]$ such that for all $\kappa \in (0, \kappa_3^*]$ the estimation of $\Delta V_1$ can be now concluded as follows:

$$\Delta V_1 \leq -\kappa \frac{2+\tilde{\kappa}}{2} |x|^p + 2\psi_1 \kappa \frac{2+\tilde{\kappa}}{2} |x|^{p-1} |P(\kappa)B|x'| \sigma(\tilde{x}) + M_1 \kappa \frac{2+\tilde{\kappa}}{2} \max\{\kappa, \kappa^{p-1}\} |x|^p + M_2 \kappa |u|^p,$$

where $M_1 > 0, M_2 > 0$ with $M_1$ independent of $\kappa$ are defined in an obvious way. In deriving (44), we have also used the fact that $|x|^{p-2} < (|B\sigma(\tilde{x})|/\eta)^{p-2}$ for $p < 2$ and $B\sigma(\tilde{x}) \neq 0$.

**Case 2:** $|x| \leq \eta |B\sigma(\tilde{x})|$.

By using Lemma 2, Lemma 3 and Lemma 4, $\Delta V_1$ along the trajectories of (35) is bounded as follows:

$$\Delta V_1 = \left[ \frac{1}{2} |x|^p + 2\psi_1 \kappa \frac{2+\tilde{\kappa}}{2} |x|^{p-1} |P(\kappa)B|x' \sigma(\tilde{x}) + M_1 \kappa \frac{2+\tilde{\kappa}}{2} \max\{\kappa, \kappa^{p-1}\} |x|^p + M_2 \kappa |u|^p, \right.$$  

where $\chi_2 > 0$ and $\zeta > 0$ are as defined in Lemma 2 and Lemma 3 respectively, and $M_1 > 0, M_2 > 0$ are constants with $M_1$ being independent of $\kappa$.

Summarizing, we may combine Case 1 with Case 2, to obtain:

$$\Delta V_1 \leq \begin{cases} 
-\kappa \frac{2+\tilde{\kappa}}{2} |x|^p + 2\psi_1 \kappa \frac{2+\tilde{\kappa}}{2} |x|^{p-1} \sigma(\tilde{x}) + M_1 \kappa \frac{2+\tilde{\kappa}}{2} \max\{\kappa, \kappa^{p-1}\} |x|^p + M_2 \kappa |u|^p, & \text{if } |x| > \eta |B\sigma(\tilde{x})|, \\
-\kappa \frac{2+\tilde{\kappa}}{2} |x|^p + M_1 \kappa \frac{2+\tilde{\kappa}}{2} \kappa^{p-1} |x|^p + M_2 \kappa |u|^p, & \text{if } |x| \leq \eta |B\sigma(\tilde{x})|,
\end{cases}$$

where $M_1 = \max\{M_1, M_1 \} \text{ and } M_2 = \max\{M_2, M_2 \}$.

For system (34), we next define another function:

$$V_0(x) = |x|^{p+1}.$$  

An estimation of its increments along the trajectories of (34) can also be carried out by separately considering each of the cases $|x| > \eta |B\sigma(\tilde{x})|$ and $|x| \leq \eta |B\sigma(\tilde{x})|$.

**Case 1:** $|x| \leq \eta |B\sigma(\tilde{x})|$.

By Lemma 4, for any $\kappa \in (0, \kappa_3^*)$,

$$\Delta V_0 = |Ax + B\sigma(\tilde{x})|^{p+1} - |x|^{p+1} \leq |Ax + B\sigma(\tilde{x})|^{p+1}$$

$$\leq \left( |x| + |B\sigma(\tilde{x})| \right)^{p+1} \leq (\eta + 1) |B\sigma(\tilde{x})|^{p+1} \leq \kappa N_1 |x|^p + N_2 \kappa |u|^p,$$

for some positive constants $N_1$ and $N_2$ independent of $\kappa$. In deriving (48), we have used the fact that both $\sigma$ and $\kappa$ are bounded.

**Case 2:** $|x| > \eta |B\sigma(\tilde{x})|$.
By Lemma 5, Lemma 4 and Lemma 1, there exists \( \kappa^*_4 \in (0, \kappa^*_3] \) such that for any \( \kappa \in (0, \kappa^*_4] \),

\[
\Delta V_0 = |Ax + B\sigma(\tilde{x})|^p + 1 - |x|^p + 1 \leq |x|^p + (p + 1)|x|^{p-1}x'A'B\sigma(\tilde{x}) + N_{1b}|B\sigma(\tilde{x})|^2|x|^{p-1} - |x|^p + 1 \\
\leq \frac{p + 1}{\kappa} |x|^{p-1}\tilde{x}'\sigma(\tilde{x}) + \kappa N_{1c}|x|^p + N_{2c}(\kappa)|u|^p 
\]

(49)

where \( N_{1c}, N_{1b} > 0, N_{2c}(\kappa) > 0 \) are constants, and \( N_{1b}, N_{1c} \) are independent of \( \kappa \). In deriving (49), the first inequality by Lemma 5, the second inequality is the consequence of the fact that \( \sigma \) is bounded and Lemmas 4 and 1.

Combining Case 1 with Case 2, we have, for any \( \kappa \in (0, \kappa^*_4] \):

\[
\Delta V_0 \leq \begin{cases} \\
\frac{p + 1}{\kappa} |x|^{p-1}\tilde{x}'\sigma(\tilde{x}) + \kappa N_{1}|x|^p + N_{2}(\kappa)|u|^p, & \text{if } |x| > \eta|B\sigma(\tilde{x})|, \\
\kappa N_{1}|x|^p + N_{2}(\kappa)|u|^p, & \text{if } |x| \leq \eta|B\sigma(\tilde{x})|, 
\end{cases}
\]

(50)

where \( N_1 = \max\{N_{1a}, N_{1c}\} \) and \( N_{2}(\kappa) = \max\{N_{2a}, N_{2c}(\kappa)\} \).

Finally, we define the following Lyapunov (or “storage”) function:

\[
V(x) = V_1(x) + \varpi V_0(x),
\]

(51)

where

\[
\varpi = \frac{2}{p + 1} \kappa^\frac{p - 2}{2} \psi_1 |P(\kappa)B|.
\]

It is straightforward to verify that there exists some \( \kappa^* \in (0, \kappa^*_3] \) such that

\[
\Delta V(x) \leq -\kappa^\frac{2-p}{2} \alpha|x|^p + \beta(\kappa)|u|^p, \quad \forall \kappa \in (0, \kappa^*]
\]

(52)

for some \( \alpha \in (0, \zeta) \) and \( \beta(\kappa) > 0 \).

Now consider an arbitrary initial state \( x(0) \) and control \( u \), and the ensuing trajectory \( x \). Summing both sides of (52) from \( t = 0 \) to \( \infty \) and using the fact that \( V \) is nonnegative, we conclude that:

\[
\kappa^\frac{2-p}{2} \alpha \|x\|_p^p \leq \beta(\kappa)\|u\|_p^p + \theta_{p0}(|x(0)|),
\]

(53)

where \( \theta_{p0}(r) = \varpi r^{p+1} + \left(\frac{\lambda_2 r^2}{\kappa}\right)^{p/2} \). This implies that

\[
\|x\|_p \leq \gamma_p\|u\|_{I_p} + \theta_p(|x(0)|),
\]

(54)

where \( \gamma_p = \left(\kappa^\frac{p-2}{2} \beta(\kappa)/\alpha\right)^{\frac{1}{2}} \), and \( \theta_p(r) = \left(\kappa^\frac{p-2}{2} \theta_{p0}(r)/\alpha\right)^{\frac{1}{2}} \).

Proof for \( p = \infty \).

From (52) we get for \( p = 2 \),

\[
\Delta V(x) \leq -\alpha|x|^2 + \beta(\kappa)|u|^2.
\]

(55)

Hence, \( \Delta V(x) \) is negative outside the ball of radius \( (\beta(\kappa)/\alpha)^{\frac{1}{2}}\|u\|_{I_\infty} \) centered at the origin, from which it follows that, for any state \( x(t) \) in the trajectory:

\[
V(x(t)) \leq \left(\frac{\varpi(\beta(\kappa)/\alpha)^{\frac{1}{2}}}{\alpha^\frac{1}{2}}\|u\|_{I_\infty} + \frac{\lambda_2 \beta(\kappa)/\alpha}{\kappa^{\frac{1}{2}}}\right)\|u\|_{I_\infty}^2 + \theta_{\infty0}(|x(0)|),
\]

(56)

where \( \theta_{\infty0}(r) = \frac{\lambda_2 r^2}{\kappa} + \varpi r^3 \). If \( \|u\|_{I_\infty} \leq 1 \), we have:

\[
\frac{\lambda_1}{\kappa} |x(t)|^2 \leq x(t)'P(\kappa)x(t) \leq V(x(t))
\]

(57)
and
\[
V(x(t)) \leq \left( \frac{\varpi \beta^2(\kappa)}{\alpha^2} + \frac{\chi_2 \beta(\kappa)}{\alpha \kappa} \right) \|u\|_\infty^2 + \theta_{\infty 0}(\|x(0)\|),
\]
which implies the following estimate for the entire trajectory:
\[
\|x\|_\infty \leq \left( \frac{\kappa \varpi \beta^2(\kappa)}{\alpha^2} + \frac{\chi_2 \beta(\kappa)}{\alpha \kappa} \right)^{\frac{1}{2}} \|u\|_\infty + \theta_{\infty 1}(\|x(0)\|),
\]
where \( \theta_{\infty 1}(r) = \left( \frac{u_{\infty 0}(r)}{\chi_1} \right)^{\frac{1}{2}} \). If, instead, \( \|u\|_\infty > 1 \), we have:
\[
\varpi|x|^3 \leq V(x) \leq \left( \frac{\varpi \beta^2(\kappa)}{\alpha^2} + \frac{\chi_2 \beta(\kappa)}{\alpha \kappa} \right) \|u\|_\infty^3 + \theta_{\infty 1}(\|x(0)\|),
\]
from which we get that
\[
\|x\|_\infty \leq \left( \frac{\beta^2(\kappa)}{\alpha^2 \varpi} + \frac{\chi_2 \beta(\kappa)}{\alpha \kappa} \right)^{\frac{1}{2}} \|u\|_\infty + \theta_{\infty 2}(\|x(0)\|),
\]
where \( \theta_{\infty 2}(r) = \left( \frac{\varpi u_{\infty 0}(r)}{\alpha^2 \varpi} \right)^{\frac{1}{2}} \). Letting
\[
\gamma_{\infty} = \max \left\{ \left( \frac{\kappa \varpi \beta^2(\kappa)}{\alpha^2 \chi_1} + \frac{\chi_2 \beta(\kappa)}{\alpha \kappa} \right)^{\frac{1}{2}}, \left( \frac{\beta^2(\kappa)}{\alpha^2 \varpi} + \frac{\chi_2 \beta(\kappa)}{\alpha \kappa} \right)^{\frac{1}{2}} \right\}
\]
and \( \theta_{\infty} = \max\{\theta_{\infty 1}, \theta_{\infty 2}\} \), we have, finally, the required conclusion:
\[
\|x\|_\infty \leq \gamma_{\infty} \|u\|_\infty + \theta_{\infty}(\|x(0)\|)
\]
for \( p = \infty \) as well. \( \square \)

We are now ready to prove Theorem 1.

**Proof of Theorem 1.** Without loss of generality, making a change of coordinates if required, we may assume that the system (1) has the following partitioned form:
\[
\begin{align*}
\dot{x}_1^+ &= A_1 x_1 + B_1 \sigma(u + u_1) \\
\dot{x}_0^+ &= A_0 x_0 + B_0 \sigma(u + u_1) \\
y &= C x + u_2
\end{align*}
\]
where \( A_1 \) is orthogonal and \( A_0 \) is asymptotically stable, and
\[
A = \begin{bmatrix} A_1 & 0 \\ 0 & A_0 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_0 \end{bmatrix}.
\]

We construct the output feedback law in the form of (3) with \( F = [-\kappa B_1^T A_1 \ 0] \), the matrix \( L \) being chosen such that \( A + LC \) is asymptotically stable. Using this feedback, the closed-loop system is:
\[
\begin{align*}
\dot{x}_1^+ &= A_1 x_1 + B_1 \sigma(-\kappa B_1^T A_1 \hat{x}_1 + u_1) \\
\dot{x}_0^+ &= A_0 x_0 + B_0 \sigma(-\kappa B_1^T A_1 \hat{x}_1 + u_1) \\
\dot{\hat{x}}^+ &= A \hat{x} + B \sigma(-\kappa B_1^T A_1 \hat{x}_1) - L(C x - C \hat{x} + u_2).
\end{align*}
\]
Let $e = \begin{bmatrix} e_1' \\ e_0' \end{bmatrix}$, where $e_1 = x_1 - \hat{x}_1$ and $e_0 = x_0 - \hat{x}_0$. Here we have partitioned $\hat{x} = \begin{bmatrix} \hat{x}_1' \\ \hat{x}_0' \end{bmatrix}$ accordingly. In the new states $(x, e)$, (64) can be written as follows,

$$
\begin{align*}
\dot{x}_1^+ &= A_1 x_1 + B_1 (\sigma(-\kappa B_1' A_1 x_1 + \kappa B_1' A_1 e_1 + u_1) + B \sigma(-\kappa B_1' A_1 x_1 + \kappa B_1' A_1 e_1 + u_1) - \sigma(-\kappa B_1' A_1 x_1 + \kappa B_1' A_1 e_1)) + u_1, \\
\dot{x}_0^+ &= A_0 x_0 + B_0 (\sigma(-\kappa B_1' A_1 x_1 + \kappa B_1' A_1 e_1 + u_1) + B \sigma(-\kappa B_1' A_1 x_1 + \kappa B_1' A_1 e_1 + u_1) - \sigma(-\kappa B_1' A_1 x_1 + \kappa B_1' A_1 e_1)) + u_1, \\
\dot{e}^+ &= (A + LC) e + B \sigma(-\kappa B_1' A_1 x_1 + \kappa B_1' A_1 e_1 + u_1) - \sigma(-\kappa B_1' A_1 x_1 + \kappa B_1' A_1 e_1) + Lu_2.
\end{align*}
$$

(65)

Since $\sigma$ is global Lipschitz with a Lipschitz constant 1,

$$
|\sigma(-\kappa B_1' A_1 x_1 + \kappa B_1' A_1 e_1 + u_1) - \sigma(-\kappa B_1' A_1 x_1 + \kappa B_1' A_1 e_1)| \leq |u_1|.
$$

(66)

Noting that $A + LC$ is asymptotically stable and viewing

$$
\sigma(-\kappa B_1' A_1 x_1 + \kappa B_1' A_1 e_1 + u_1) - \sigma(-\kappa B_1' A_1 x_1 + \kappa B_1' A_1 e_1) + Lu_2
$$
as an $l_p$ input to the $e$-subsystem, we have that, for some constant $\gamma_{pe} > 0$,

$$
\|e\|_{l_p} \leq \gamma_{pe}(\|u_1\|_{l_p} + \|u_2\|_{l_p} + |e(0)|).
$$

(67)

Next, applying Proposition 1 to the $x_1$-subsystem, and viewing $\kappa B_1' A_1 e_1 + u_1$ as an $l_p$ input to this subsystem, we have,

$$
\|x_1\|_{l_p} \leq \gamma_{p1}(\|u_1\|_{l_p} + \|u_2\|_{l_p} + |e(0)|) + \theta_{p1}(|x_1(0)|)
$$

for some $\gamma_{p1} > 0$ and $\theta_{p1}$ of class $K$.

On the other hand, viewing $\sigma(-\kappa B_1' A_1 x_1 + \kappa B_1' A_1 e_1 + u_1)$ as an $l_p$ input to the $x_0$-subsystem, we have the estimate:

$$
\|x_0\|_{l_p} \leq \gamma_{p0}(\|x_1\|_{l_p} + \|e\|_{l_p} + \|u_1\|_{l_p} + |x_0(0)|),
$$

for some $\gamma_{p0} > 0$.

In conclusion, we have,

$$
\|x\|_{l_p} \leq \|x_1\|_{l_p} + \|x_0\|_{l_p} \leq \gamma_p(\|u_1\|_{l_p} + \|u_2\|_{l_p} + \|e\|_{l_p} + \|u_1\|_{l_p} + \|x_0(0)|)
$$

(68)

where $\gamma_p > 0$ is some constant and $\varphi_p$ is a suitable class-$K$ function. Together with (67), and changing back to the original coordinates, we also conclude that an estimate like the one in (5) holds.

\[\square\]

4 Conclusions

In this paper, we have established that a discrete-time, neutrally stable, stabilizable, and detectable linear system, when subject to actuator saturation, is finite gain $l_p$ stabilizable by linear feedback, for any $p \in (1, \infty]$. A linear output feedback law which simultaneously achieves $l_p$ stabilization and global asymptotic stabilization was constructed.

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References


