Zeros of nonlinear systems with input invariances

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\textbf{A R T I C L E I N F O}

\begin{abstract}
A nonlinear system possesses an invariance with respect to a set of transformations if its output dynamics remain invariant when transforming the input, and adjusting the initial condition accordingly. Most research has focused on invariances with respect to time-independent pointwise transformations like translational-invariance \(u(t) \mapsto u(t + p, p \in \mathbb{R})\) or scale-invariance \(u(t) \mapsto pu(t, p \in \mathbb{R}_{>0})\). In this article, we introduce the concept of S\textsubscript{0}-invariances with respect to continuous input transformations exponentially growing/decaying over time. We show that S\textsubscript{0}-invariant systems not only encompass linear time-invariant (LTI) systems with transfer functions having an irreducible zero at \(s_0 \in \mathbb{R}\), but also that the input/output relationship of nonlinear \(s_0\)-invariant systems possesses properties well known from their linear counterparts. Furthermore, we extend the concept of \(s_0\)-invariances to second- and higher-order \(s_0\)-invariances, corresponding to invariances with respect to transformations of the time-derivatives of the input, and encompassing LTI systems with zeros of multiplicity two or higher. Finally, we show that n-th-order 0-invariant systems realize – under mild conditions – n-th-order nonlinear differential operators: when excited by an input of a characteristic functional form, the system’s output converges to a constant value only depending on the n-th (nonlinear) derivative of the input.

\end{abstract}

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1. Introduction

Systems invariant with respect to a set of pointwise input transformations (see e.g. Adler, Mayo, & Alon, 2014; Goentoro, Shoval, Kirschner, & Alon, 2009; Hironaka & Morishita, 2014; Shoval, Alon, & Sontag, 2011; Shoval et al., 2010) show the same output dynamics when applying a transformation to the systems’ input, and adjusting the initial conditions appropriately (see Section 3 for precise definitions). For example, linear time-invariant (LTI) systems with a zero at the origin are translational invariant, that is, invariant with respect to translations \(u(t) \mapsto u(t) + p\) of the input, with \(p \in \mathbb{R}\). Similarly, scale-invariance (also referred to as fold-change detection Shoval et al., 2010) is defined with respect to geometric scaling \(u(t) \mapsto pu(t)\) of the input, with \(p \in \mathbb{R}_{>0}\). In the context of invariant systems, the major result in Shoval et al. (2011) is of specific importance, showing that nonlinear systems are invariant with respect to a certain set of input transformations if and only if they are equivariant with respect to the same transformations (see Section 3 for details). Different to invariance, equivariance is a “memoryless” structural property only depending on the current state and input of the system; that a given system is equivariant is, thus, typically easier to prove.

Recently, we have shown that – under mild conditions – invariant systems realize first-order nonlinear differential operators (Lang & Sontag, 2016). That is, there exists a set of characteristic inputs for which the output of an equivariant system remains constant (in general nonzero) when initialized appropriately. Importantly, the constant value of the output only depends on the (nonlinear) derivative of the characteristic input, with the functional form of the derivative defined by the invariance itself. For example, translational invariant systems can realize the differential operator \(\frac{d}{dt} (i.e., \bar{y} = \alpha \frac{d}{dt} u(t))\), with \(u\) a characteristic input, \(\bar{y}\) the constant output, and \(\alpha\) some nonlinear map) in agreement with the known property that the output of Hurwitz LTI systems with a zero at the origin excited by ramps converge to constant values proportional to the slope of the ramp. Similarly, scale-invariant systems can realize the nonlinear differential
operator $\frac{d}{dt} \log$ (i.e., $\dot{y}^t = \alpha \left( \frac{d}{dt} \log(u(t)) \right)$), with the characteristic inputs given by exponential functions (Lang & Sontag, 2016).

In this article, we introduce two mutually compatible generalizations of invariance: (i) for any given $s_0 \in \mathbb{R}$, $s_0$-invariant systems are invariant with respect to continuous input transformations exponentially growing/decaying over time, and comprise LTI systems with a zero at $s_0$; and (ii) second-order $s_0$-invariant systems are invariant with respect to transformations of the time-derivative of the input, and comprise LTI systems possessing zeros with multiplicity two. Additionally, we show how the latter can be generalized to arbitrary-order invariances. For each of the two generalizations of invariance, we derive the corresponding generalized invariance without having to consider state trajectories or past inputs. Finally, based on the definition of a characteristic model of an $s_0$-invariant system – a concept related to pole-zero cancellation of LTI systems test if a given system possesses a generalized invariance without generalizationsofinvariance, we derive the corresponding generalized $s_0$-invariant system to have a zero at $s_0$, where $\bar{z}$ is an external input, with $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$. The dynamics are given by the vector field $\dot{z} = \bar{z} \cdot \mathbb{R}^n$, the initial conditions by $z(0) = \bar{z}$, and the output by $y(t) \in \mathbb{R}$, with $h : \mathbb{R} \to \mathbb{R}$. We assume that $f$ and $h$ are analytic, and that for each initial condition $z \in \mathbb{R}$ and each input $u \in \mathbb{U}$ there exists a unique, piecewise differentiable and continuous solution of Eq. (3), which we denote by $\xi : \mathbb{R} \to \mathbb{R}$, $\xi(t, z, u) = z(t)$.

### 3. First-order $s_0$-invariance and $s_0$-equivalence

Throughout the rest of this article, we consider nonlinear systems given by ODEs of the form

\[
\begin{align*}
\frac{d}{dt} z(t) &= f(z(t), u(t)), \\
y(t) &= h(z(t)).
\end{align*}
\]

The vector $z(t) \in \mathbb{R} \subseteq \mathbb{R}^n$ represents the state of the system at time $t \in \mathbb{R}_{>0}$, and $u \in \mathbb{U} \subseteq \mathbb{P}^I(\mathbb{R}_{>0}, U)$ a piecewise-continuous (external) input, with $u : \mathbb{R}_{>0} \to U \subseteq \mathbb{R}$. The dynamics are given by the vector field $f : \bar{z} \times U \to \mathbb{R}^n$, the initial conditions by $z(0) \in \mathbb{R}$, and the output by $y(t) \in \mathbb{R}$, with $h : Z \to \mathbb{R}$. We assume that $f$ and $h$ are analytic, and that for each initial condition $z \in \mathbb{R}$ and each input $u \in \mathbb{U}$ there exists a unique, piecewise differentiable and continuous solution of Eq. (3), which we denote by $\xi : \mathbb{R} \to \mathbb{R}$, $\xi(t, z, u) = z(t)$.

In the previous section, we have shown that the output of an LTI system is invariant with respect to the input transformations $\pi_p : U \to U_p$ by a parameter $p \in \mathbb{R}$, corresponding to translations of the input (2). In the following, we define an equivalent property for nonlinear systems – referred to as $s_0$-invariance – with respect to input transformations not necessarily corresponding to translations. Different to previous work (Shoval et al., 2011), we restrict ourselves to input transformations forming a one-parameter Lie group under function composition $\circ$ as defined in Blum and Kumei (1989). This implies that we can parametrize the input transformations $P = \pi_p : U \to U_p$, by a parameter $p \in \mathbb{P} \subseteq \mathbb{R}$ such that $\pi_p$ is differentiable in $U$ and analytic in $P$ (Blum & Kumei, 1989, p. 34). Furthermore, by the first fundamental theorem of Lie (Blum & Kumei, 1989, p. 37), $\pi_p$ can always be parametrized such that $P = \mathbb{R}$, such that the law of composition becomes additive ($\pi_{p_1} \circ \pi_{p_2} = \pi_{p_1 + p_2}$), and such that $p = 0$ corresponds to the identity transformation ($\pi_0(u) = u$ for all $u \in U$). In the following, we assume that every Lie group is parametrized as described above.

**Definition 1** (First-order $s_0$-invariance). Consider the system (3) and a one-parameter Lie group of input transformations $\mathcal{P} = \{\pi_p : U \to U\}_{p \in \mathbb{R}}$. Then, the system is first-order $s_0$-invariant with
respect to \(\mathcal{P}\) (in short, is \(\mathcal{P}[S_0]\)-invariant), if for all \(p \in \mathbb{R}\) and \(\tilde{z} \in \mathcal{Z}\) there exists a \(\tilde{z}^* \in \mathcal{Z}\) such that
\[
h(\xi(t, \tilde{z}, u)) = h(\xi(t, \tilde{z}', t \mapsto \pi_{\mathcal{P}[S_0]}(u(t)))) = h(\tilde{z}'), \quad \text{for all } u \in \mathcal{U} \text{ and } t \geq 0.
\]

**Remark 2.** For \(l = 1, 2\) is a special case of \(4\), implying that LTI systems with an (ir)reducible zero at \(s_0 \in \mathbb{R}\) are \(\mathcal{P}[S_0]\)-invariant, with \(\mathcal{P} = \{\pi_p(\tilde{u}) = \tilde{u} + p\} \subset \mathbb{R}\).

**Remark 3.** When defining a new output \(\hat{y}(t) = h(z(t)) - h(\xi(t, \tilde{z}, u))\), the tuple \((\pi_{\mathcal{P}[S_0]}(u(t)), \tilde{z}')\) zeros the output \(\hat{y}(t)\). For \(S_0\)-invariance, we additionally require that \(\tilde{z}'\) does not depend on \(u(t)\), and that \(\pi_{\mathcal{P}[S_0]}(u(t))\) is a zero input for all \(u\) and \(\tilde{z}'\).

**Remark 4.** Invariance as defined in Shoval et al. (2011) closely resembles \(S_0\)-invariance for \(S_0 = 0\). However, in Shoval et al. (2011) the input transformations are not required to form a Lie group. Furthermore, invariance (Shoval et al., 2011) requires that the system possesses a globally asymptotic stable (GAS) steady-state \(\sigma(\tilde{u}) \in \mathcal{Z}\) for constant inputs \(\tilde{u} \in \mathcal{U}\) (i.e., \(\xi(t, \tilde{z}, \tilde{u}) \rightarrow \sigma(\tilde{u})\) for all \(\tilde{z} \in \mathcal{Z}\), and that \(\tilde{z} = \sigma(\tilde{u})\) and \(\tilde{z}' = \sigma(\pi_p(\tilde{u}))\). Thus, for the invariance as defined in Shoval et al. (2011), (4) only has to hold for \(\tilde{z}'\) reachable from a GAS steady-state.

It is typically not trivial to prove that a given system is \(S_0\)-invariant by directly applying Definition 1. However, \(S_0\)-invariance is closely related to the “memoryless” structural property \(S_0\)-equivariance (compare Shoval et al., 2011).

**Definition 5 (First-order \(S_0\)-equivariance).** Consider the system (3) and a one-parameter Lie group of input transformations \(\mathcal{P} = \{\pi_p : U \rightarrow U\}_{p \in \mathbb{R}}\). Then, the system is first-order \(S_0\)-equivariant with respect to \(\mathcal{P}\) (in short, \(\mathcal{P}[S_0]\)-equivariant), if there exists a one-parameter Lie group of state transformations \(\mathcal{R}[S_0] = \{\rho_p : Z \rightarrow \tilde{Z}\}_{p \in \mathbb{R}}\) such that
\[
f(\rho_p(z), \pi_p(\tilde{u})) = (\partial_\rho \rho_p)(z) \pi_p(\tilde{u}) + (\partial_\tilde{u} \pi_p)(z) f(z, \tilde{u})
\]
and
\[
h(\rho_p(z)) = h(z)
\]
for all \(z \in \mathcal{Z}\), \(\tilde{u} \in \mathcal{U}\) and \(\rho_p \in \mathcal{P}\), with \((\partial_\rho y) = \frac{\partial}{\partial \rho} y(x)\) the Jacobian of \(y\) with respect to \(x\).

**Remark 6.** Equivariance as defined in Shoval et al. (2011) closely resembles \(0\)-equivariance as defined by us, except that in Shoval et al. (2011) the input and state transformations do not have to form Lie groups. To our knowledge, no general method exists to find all input and state transformations with respect to which a system is \(S_0\)-equivariant, but they can often be easily “guessed” as described in Shoval et al. (2011).

The following theorem (compare Theorem 1 in Shoval et al., 2011) establishes a close relationship between \(S_0\)-equivariance and \(S_0\)-invariance. In this theorem, \(\text{observability}\) refers to the property of a system (3) that for every two different initial conditions the output dynamics must be different for some input, i.e., that \(\forall t \geq 0, \forall u \in \mathcal{U}: h(\xi(t, \tilde{z}, u)) = h(\xi(t, \tilde{z}, u)) \Rightarrow \tilde{z}_1 = \tilde{z}_2\) (Sussmann, 1977).

**Theorem 7.** An analytic and observable system (3) is \(\mathcal{P}[S_0]\)-invariant, if and only if it is \(\mathcal{P}[S_0]\)-equivariant.

**Proof.** Sufficiency: Suppose that \(z(t) = \xi(t, \tilde{z}, u)\) is the solution of (3) for initial conditions \(\tilde{z}\) and input \(u\). Let \(z_p(t) = \rho_p(\pi_{\mathcal{P}[S_0]}(z(t)))\). Then,
\[
\frac{d}{dt} z_p(t) = ((\partial_\rho \rho_p)_{\mathcal{P}[S_0]}(z(t)) \pi_p \exp(S_0 t) + (\partial_\rho \rho_p)_{\mathcal{P}[S_0]}(z(t)) f(z(t), u(t)) = f(z(t), \pi_{\mathcal{P}[S_0]}(u(t)));
\]
i.e., \(z_p(t) = \xi(t, \tilde{z}', t \mapsto \pi_{\mathcal{P}[S_0]}(u(t))\)) with \(\tilde{z}' = \rho_p(\tilde{z})\). Furthermore, \(h(\xi(t, \tilde{z}', t \mapsto \pi_{\mathcal{P}[S_0]}(u(t))))) = h(\rho_p(\pi_{\mathcal{P}[S_0]}(\xi(t, \tilde{z}, u)))) = h(\xi(t, \tilde{z}, u))\).

**Remark 8.** In principle, one could generalize \(S_0\)-invariance by changing (4) to \(h(\xi(t, \tilde{z}, u)) = h(\xi(t, \tilde{z}', t \mapsto \pi_{\mathcal{P}[S_0]}(u(t))), with\(\alpha_0 : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}\) analytic. However, only for \(\alpha_0(t) = e^{st}\) we obtain a relationship between zeros of LTI systems and \(S_0\)-invariance. Furthermore, it is unclear for which other functional forms of \(\alpha_0\) there exists a memoryless equivalence of \(S_0\)-equivariance.

### 3.1. Example 1—LTI system with a zero

Consider the LTI system
\[
\frac{d}{dt} x(t) = y(t) - k_1 x(t)
\]
\[
\frac{d}{dt} y(t) = u(t) - x(t) - k_2 y(t),
\]
with parameters \(k_1, k_2 \in \mathbb{R}_{\geq 0}\), input \(u(t) \in \mathbb{R}\), internal state \(x(t) \in \mathbb{R}\), and output \(y(t) \in \mathbb{R}\). The transfer function of the system is
\[
G(s) = \frac{1}{s^2 + k_1 s + k_2}, \quad \text{with } s \in \mathbb{C}\text{ the Laplace variable.}
\]
and curves). The case Furthermore, these input transformations can be combined (red Fig. 1(a), (b)), or by an exponentially decaying value (green curves).

![Fig. 1. Input/output dynamics of the multiple-equivariances example (Section 3.2). (a) For $b = 0.04$, the system is excited by the inputs $u_1(t) = \pi_{\delta_j}^{g_{y_2}, \exp_{-\delta_j}}(u(t))$, with $p_1 = p_2 = 0$ (black, solid), $p_1 = 1$ and $p_2 = 0$ (blue, dash-dotted), $p_1 = 0$ and $p_2 = 1$ (green, dashed), or $p_1 = p_2 = 1$ (red, dotted), with $u(t)$ an arbitrary reference input. (b) Output dynamics $y_1(t)$ for the inputs depicted in (a), when the system is initialized at $\bar{k}_1 = \sigma_1(\bar{u}) – \frac{1}{2} p_1 k_1 = e^{\delta_1 + \delta_j} \sigma_1(\bar{u})$ and $\bar{y}_1(t)$, with $\sigma_1(\bar{u}) = 0$. $\sigma_1(\bar{u}) = \frac{1}{\delta_j}$ and $\sigma_1(\bar{u}) = y_0$ the steady-state of the system for the constant input $u = 2$. (c) For $b = 0$, the system is excited by $u_2(t) = u_{b_1,b_2}(t) = \exp\left(\frac{1}{2} k_1 t + k_2\right)$ with $k_2 = 0.5$, $k_1 = -2$ and $k_2 = 4$. (d) Output dynamics $y_2(t)$ for the input depicted in (c), when the system initialized at $(k_1^*, x_2^*, y_1^*)$ (black, dash-dotted), or at $(1, 1, 1)^T$ (blue, solid). The common parameters for (a-d) were set to $a = d = 0.5$, $c = 5$, $e = 3$ and $y_0 = 1$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Since the system has a zero at $s_0 = -k_1$, it is $\mathcal{P}[-k_1]$-equivariant (Theorem 7 and Remark 2), with input and state transformations

$$\mathcal{P}[-k_1] = \{ \pi_\sigma(\bar{u}) = \bar{u} + p \}_{p \in \mathbb{R}}$$

$$\mathcal{R}[-k_1] = \{ \rho_\sigma(x, y) = [x + p, y]^T \}_{p \in \mathbb{R}}.$$  

### 3.2. Example 2—multiple invariances

Consider the system (compare Shoval et al., 2011, Figure 1c)

$$\frac{d}{dt} x_1(t) = a(y(t) - y_0) - b x_1(t)$$

$$\frac{d}{dt} x_2(t) = c x_2(t) x_1(t) + y(t) - y_0$$

$$\frac{d}{dt} y(t) = d u(t) - e y(t),$$

with parameters $a, c, d, e \in \mathbb{R}_{> 0}$ and $b \in \mathbb{R}_{\geq 0}$, internal states $x_1(t) \in \mathbb{R}$ and $x_2(t) \in \mathbb{R}_{> 0}$, output $y(t) \in \mathbb{R}_{< 0}$, and input $u(t) \in \mathbb{R}_{\geq 0}$. The system is $\mathcal{P}[0]$-equivariant and, for $b \neq 0$, additionally $\mathcal{P}[-b]$-equivariant, with input and state transformations given for both cases by

$$\mathcal{P}[s_0] = \{ \pi_\rho(\bar{u}) = e^{\rho} \bar{u} \}_{\rho \in \mathbb{R}}$$

$$\mathcal{R}[s_0] = \{ \rho_\sigma(x_1, x_2, y) = \left( x_1 + \frac{s_0}{c} p, e^{\rho} x_2, y \right)^T \}_{p \in \mathbb{R}}.$$  

Thus, for $b > 0$, the output dynamics are invariant with respect to geometrically scaling the input by some fixed value (blue curves in Fig. 1(a), (b)), or by an exponentially decaying value (green curves). Furthermore, these input transformations can be combined (red curves). The case $b = 0$ will be further discussed in Section 7.1.

### 4. The characteristic model

In this section, we derive the characteristic model of an $s_0$-equivariant system important for the generalization of $s_0$-equivariance and $s_0$-invariance to higher orders (Section 6). For this, consider a $\mathcal{P}[s_0]$-equivariant system (3) excited by the input $u(t) = \pi_{\delta_i}(t^b(t))$, with $u^0(t) \in \mathcal{U}$ an arbitrary external input, and $p : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ some yet not specified differentiable function. Setting $\tilde{z}(t) = \rho_{-\pi_{\delta_i}}(z(t))$, differentiating by time and using the definition of $s_0$-equivariance applied to $-p(t)$, we obtain

$$\frac{d}{dt} \tilde{z}(t) = (\partial_\rho - \partial_{p(t)}) \tilde{z}(t) d (\pi_{\delta_i}(u^0(t)))$$

$$= f(\tilde{z}(t), u^0(t)) - \eta(\tilde{z}(t)) \left( \frac{d}{dt} p(t) - p(t) s_0(t) \right),$$

with $\eta(\tilde{z}) := (\partial_\rho - \partial_z) \tilde{z}$ the infinitesimals of $\rho_\delta$ (Bluman & Kumei, 1989, p. 37). Furthermore, $h(t) = h(\rho_{\pi_{\delta_i}}(z(t))) = h(\tilde{z}(t))$.

In the following, we refer to

$$\frac{d}{dt} \tilde{z}(t) = f(\tilde{z}(t), u^0(t)) - \eta(\tilde{z}) u^1(t),$$

$$\tilde{z}(0) = \rho_{-\pi_{\delta_i}}(\tilde{z})$$

$$y(t) = h(\tilde{z}(t))$$

as the first-order characteristic model of the system (3) with respect to its $\mathcal{P}[s_0]$-equivariance (in short, the $\mathcal{P}[s_0]$-characteristic model), with the additional input $u^1(t) = \frac{d}{dt} p(t) - s_0 p(t)$, and $p(0) = \tilde{p} \in \mathbb{R}$.

Given $u^0$ and $u^1$, the corresponding input $u$ to the $\mathcal{P}[s_0]$-equivariant system (3) is generated by the input module of the system with respect to $\mathcal{P}[s_0]$ (in short, the $\mathcal{P}[s_0]$-input module) given by (Fig. 2)

$$\frac{d}{dt} p(t) = s_0 p(t) + u^1(t), \quad p(0) = \tilde{p}$$

$$u(t) = \pi_{\delta_i}(u^0(t)).$$

**Lemma 9.** The input/output dynamics of a model composed of the input module of a $\mathcal{P}[s_0]$-equivariant system (3) and the
\[ \frac{d}{dt} \frac{d}{dz} = f(z, u) - \eta(z)u^1 \]

\[ y = h(\hat{x}) \]

\[ y_0(t) = \frac{u(\hat{x}_2(t) - \sqrt{1 + u(t)u(t)})}{y} - y(t) \]

\[ \rho_p \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) = \begin{array}{c} \cosh(p)x_1 + \sinh(p)x_2 \\ \sinh(p)x_1 + \cosh(p)x_2 \end{array} \]  for \( p \in \mathbb{R} \).

\[ \frac{d}{dt} p(t) = u^1(t), \quad p(0) = \bar{p} \]

\[ u(t) = \sinh \left( \frac{p(t) + \sinh(u^0(t))}{10} \right) \]

5. First-order nonlinear differential operators

In Section 2, we have shown that the output of a Hurwitz LTI system with a zero at the origin excited by a ramp input \( u_{k_1, k_0}(t) = k_1 t + k_0 \) converges to a constant value \( y(t) \to \bar{y}^* \propto \frac{1}{2} u_{k_1, k_0}(t) \) proportional to the time-derivative (i.e. the slope) of the ramp input. In this section, we shortly summarize our previous results (Lang & Sontag, 2016) generalizing this property to \( 0 \)-equivariant systems:

**Definition 11** (First-order Differential Operators, Lang & Sontag, 2016). Consider a nonlinear system \( 3 \) and an indexed family of inputs \( \mathcal{U}_g = \{ u_{k_1, k_0} : [0, \infty) \to \mathbb{R} \mid u_{k_1, k_0}(t) = g(k_1 t + k_0) \}_{k_1, k_0 \in \mathbb{R}} \) defined by a non-constant piecewise-continuous “prototype” function \( g : \mathbb{R} \to \mathbb{R} \). Then, the system realizes the (nonlinear) differential operator \( D_g : \mathcal{U}_g \to \mathcal{Y}_g \), if there exists a specific input \( u_{k_1, k_0} \in \mathcal{U}_g \) with \( (k_1, k_0) \in K_1 \times K_0 \) such that \( \forall t \in [0, \infty) \), the system output \( y(t) \to \bar{y}^* \propto \frac{1}{2} u_{k_1, k_0}(t) \) converges to a constant value \( \bar{y}^* \propto \frac{1}{2} u_{k_1, k_0}(t) \) as \( t \to \infty \).
Consider a $\mathcal{P}[0]$-equivariant system (3) excited by the input $u_k(t) = \pi_k(t-k_0)(\bar{u}^0)$, with $k_0, k_1 \in \mathbb{R}$ and $\bar{u}^0 \in U$. Since $u_k, k_1$ corresponds to the output of the input module (7) initialized at $\vec{p} = k_0$, and excited by $u^0(t) = \bar{u}^0$ and $u^1(t) = k_1$, the output dynamics of the system equal the output dynamics of the characteristic model
\[
\frac{d}{dt} \hat{x}(t) = f(\hat{x}(t), \bar{u}^0) - \eta(\hat{x})k_1, \\
\hat{x}(0) = \rho_{-k_0}(\bar{z}),
\]
(8a)
\[
y(t) = h(\hat{x}(t)).
\]
(8b)

Note, that the dynamics of this autonomous system of ODES (8) only depend on $k_1$, but not on $k_0$.

**Theorem 12** (First-order Differential Operators, Lang & Sontag, 2016). Consider a $\mathcal{P}[0]$-equivariant system (3). If there exists a set $\mathcal{K} \times \mathcal{K} \subseteq \mathbb{R}^n$ with an empty interior such that (8) has at least one steady-state $\bar{x} = \rho_{-k_0}(\bar{z})$ for each $(k_1, k_0) \in \mathcal{K} \times \mathcal{K}$, the system realizes the (nonlinear) differential operator $D_{\pi_{k_1}(\bar{u}^0)}$ with respect to the characteristic inputs $\pi_{k_1}(\bar{u}^0)$, defined by the prototype function pair $(\bar{u}^0)$, with $\bar{u}^0 \in U$ and $\pi_{k_1} \in \mathcal{P}$. The characteristic inputs $\pi_{k_1+k_0}(\bar{u}^0)$, with $(k_1, k_0) \in \mathcal{K} \times \mathcal{K}$ are proper. If for a given $(k_1, k_0) \in \mathcal{K} \times \mathcal{K}$ the steady-state of (8) is (globally) asymptotically stable, the system is (globally) convergent with respect to $\pi_{k_1+k_0}(\bar{u}^0)$.

**Proof.** The proof is given in Lang and Sontag (2016), Theorem 1.

**Remark 13.** If for a constant input $u(t) = \bar{u} \in U$, a $\mathcal{P}[0]$-equivariant system has an exponentially stable steady-state $\bar{x} = \rho_{-k_0}(\bar{z})$ in the interior of $Z$, and $\eta(z) = (\partial_\rho \rho_0)(z)$ is continuously differentiable in a neighborhood of $Z$, we can apply the implicit function theorem to show that the system realizes $D_{\pi_{k_1}(\bar{u}^0)}$.

5.1. Example 3 (continued)

For $u^0(t) = 0$ and $u^1(t) = k_1 \in \mathbb{R}$, the characteristic model of the hyperbolic example from Section 4.2 has an infinite number of steady-states $\bar{x} = \bar{u} + k_0 + k_1^t \in \mathbb{R}$, with $r \in \mathbb{R}$. The system realizes the differential operator $D_{\bar{u}(t)} = \frac{d}{dt}$ ar sinh for the characteristic inputs $u(t) = \pi_{k_1+k_0}(0) = \sinh(k_1t + k_0)$ (Fig. 3(a), (b)).

When exciting the system by differentiable inputs $u(t)$ not conforming to characteristic inputs, the output of the system is in general different from $D_{\bar{u}(t)}$. However, if the input can locally be approximated sufficiently long by characteristic inputs with respect to which the system is convergent, the output still approximately performs the correct “differential” operation, i.e. stays close to $\alpha_{\bar{u}}(D_{\bar{u},u(t)}(\bar{u})) = y_0 + \frac{1}{4} \sin(t)\ar sinh(u(t))$: (Fig. 3(c), (d)).

6. Second-order invariances & equivalences

Recall that LTI systems can not only have several different zeros, but also zeros with multiplicities greater than one, resulting in additional dynamic properties of their input/output relationship (Section 2). In this section, we show that similar holds for second-order $\mathcal{S}_0$-equivariant systems, with the additional dynamic properties described by second-order $\mathcal{S}_0$-invariances. The generalization to arbitrary-order invariances and equivalences is shortly discussed at the end of Section 7.

Intuitively, we define second-order $\mathcal{S}_0$-invariances with respect to continuous transformations of the time-derivative of the input. The mathematical definition, however, is slightly more involved since it also copes with inputs whose time-derivative is not always defined, and is compatible with causal systems:
follows is referred to the web version of this article.)

When initialized at $z(0) = [0, 0.05, 0]^T$ (blue, solid), the output converges to the constant value. (c, d) When initializing the system at $z(0) = \rho \cdot [(-1 - \frac{\pi}{2}, 2, y_0 + k_1)]^T$ (d, black dotted) or $z(0) = [0, 0.05, 0]^T$ (d, blue solid) and exciting it by $u_2(t) = \sinh (\frac{1}{4} \pi t^2 + k_1 t + k_0)$ (c), with $k_0 = 3, k_1 = -0.4$, and $\hat{t} \geq 0$, the output (d) closely follows $\frac{\pi}{2} \arcsin(u_2(t)) + y_0 = k_1 + \frac{1}{4} \pi t + y_0$ (red, dotted). Note, that the input $u_2(t) = \sinh (k_1 + \frac{1}{4} \pi t^2 + k_0 - \frac{1}{4} \pi t^2) + \frac{1}{4} \pi t (t - \hat{t}^2)$ can be well approximated around every $\hat{t} \geq 0$ by a characteristic input if $\frac{1}{4} \pi \leq 1$. (a-d) The parameter $y_0$ was set to 0.2. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Necessity: Let $Z_0(t) = \xi \begin{pmatrix} t, \bar{z}, t \mapsto \pi_1^{(\bar{y}(0)(t))}(\bar{w}^0) \\ \bar{z} \end{pmatrix}$ and $Z_0(t) = \xi \begin{pmatrix} t, \bar{z}, t \mapsto \pi_1^{(\bar{w}(0)(t))}(\bar{w}^0) \\ \bar{z} \end{pmatrix}$, with $\bar{w}^0 \in U$ and $\bar{w} \in \mathcal{P}(\mathbb{R}_{\geq 0}, \mathbb{R})$. By Theorem 7, the system is $\mathcal{P}^{[s_0]}$-equivariant with state transformations given by $\mathcal{P}^{[s_0]} = \{ \rho^x: \mathbb{Z} \to \mathbb{Z} \}_{x \in \mathbb{R}}$. We set $Z_0(t) = \rho^{(\bar{w}(0)(t))}(Z_0)$ and $Z_0(t) = \rho^{(\bar{w}(0)(t))}(Z_0)$. Differentiating by time, the ODEs for $Z_0$ and $\bar{z}_0$ correspond to the $\mathcal{P}^{[s_0]}$-characteristic model (6) of the system excited by $\bar{w}_0(t) = \bar{w}^0$ and $u^0(t) = u(t)$, respectively $u^0(t) = \bar{w}^0$ and $u^0(t) = \pi_{eq}(u^0(t))$ (7), and initialized at $\bar{z}$, respectively $\bar{z}_0$. Then, $\hat{h}(Z_0(t))$ implies that $h^{\bar{z}}(t, \bar{z}, \bar{w}, u) = h^{\bar{z}}(t, \bar{z}, \bar{w}, u) \mapsto \pi_2^{\bar{w}(0)(t)}(u(t)))$, with $\hat{z}(t, \bar{z}, \bar{u}, u^0(t) = \bar{z}(t)$ the solution of the $\mathcal{P}^{[s_0]}$-characteristic model (6) initialized at $\bar{z}$, and excited by $u^0$ and $u^0$. Thus, the $\mathcal{P}^{[s_0]}$-characteristic model of the system has to be $\mathcal{P}^{[s_0]}$-invariant with respect to its input $u^0$. The $\mathcal{P}^{[s_0]}$-characteristic model is analytic, and by Lemma 9 - observable. It follows from Theorem 7 that the $\mathcal{P}^{[s_0]}$-characteristic model is $\mathcal{P}^{[s_0]}$-equivariant. By Remark 19, this implies that (3) is $\mathcal{P}^{[s]} \times \mathcal{P}^{[s_0]}$-equivariant.

7. Second-order characteristic models & differential operators

In this section, we introduce the notion of second-order characteristic models, corresponding to the characteristic models of first-order characteristic operators, as well as systems realizing second-order characteristic operators. The definitions and results closely follow the ones in Sections 4 and 5, and are mainly stated for completeness. Furthermore, we outline how to extend our theory to arbitrary-order $\mathcal{P}^{[s_0]}$-invariances and $\mathcal{P}^{[s_0]}$-equivariances.

Consider a $\mathcal{P}^{[s]} \times \mathcal{P}^{[s_0]}$-equivariant system (3) excited by $u(t) = \pi_{eq}(\rho^{(\bar{y}(0)(t))}(\bar{w}^0(t)))$, with $v(t) = \pi_{eq}(\rho^{(\bar{y}(0)(t))}(\bar{w}^0(t)))$ (u^0(t)), external inputs $u_2(t) \in \mathcal{U}$ and $u^1, u^2 \in \mathcal{P}(\mathbb{R}_{\geq 0}, \mathbb{R})$, and $\rho^{(\bar{y}(0)(t))} \in \mathcal{P}^{[s_0]}$. Setting $z(t) = \rho^{(\bar{y}(0)(t))}(\bar{w}^0(t))$ and $\bar{z}(t) = \rho^{(\bar{y}(0)(t))}(\bar{w}^0(t))$, with $\bar{z}(t) = \rho^{(\bar{y}(0)(t))}(\bar{w}^0(t))$, and differentiating by time, we obtain the second-order characteristic model of (3) with respect to its $\mathcal{P}^{[s]} \times \mathcal{P}^{[s_0]}$-equivariance (in short, the $\mathcal{P}^{[s]} \times \mathcal{P}^{[s_0]}$-characteristic model):

\[
\frac{d}{dt} z(t) = f(z(t), u(t)) - \eta(z(t))u^1(t) - \eta^2(z(t))u^2(t)
\]

with $\eta^2(z(t)) = (\partial_{\bar{y}}\rho^2)(z(t))$.

In our definition of systems realizing second-order differential operators, we restrict ourselves to operators being strictly of a given order, that is, “mixed-order” differential operators like $\frac{d^2}{dt^2} \log + \frac{2}{t}$ are not considered.

**Definition 21 (Second-order Differential Operators).** Consider the system (3) and an indexed family of inputs

\[
\mathcal{U}_{\mathcal{R}, \mathcal{S}_1} = \left\{ u_{k_2, k_1, k_0} : [0, \infty) \to \mathbb{R} \mid u_{k_2, k_1, k_0}(t) \right\}
\]

defined by two non-constant piecewise-continuous “prototype” functions $g_1, g_2 : \mathbb{R} \to \mathbb{R}$. Then, the system realizes the second-order (nonlinear) differential operator $D_{\mathcal{R}, \mathcal{S}_1} = D_{\mathcal{R}, \mathcal{S}_1} : \mathcal{U}_\mathcal{S}_1 \to \mathbb{R}$, if there exists a set $K_2 \times K_1 \times K_0 \subseteq \mathbb{R}^3$ with non-empty interior, such that for all inputs $u_{k_2, k_1, k_0} \in \mathcal{U}_{\mathcal{R}, \mathcal{S}_1}$ with $(k_2, k_1, k_0) \in K_2 \times K_1 \times K_0$ there exists an initial condition $z^* \in \mathbb{Z}$ for which the output is constant and independent of $k_1$ and $k_0$, i.e., $y(t) = h(z(t), u_{k_2, k_1, k_0}(t)) = g_{\mathcal{R}, \mathcal{S}_1}(k_2) = g_{\mathcal{R}, \mathcal{S}_1}(D_{\mathcal{R}, \mathcal{S}_1} u_{k_2, k_1, k_0})$ for all $t \geq 0$, with $g_{\mathcal{R}, \mathcal{S}_1} : K_2 \to \mathbb{R}$ a function which might depend on the specific system.
For systems realizing second-order differential operators, (proper) characteristic inputs, (global) convergence and degeneracy can be defined similarly as in Definition 11.

**Lemma 22** (Second-order Differential Operators). Consider a $P^{1} \times P^{2}[0]$-equivariant system (3). If there exists a set $K_{0} \times K_{1} \times K_{2} \in \mathbb{R}^{3}$ with non-empty interior such that the $P^{1} \times P^{2}[0]$-characteristic model, with $\bar{p}_{1} = k_{0}$ and $\bar{p}_{2} = k_{1}$, excited by $u^{\dagger}(t) = \bar{u} \in U$, $u^{\dagger}(t) = \bar{u} \in \mathbb{R}$, and $u^{\dagger}(t) = k_{2}$ has at least one steady-state $z^{\ast} \in \rho^{+}_{2} \left( z^{\ast} \right)$ for all $(k_{0}, k_{1}, k_{2}) \in K_{0} \times K_{1} \times K_{2}$, the system realizes the (nonlinear) differential operator $D_{\rho^{2}(\bar{u})}^{-1} D_{\rho^{1}(\bar{u})}$ with respect to the characteristic inputs $U_{\rho^{2}(\bar{u}), \rho^{1}(\bar{u})}$ defined by the prototype functions $\pi_{1}(\bar{u})$ and $\pi_{2}(\bar{u})$.

**Proof.** For all $(k_{0}, k_{1}, k_{2}) \in K_{0} \times K_{1} \times K_{2}$, the output of the characteristic model initialized at $z^{\ast}$ and excited by $u^{\dagger}(t) = \bar{u} \in U$, $u^{\dagger}(t) = \bar{u} \in \mathbb{R}$ and $u^{\dagger}(t) = k_{2}$ is constant and does not depend on $k_{0}$ or $k_{1}$. By definition of the characteristic model, these constant output dynamics equal the output dynamics of the original system (3) initialized at $z^{\ast} \in \rho^{+}_{2} \left( z^{\ast} \right)$ and excited by $u_{2k_{1}, k_{2}}(t) = \pi_{1}^{+} \left( \bar{u} \right)$ and $\pi_{2}^{-} \left( \bar{u} \right)$. □

Finally, let us outline how $S_{0}$-equivariance, $S_{0}$-invariance and systems realizing differential operators can be generalized to arbitrary order. Recall, that a system is second-order $P^{1} \times P^{2}[S]_{0}$-equivariant and if and only if it is first-order $P^{1}[S]_{0}$-equivariant, and if its $P^{1}[S]_{0}$-characteristic model is $P^{2}[S]_{0}$-equivariant with respect to its input $u^{\dagger}$ (Remark 19). Then, we can recursively define higher-order equivariances and invariances as follows: A system is $P^{n+1} \times \cdots \times P^{1}[S]_{0}$-equivariant if and only if it is $P^{n} \times \cdots \times P^{1}[S]_{0}$-equivariant and if its $P^{n} \times \cdots \times P^{1}[S]_{0}$-characteristic model is $P^{n+1}[S]_{0}$-equivariant with respect to the input $u^{\dagger}$. Thereby, the $P^{n} \times \cdots \times P^{1}[S]_{0}$-characteristic model is recursively defined as the $P^{n+1}[S]_{0}$-characteristic model of the $P^{n+1} \times \cdots \times P^{1}[S]_{0}$-characteristic model with respect to the input $u^{\dagger}$. Similarly, we can recursively define $P^{n+1} \times \cdots \times P^{1}[S]_{0}$-invariance by requiring that the system is $P^{n} \times \cdots \times P^{1}[S]_{0}$-invariant, and that it is invariant with respect to transformations of the nth time-derivative of the input in the sense of Definition 14. The relationship between $P^{n} \times \cdots \times P^{1}[S]_{0}$-equivariance and -invariance can be established following the methods developed in the proof of Theorem 20. Systems realizing nth-order differential operators can be defined according to Definition 21 using n different prototype functions.

Recall that an observable LTI system with an irreducible zero at $S_{0}$ is first-order $S_{0}$-equivariant (Remark 2 and Theorem 7), that the transfer function of its first-order $S_{0}$-equivariant model corresponds to its transfer function after canceling one occurrence of the zero at $S_{0}$ (Remark 10), and that a system is ($n+1$)-th-order $S_{0}$-equivariant only if its nth-order $S_{0}$-characteristic model is $S_{0}$-equivariant. Thus, while we only provide detailed examples for second-order invariant systems in the following, we remark that observable LTI systems possessing irreducible zeros with multiplicity $m_{0} \geq 2$ are examples for higher-order ($m_{0}$th-order) $S_{0}$-invariant systems.

7.1. Example 2 (continued)

For $b = 0$, we have already shown that the example system in Section 3.2 is $P^{1}[0]$-equivariant and $P^{1}[-b]$-equivariant. For $b \to 0$, the two equivariances collapse, such that for $b = 0$ the system possesses only one (first-order) 0-equivariance. However, for $b = 0$ the system is second-order $P^{1} \times P^{2}[0]$-equivariant, with $P^{1}$ and $P^{2}$ as given in Section 3.2, and $P^{1} = \{ p_{2}(\bar{u}) = \bar{u} + p \} \in \mathbb{R}$ and $P^{2} = \{ p_{2}(\bar{y}) = \bar{y} + \} \in \mathbb{R}$. The second-order characteristic model is given by

$$
\begin{align*}
\frac{d}{dt} \dot{x}_{1}(t) &= a \left( y(t) - y_{0} \right) - \frac{u^{\dagger}(t)}{c} \\
\frac{d}{dt} \dot{x}_{2}(t) &= c \left( \dot{x}_{2}(t) - \left( x_{1}(t) - \frac{u^{\dagger}(t)}{c} \right) + \dot{y}(t) - y_{0} \right) \\
\frac{d}{dt} \dot{y}(t) &= d \left( u^{\dagger}(t) - \dot{x}_{2}(t) - \dot{y}(t) \right).
\end{align*}
$$

For $u^{\dagger}(t) = 1$, $u^{\dagger}(t) = 0$ and $u^{\dagger}(t) = k_{2} > K_{2}^{2} = c y_{0}$. The characteristic model has a steady-state at $[\dot{x}_{1}, \dot{x}_{2}, \dot{y}]^{T} = \left( \frac{-k_{2} y_{0}}{c}, \frac{-k_{2} y_{0}}{c}, y_{0} + \frac{k_{2} y_{0}}{c} \right)$. Thus, for $k_{2} > K_{2}^{2}$, the system realizes the second-order differential operator $\frac{d^{2}}{dt^{2}}$ (Fig. 1(c), (d)), with the characteristic inputs given by $u_{k_{2}, k_{1}, k_{2}}(t) = e^{k_{2} x_{1}^{2} + k_{1} x_{1} + k_{0}}$.

7.2. Example 4—realizing $\frac{d}{dt} \log(\frac{1}{1+\frac{1}{2}})$ (\(\mathcal{L}(\frac{1}{1+\frac{1}{2}}) + v_{0}\))

Consider the system

$$
\begin{align*}
\frac{d}{dt} x_{1}(t) &= x(t) - x_{1}(t) \\
\frac{d}{dt} x_{2}(t) &= -ax_{2} + b \left( x(t) - x_{1}(t) + v_{0} \right) \\
\frac{d}{dt} y(t) &= c \left( x(t) - x_{2}(t) + v_{0} - dy(t) \right).
\end{align*}
$$

The system is second-order 0-equivariant, with $\pi_{2}^{+}(\bar{u}) = \bar{u} + p$, and $\pi_{2}^{-}(\bar{u}) = \bar{u} + p$. The characteristic model has a steady-state at $[\dot{x}_{1}, \dot{x}_{2}, \dot{y}]^{T} = \left( 1, 0, 0 \right)$. Thus, for $k_{2} > K_{2}^{2}$, the system realizes the second-order differential operator $\frac{d^{2}}{dt^{2}}$ (Fig. 4), with the characteristic inputs given by $u_{k_{2}, k_{1}, k_{2}}(t) = e^{k_{2} x_{1}^{2} + k_{1} x_{1} + k_{0}}$.

8. Discussion

Systems invariant with respect to sets of pointwise and time-invariant input transformations have been widely studied in the literature (Adler et al., 2014; Geonotoro et al., 2009; Hironaka & Morishita, 2014; Shoval et al., 2011, 2010) with arguably the most prominent examples being scale-invariant systems ($u(t) \mapsto e^{t}u(t)$, with $p \in \mathbb{R}_{>0}$) and translational-invariant systems ($u(t) \mapsto p + u(t)$, with $p \in \mathbb{R}$). In this article, we introduced the concept of $S_{0}$-invariant systems, corresponding to invariances with respect to input transformations exponentially growing/decaying over time. We have shown the close relationship between $S_{0}$-invariance and $S_{0}$-equivariance, with the latter being a memoryless property only
depending on the current state and input of the system. We then extended our framework to second-order $S_0$-invariant systems, that is, systems invariant with respect to transformations of the time-derivatives of the input. Since this extension is based on a so-called characteristic model of an $S_0$-equivariant system, and since it is possible to recursively define characteristic models of characteristic models, our theory is easily extendable to arbitrary-order $S_0$-invariant systems. Finally, we introduced the concept of systems realizing first- and higher-order differential operators, that is, systems whose output remains at a constant value only depending on the (first- or higher-order) time-derivative of the input when excited by a characteristic input and initialized appropriately.

First- and higher-order $S_0$-equivariant systems not only encompass LTI systems possessing zeros at $S_0 \in \mathbb{R}$ with single or higher multiplicities, but their input/output dynamics also show properties generalizing those of LTI systems with transfer function zeros. We expect that in future research, our theoretical framework might be extended to also govern complex conjugated pairs of zeros, e.g., systems invariant with respect to exponentially decaying or increasing oscillatory input-transformations. It might also be possible to generalize certain linear controller or observer design techniques to $S_0$-equivariant systems.

Practically, we expect our framework to prove valuable in trajectory tracking applications and in the analysis of natural or engineered (biological) systems. For example, the tracking error of type-N servomechanisms converges to zero for polynomial inputs of order $N = 1$. If such a servomechanism would additionally be $N$th order 0-invariant, its transient error dynamics would become independent of the current position for $N \geq 1$, of the current velocity ($N \geq 2$), and of the current acceleration ($N \geq 3$). On the other hand, several naturally evolved biomolecular networks are known to possess 0-invariances. For example, bacterial chemotaxis was not only shown to be scale-invariant, but also to realize the differential operator $\frac{d}{dt} \log(\frac{1}{d} (f(x) + v_0))$ (Section 7.2) when excited by the characteristic input $u_{0,k_1,k_2}(t) = \frac{k_2}{k_1} (\exp(k_2 t + k_1) - \exp(k_1)) - v_0 t + k_0$ (a) with $k_2 = 0.4$, $k_1 = -4$ and $k_0 = 6$.

(b) The black curve represents the output dynamic when the system is initialized at $(\bar{x}_1, \bar{x}_2, \bar{y})^T$, while the blue curve corresponds to the initial condition $(1, 1, 1)^T$. The parameters were set to $a = b = c = d = v_0 = 1$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Fig. 4. (a, b) Input/output dynamics of the system realizing the differential operator $\frac{d}{dt} \log(\frac{1}{d} (f(x) + v_0))$ (Section 7.2) when excited by the characteristic input $u_{0,k_1,k_2}(t) = \frac{k_2}{k_1} (\exp(k_2 t + k_1) - \exp(k_1)) - v_0 t + k_0$ (a) with $k_2 = 0.4$, $k_1 = -4$ and $k_0 = 6$.

References


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