modern optimal control

Edited by
Emilio O. Roxin
Nonlinear Discrete-Time Systems

Accessibility Conditions

Bronislaw Jakubczyk¹, Professor, Institute of Mathematics, Polish Academy of Sciences, Warsaw, Poland
Eduardo D. Sontag², Professor, Department of Mathematics, Rutgers University, New Brunswick, New Jersey

1 INTRODUCTION

In this paper we study the problem of positive-time accessibility for control systems defined by difference equations. Lie-theoretic conditions are given for this property to hold, and it is related to transitivity (positive and negative time control).

Specifically, we are interested in nonlinear discrete time systems of the form

\[ x(t + 1) = f(x(t), u(t)), \quad x(t) \in X, u(t) \in U, \]

where \( X \) is \( \mathbb{R}^n \) or more generally a connected differentiable manifold of dimension \( n \) and \( U \) is a subset of \( \mathbb{R}^m \) (or more generally a manifold itself). We assume that \( X \) is of class \( C^k \), \( k = \infty \) (the smooth case) or, when dealing with analytic systems, \( k = \omega \). The function

\[ f : X \times U \to X \]

is also assumed to be \( C^k \), in a sense to be made more precise later. The time \( t \) takes values in the set of nonnegative integers \( \mathbb{Z}_+ \) (forward case), the set of nonpositive integers \( \mathbb{Z}_- \) (backward case), or in the set of all integers \( \mathbb{Z} \) (forward-backward case).

It is recognized that a better understanding of discrete time control systems is essential in the modeling of physical systems under digital control. However, discrete time systems have been studied much less than their continuous counterparts, and their properties may diverge considerably from those of the latter. The paper [15] and the monograph [16] considered various aspects of discrete systems defined by polynomial evolution equations. However, the general theory remained until recently much weaker than that possible in

¹Work was done while a Visiting Professor at Rutgers, The State University of New Jersey.
²Research supported in part by US Air Force Grants 85-0247 and 88-0235
the more classical continuous time case, for which a large body of knowledge (see for example [1], [4], [5], [23]) is now available. One difficulty is that noninvertibility of the transition maps \( f(\cdot, u) \) means that semigroups appear where groups would appear in the continuous case, so less algebraic structure is available. Another problem is that no analogue is available for difference equations of the infinitesimal information obtained by taking derivatives with respect to time. One way to try to avoid the first of these difficulties is to restrict attention to invertible systems, for which by assumption the above transition maps are indeed invertible for each fixed control value \( u \). For the second problem, the lack of infinitesimal information, one may substitute derivations with respect to control values, provided that one restricts attention to situations in which enough differentiable structure exists in the control set \( U \).

Fortunately, both of the above restrictions are very reasonable. Although invertibility is a priori a strong assumption in the context of general discrete time systems, it is the case that for systems that result from sampling, this assumption is always satisfied. Recall that sampling is the process under which the state of a continuous time system is measured at discrete instants, and control actions are taken also at discrete instants. Under such a process, the obtained transition maps as observed at the sampling times turns out to give rise to an invertible model. (Analogously, in classical dynamical systems one studies time-one diffeomorphisms and Poincaré maps associated to differential equations.) The second restriction, having enough directions in the control set, is satisfied in any situation in which controls can take unrestricted values, or even if they are subject to magnitude or various other types of algebraic constraints. Invertible discrete time systems are often also obtained in numerical schemes for discretizing continuous time models.

The paper [6] introduced the idea of studying invertible discrete nonlinear control systems, and developed a realization theory which parallels much of the continuous situation. A major related development was the discovery by M.Fliess and D.Normand-Cyrot ([2], [14]) that a large number of concepts from the continuous time theory admit generalizations to invertible discrete time systems, provided that appropriate objects -vector fields, distributions, and Lie algebras- be defined; further work along these lines was carried out in [8], [20], [12], [17], and related papers.

In [10] a systematic and rigorous study of the structure needed for studying controllability problems for discrete time invertible systems is presented. This paper summarizes and discusses some of the results given there, in the special case where controls are assumed to be scalar valued and unrestricted, and where \( X = \mathbb{R}^n \) rather than a general manifold. This will result in a simpler exposition than that in [10], which should be consulted for all proofs.

## 2 BASIC DEFINITIONS

We start by introducing basic notation and definitions. As stated previously, time takes integer values, \( t \in \mathbb{Z} \) and we consider only real-valued, unrestricted controls, and state spaces which are Euclidean.

We introduce the following notations for the effect of shift operators:

\[ x^+(t) = x(t + 1) \quad \text{and} \quad x^-(t) = x(t - 1). \]
In this way we can write equation (1) in the more compact form

\[ x^+ = f(x, u), \quad x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}. \]

The system is of class \( C^k \) if the function \( f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \) is of class \( C^k \). By smooth we mean of class \( C^{\infty} \).

Associated to each such system there is a family of maps

\[ f_u = f(\cdot, u) : \mathbb{R}^n \to \mathbb{R}^n, \quad u \in \mathbb{R}. \]

**DEFINITION 2.1** The system (1) is invertible if for each \( u \in \mathbb{R} \) the map \( f_u \) is a global diffeomorphism of \( \mathbb{R}^n \).

To any invertible system one can associate an inverse or reversed-time system with equations

\[ x^- = f^-(x, u), \]

where \( f^-(x, u) = f_u^{-1}(x) \). By the implicit mapping theorem, this is again of class \( C^k \), and its inverse is the original system.

Unless otherwise stated, every system appearing in this paper will be assumed to be invertible.

The maps \( f_u \) and their inverses \( f_u^{-1} \) can be considered as “one step forward maps” (respectively, “one step backward maps”). If we apply a sequence of controls \( u_1, \ldots, u_k \) then we obtain the composition of these maps denoted by

\[ f_{u_k \cdots u_1} = f_{u_k} \circ \cdots \circ f_{u_1}. \]

Allowing backward as well as forward steps we obtain a larger family of maps

\[ f_{u_k \cdots u_1}^{\epsilon_1 \cdots \epsilon_k} = f_{u_k}^{\epsilon_k} \circ \cdots \circ f_{u_1}^{\epsilon_1}, \]

where each of \( \epsilon_1, \ldots, \epsilon_k \) takes a value \( \pm 1 \).

We shall denote by \( A_k^+(x) \) the set of points attainable from \( x \) in \( k \) forward steps, and by \( A^+(x) \) the set of points attainable from \( x \) in any nonnegative number of forward steps. Replacing forward steps by backward steps we obtain other sets, \( A_k^-(x) \) and \( A^-(x) \), which consist of points controllable to \( x \) in \( k \) steps, and controllable to \( x \) in any nonnegative number of steps, respectively. Finally, the set of points attainable from \( x \) in any number of positive and negative steps is called the orbit of \( x \), and is denoted by \( A(x) \).

**DEFINITION 2.2** The system (1) is forward (backward) accessible from \( x \) if its attainable set \( A^+(x) \) (respectively, \( A^-(x) \)) has a nonempty interior. It is called transitive from \( x \) (or forward-backward accessible from \( x \)) if its orbit \( A(x) \) has a nonempty interior (and so it is necessarily open).

Finally, the system is forward (backward) accessible if it is forward (backward) accessible from any \( x \in \mathbb{R}^n \), and it is called transitive if it is transitive from any \( x \in \mathbb{R}^n \).

There is a straightforward criterion for accessibility of the discrete time system, based on the rank of the following map. For each fixed state \( x \) and integer \( k \) define

\[ \psi_{k,x}(u) := f_{u_k \cdots u_1}(x), \]

where \( u = (u_1, \ldots, u_k) \) takes values in the \( k \)-th Cartesian product \( \mathbb{R}^k \). Notice that the attainable set \( A_k^+(x) \) is by definition equal to the image of this map. By Sard’s theorem and the implicit function theorem one has the following:
PROPOSITION 2.3 Consider a smooth system (1). For any fixed \( x \) and \( k \), the interior of the attainable set \( A_k^+(x) \) is nonempty if and only if
\[
\sup \{ \text{rank} \frac{\partial}{\partial u} \psi_{k,z}(u), u \in \mathbb{R}^k \} = n
\]
and thus
\[
\sup \{ \text{rank} \frac{\partial}{\partial u} \psi_{k,z}(u), u \in \mathbb{R}^k, k \geq 1 \} = n
\]
is necessary and sufficient for forward accessibility of system (1) from \( x \).

Remark 2.4 The orbit \( A(x) \) is the (countable) union of the images of the maps (4), so we can use an analogous argument to give a criterion for transitivity from \( x \), using the maps (4) rather than (3) in order to define a family of maps playing the role of the \( \psi_{k,z} \)'s.

The above proposition and remark might appear to give satisfactory criteria for forward accessibility and transitivity. Unfortunately, this is not the case. Although for simple systems they may be used to decide whether a given system is forward accessible or not, for more complicated systems explicitly computing the functions \( \psi_{k,z} \) may be highly non-trivial, since composition is hard to deal with computationally. As an example, consider for instance the problem of obtaining a general formula for the \( n \)-th composition of the quadratic function \( g(x) = ax^2 + bx + c \) with itself or that of computing the function \( \psi_{k,z} \) if \( f(x, u) = g(x) + xu \). The problem becomes even more serious in the case of deciding the transitivity of the system, as this requires also finding the inverse maps \( f_u^{-1} \) needed for computing the composed maps (4). Even for classes like that of bilinear systems, proposition 2.3 doesn't seem to provide much useful information.

Also from a purely theoretical point of view proposition 2.3 is of little interest. This is because it gives too limited an insight into the geometry of our systems and it provides an even more limited tool for their study. The maps appearing in the criteria do not have much algebraic and geometric structure.

The main aim of the next section is to introduce a sort of "infinitesimal description" of the discrete time system. This is done by introducing certain vector fields associated to it. By doing so we immediately get a powerful tool and a rich algebraic and geometric structure based on the Lie product of vector fields. In particular, the accessibility properties of the system can be studied using natural Lie algebras of vector fields associated to the system. The idea of introducing vector fields corresponding to infinitesimal perturbations of control values is a natural generalization of the concept of actions of Lie groups, and it was originally proposed in the context of nonlinear control in [2]. These vector fields also find natural applications in the study of controllability properties and the feedback linearizability of sampled systems ([17], [9]).

2.1 Vector Fields Associated to the System

We associate the following four families of vector fields to our discrete time system (1), one vector field for each \( u \in \mathbb{R} \):

\[
X_u^+ (x) = \left. \frac{\partial}{\partial v} \right|_{v=0} f_u^{-1} \circ f_{u+v}(x),
\]
\[
X_u^-(x) = \left. \frac{\partial}{\partial v} \right|_{v=0} f_u \circ f_u^{-1}(x),
\]
\[
Y_u^+(x) = \left. \frac{\partial}{\partial v} \right|_{v=0} f_u^{-1} \circ f_u(x),
\]
\[
Y_u^-(x) = \left. \frac{\partial}{\partial v} \right|_{v=0} f_{u+v} \circ f_u^{-1}(x).
\]

The interrelations between these vector fields are explained in the next proposition.

The special case in which the function \(f\) happens to correspond to the flow of a (complete) vector field \(Z\), that is \(f(x, u) = \exp(uZ)\), is important when discussing continuous time systems within our framework. In that case all of the above vector fields are in fact independent of \(u\), and they provide the same information about the system. This is because by the semigroup property of flows it holds that \(f_{u+v} = f_u \circ f_v = f_v \circ f_u\), so that \(X_u^+ = X_u^- = Z = Y_u^+ = Y_u^-\). These equalities help understand why the continuous time theory is considerably simpler than the discrete one.

Note that applying these definitions to the inverse system (2) instead of system (1) gives the same vector fields except that the pluses are changed for minuses and vice versa.

Given a vector field \(Y\) and a control value \(u\), we can define another vector field from \(Y\) by applying a change of coordinates given by the diffeomorphism \(f_u\),

\[
(\text{Ad}_u Y)(x) = (df_u(x))^{-1} Y(f_u(x)).
\]

Here \(df_u\) stands for the differential of \(f_u\) with respect to \(x\). Using the diffeomorphisms (3), we may also define

\[
(\text{Ad}_{u_k \ldots u_1} Y)(x) = (df_{u_k \ldots u_1}(x))^{-1} Y(f_{u_k \ldots u_1}(x)),
\]

and, applying the even more general family of diffeomorphisms (4),

\[
(\text{Ad}_{u_k \ldots u_1}^{u_{k-1} \ldots u_1} Y)(x) = (df_{u_k \ldots u_1}^{u_{k-1} \ldots u_1}(x))^{-1} Y(f_{u_k \ldots u_1}^{u_{k-1} \ldots u_1}(x)).
\]

Clearly, the operators "Ad" so defined are linear operators acting on vector fields \(Y\), and we have that

\[
\text{Ad}_{u_k \ldots u_1}^{u_{k-1} \ldots u_1} Y = \text{Ad}_{u_1}^{u_2} \cdots \text{Ad}_{u_k}^{u_{k-1}} Y.
\]

(Note the reversal of indices.) We will use the abbreviated notation \(\text{Ad}_u^k Y\) for \(\text{Ad}_{u \cdots u} Y\) with \(u = 0\) repeated \(k\)-times, if \(k > 0\), and for \(\text{Ad}_0^{1 \cdots 0} Y\), if \(k < 0\). Additionally, \(\text{Ad}_0^0 Y = Y\). With this notation we have that

\[
(\text{Ad}_0^k X_u^+)(x) = \left. \frac{\partial}{\partial v} \right|_{v=0} f_0^{-k} \circ f_u^{-1} \circ f_{u+v} \circ f_0^k(x)
\]

and, more generally,

\[
(\text{Ad}_{u_k \ldots u_1} X_u^+)(x) = \left. \frac{\partial}{\partial v} \right|_{v=0} f_{u_k \ldots u_1}^{-1} \circ f_{u_0}^{-1} \circ f_{u_0+v} \circ f_{u_k \ldots u_1}(x).
\]

Since our system is assumed to be invertible, we could apply all definitions to the inverse system (2) instead of (1). Then all the pluses in the superscripts change for minuses and \(\text{Ad}_u\) changes for \(\text{Ad}_u^{-1}\), and vice versa. Therefore, we will have the following fact.

**Reversion principle:** Any general property of systems of the type (1) which can be expressed in terms of the above defined vector fields is preserved if we change the pluses in the superscripts for the minuses and each \(\text{Ad}_u\) for \(\text{Ad}_u^{-1}\), and vice versa.
Remark 2.5 Some of the above defined vector fields can be equivalently defined as follows:

\[ X_u^+(x) = (df_u(x))^{-1} \frac{\partial}{\partial u} f_u(x), \]

\[ (\text{Ad}_{u_1 \ldots u_k} X_{u_0}^+)(x) = (df_{u_1 \ldots u_k}(x))^{-1} X_{u_0}^+(f_{u_1 \ldots u_k}(x)). \]

Since the inverses \( f_u^{-1} \) do not appear, the right-hand sides now make sense for \textit{locally} invertible systems. Those of our results which can be stated exclusively in terms of the above vector fields will also hold for locally invertible systems. Furthermore, criteria stated in their terms can be checked without computing the inverse of any diffeomorphism; only matrix inversions are required.

The basic interrelations between the vector fields \( X_u^+, X_u^-, Y_u^+, Y_u^- \) are given by the following proposition.

PROPOSITION 2.6 The following equalities:

(a) \( X_u^+ = -Y_u^+ \), \( X_u^- = -Y_u^- \).

(b) \( X_u^+ = -\text{Ad}_u X_u^- \), \( Y_u^+ = -\text{Ad}_u Y_u^- \).

hold for each \( u \in \mathbb{R} \).

In the next proposition and in the rest of the paper we shall use the following notational convention. Given a family of vector fields \( \{ Y_\alpha | \alpha \in A \} \), we denote by \( \text{Lie} \{ Y_\alpha | \alpha \in A \} \) the Lie algebra generated by this family of vector fields and by \( \text{Lie} \{ Y_\alpha | \alpha \in A \}(x) \) the subspace of the tangent space at \( x \) generated by the vector fields in this Lie algebra.

PROPOSITION 2.7 For analytic systems,

\[ \text{Ad}_0^k X_u^+(x) \in \text{Lie} \{ \text{Ad}_0^{k+1} X_u^- | u \in \mathbb{R} \}(x) \]

and

\[ \text{Ad}_0^{-k} X_u^-(x) \in \text{Lie} \{ \text{Ad}_0^{-k-1} X_u^+ | u \in \mathbb{R} \}(x) \]

for each \( x \in \mathbb{R}^n \), each \( u \in \mathbb{R} \), and each integer \( k \).

Note that it is not claimed that, for instance, \( X_u^+ \) is in the Lie algebra generated by the vector fields \( \text{Ad}_0 X_u^- \). The statement pertains only to the equality of the associated distributions, that is, of the tangent spaces at each point.

3 ACCESSIBILITY CRITERIA

To state our criteria we shall need the following families of vector fields:

\[ \Gamma^+ = \{ \text{Ad}_{u_1 \ldots u_k} X_{u_0}^+ | k \geq 0, u_0, \ldots, u_k \in \mathbb{R} \}, \]

\[ \Gamma^- = \{ \text{Ad}^{-1 \ldots -1}_{u_1 \ldots u_k} X_{u_0}^- | k \geq 0, u_0, \ldots, u_k \in \mathbb{R} \}, \]

\[ \Gamma = \{ \text{Ad}_{u_1^{\varepsilon_1} \ldots u_k^{\varepsilon_k}} X_{u_0}^\sigma | k \geq 0, u_0, \ldots, u_k \in \mathbb{R}, \varepsilon_1, \ldots, \varepsilon_k = \pm 1, \sigma = \pm \}. \]
As previously, for a family of vector fields \( \Delta \), we denote by \( \text{Lie}\{\Delta\} \) the Lie algebra of vector fields generated by \( \Delta \), by \( \Delta(x) \) the linear space spanned by the vectors at \( x \) given by the vector fields in \( \Delta \), and by \( \text{Lie}\{\Delta\}(x) \) the linear space of tangent vectors at \( x \) given by the vector fields in the Lie algebra.

The following theorem gives criteria for accessibility of smooth systems. It will be one of the main results of this paper.

**Theorem 1** The following properties hold for any smooth system (1).

(a) The system is forward accessible if and only if any of the following two equivalent conditions hold:

\[
\dim \Gamma^+(x) = n \ \forall x \in \mathbb{R}^n, \quad \text{or} \quad \dim \text{Lie}\{\Gamma^+\}(x) = n \ \forall x \in \mathbb{R}^n.
\]

(b) The system is backward accessible if and only if any of the following two equivalent conditions hold:

\[
\dim \Gamma^-(x) = n \ \forall x \in \mathbb{R}^n, \quad \text{or} \quad \dim \text{Lie}\{\Gamma^-\}(x) = n \ \forall x \in \mathbb{R}^n.
\]

(c) The system is transitive if and only if any of the following two equivalent conditions hold:

\[
\dim \Gamma(x) = n \ \forall x \in \mathbb{R}^n, \quad \text{or} \quad \dim \text{Lie}\{\Gamma\}(x) = n \ \forall x \in \mathbb{R}^n.
\]

To state a stronger version of our result, valid for analytic systems, we need the following Lie algebras of vector fields:

\[
L^+ = \text{Lie}\{\text{Ad}^k_u X^+ | k \geq 0, u \in \mathbb{R}\},
\]

\[
L^- = \text{Lie}\{\text{Ad}^k_u X^- | k \leq 0, u \in \mathbb{R}\},
\]

\[
L = \text{Lie}\{\text{Ad}^k_u X_\sigma^e | k \in \mathbb{Z}, u \in \mathbb{R}, \sigma \in \{+,-\}\}.
\]

The following inclusions are evident:

\[
L^+ \subset \text{Lie}\Gamma^+, \quad L^- \subset \text{Lie}\Gamma^-, \quad L \subset \text{Lie}\Gamma.
\]

In terms of this data, we now state another one of our main results.

**Theorem 2** The following properties hold for any analytic system (1):

(a) The system is forward accessible if and only if

\[
\dim L^+(x) = n \quad \text{for any} \ x \in \mathbb{R}^n.
\]

(b) The system is backward accessible if and only if

\[
\dim L^-(x) = n \quad \text{for any} \ x \in \mathbb{R}^n.
\]

(c) The system is transitive if and only if

\[
\dim L(x) = n \quad \text{for any} \ x \in \mathbb{R}^n.
\]
Remark 3.1 As a consequence of proposition 2.7, if we were to take in the definition of the Lie algebra $L$ only $\sigma = +$, or alternatively only $\sigma = -$, a smaller set of vector fields may result, but the conclusions in the theorem would hold equally well. 

There is a pointwise version of the above results. An equilibrium point $x_0 \in \mathbb{R}^n$ is one such that $f(x_0, 0) = 0$.

**THEOREM 3** The following properties hold:

(a) A smooth system (1) is transitive from $x$ if and only if $\dim \Gamma(x) = n$ (equivalently, $\dim \text{Lie} \{\Gamma\}(x) = n$). An analytic system (1) is transitive from $x$ if and only $\dim L(x) = n$.

(b) An analytic system (1) is forward (respectively, backward) accessible from an equilibrium point $x_0$ if and only if $\dim L^+(x_0) = n$ (respectively, $\dim L^-(x_0) = n$).

The second part of theorem 3 will be strengthened as a consequence of the following.

**PROPOSITION 3.2** If the system is analytic and $x_0$ is an equilibrium point then

$$L^+(x_0) = L^-(x_0) = L(x_0).$$

The above theorem and proposition immediately imply the following corollary.

**COROLLARY 3.3** Assume that the system is analytic and $x_0$ is an equilibrium point. Then forward accessibility from $x_0$, backward accessibility from $x_0$, and transitivity from $x_0$ are all equivalent properties.

The above results are consequences of the following somewhat stronger sufficiency and necessity results.

Define the following families of vector fields:

$$X_{u^+}^i = \frac{\partial^i}{\partial u^i} X_u^+, \quad X_{u^-}^i = \frac{\partial^i}{\partial u^i} X_u^-.$$

**THEOREM 4** The following statements hold for any smooth system (1).

(a) If

$$\dim \text{Lie} \{\Gamma^+\}(x) = n \text{ for all } x \in \mathbb{R}^n,$$

then the system is forward accessible.

(b) If $x_0$ is an equilibrium point and if

$$\dim \text{Lie} \{\text{Ad}_{x_0}^k X_{u^+}^i \mid k \geq 0, \ i \geq 0 \} (x_0) = n,$$

then the system is forward accessible from $x_0$.

(c) The same statements hold for backward accessibility if we replace $\Gamma^+$ for $\Gamma^-$ and $X_{u^+}^i$ for $X_{u^-}^i$. 


This is a converse to theorem 4:

**Theorem 5** (a) If system (1) is of class $C^1$ and forward accessible from $x$,

$$\dim \Gamma^+(x) = n.$$  

(b) If system (1) is analytic and forward accessible from $x$,

$$\dim L^+(x) = n.$$  

(c) Analogous results hold for backward accessibility with $\Gamma^+$, $L^+$ replaced by $\Gamma^-$, $L^-$.  

## 4 NONACCESSIBLE SYSTEMS

In this section we will briefly discuss nonaccessible and, more generally, nontransitive systems. The following "orbit theorem" is crucial in understanding such systems. The theorem has a long history starting with results of Chow, Nagano, ([13]), Sussmann ([22]) and Stefan ([21]) in the continuous time case. In the discrete time case analogous results to those in continuous time where provided in [6], [20], [8], and [17], the latter containing also a proof of a more abstract result dealing with a general notion of action on manifolds. These papers should be consulted for details of the proof, which we omit.

**Theorem 6** Any orbit $A(x)$ of the smooth system (1) is an immersed submanifold of $\mathbb{R}^n$ with at most countably many connected components, whose the tangent space is given by

$$T_yA(y) = \Gamma(y)$$

at each $y \in A(x)$. In the analytic case we have that

$$T_yA(y) = L(y)$$

holds too.

As the attainable set from $x$ lies in the orbit from $x$, there is no chance for forward or backward accessibility from $x$ if there is no transitivity from $x$ (that is, the orbit is not of full dimension). In this case it is reasonable to ask whether the attainable set has a nonempty interior in the orbit. In the case of analytic continuous time systems the answer is always positive, as proved by Sussmann and Jurdjevic ([24]). The following theorem generalizes this result to discrete time systems.

**Theorem 7** If $x_0$ is an equilibrium point of an analytic system (1), then each of the attainable sets $A^+(x_0)$ and $A^-(x_0)$ has a nonempty interior in the orbit $A(x_0)$.

**Proof.** If we restrict our system to the orbit then the problem reduces to proving that the system is forward (backward) accessible from $x_0$, if it is transitive from $x_0$. But this follows immediately from Theorem 3.
Remark 4.1 The above provides an analogue of what is sometimes called the positive form of Chow's lemma for continuous time systems. In fact, the proof is related to that in the continuous time case. However, there is an interesting subtlety that appears here. Contrary to the continuous situation, it is not true now that the assumption that $x_0$ is an equilibrium state can be relaxed. In the paper [17], remark 9.15, an example is given of an analytic system with $n = 1$ and a state $x$ such that $A(x) = \mathbb{R}^n$ but the system is not forward accessible from this $x$. In fact, the system in question arises from the sampling of a continuous time system. This means that the notion of transitivity is in the discrete time case probably too weak with respect to nonequilibrium states. 

The following families of vector fields will help us to understand better the geometry of the attainable sets $A^+(x)$ and $A^-(x)$ and, in particular, to estimate their dimensions. Define

$\Delta_k^+ = \{Ad_0^i X_u^+ \mid 0 \leq i \leq k - 1, u \in \mathbb{R}\}, \quad L_k^+ = \text{Lie} \, \Delta_k^+,$

and

$\Delta_k^- = \{Ad_0^{-i} X_u^- \mid 0 \leq i \leq k - 1, u \in \mathbb{R}\}, \quad L_k^- = \text{Lie} \, \Delta_k^-.$

For any family of vector fields $\Delta$, let $\text{Orb}_\Delta(x)$ denote the orbit of this family passing through $x$. This orbit has a natural structure of immersed second countable submanifold ([22], [21]). Further, the orbit of $\text{Lie} \, \Delta$ coincides with the orbit of $\Delta$.

PROPOSITION 4.2 For any smooth system,

$A_k^+(x) \subset \text{Orb}_{\Delta_k^-}(y), \text{for any } y \in A_k^+(x),$

and

$A_k^-(x) \subset \text{Orb}_{\Delta_k^+}(y), \text{for any } y \in A_k^-(x).$ 

From the above proposition we immediately conclude the following necessary conditions for accessibility.

COROLLARY 4.3 If an analytic system is forward accessible from $x$ then

$\dim L^-(y) = n \text{ for any } y \in A^+(x).$

Similarly, if it is backward accessible from $x$, then

$\dim L^+(y) = n \text{ for any } y \in A^-(x).$ 

5 AN EXAMPLE

Consider the following invertible polynomial system with $n = 3$.

\begin{align*}
x^+ &= x(z^2 + 1)^2 \\
y^+ &= y(z^2 + 1)^3 \\
z^+ &= z + u,
\end{align*}
where we are using the superscript $+$ to denote time shift, and we denote coordinates as $(x, y, z)$. Calculating, one obtains that $X_u^+ = -2z(z^2 + 1)^{-1}Z - X_u^-$ and $X_u^- = (0, 0, -1)^t$, where $Z$ is the vector field

$$
egin{pmatrix}
2x \\
3y \\
0
\end{pmatrix},
$$

for each $u \in \mathbb{R}$. Since the basic vector fields $X_u^+$ and $X_u^-$ turn out to be independent of $u$ in this example, we drop the subscripts $u$ from now on. Further,

$$
\text{Ad}_0 X^+ = \begin{pmatrix}
-8xz(z^2 + 1)^{-1} \\
-12yz(z^2 + 1)^{-1} \\
1
\end{pmatrix} = 2X^+ + X^-,
$$

from which it follows that

$$\text{span} \{X^+, \text{Ad}_0 X^+\} = \text{span} \{X^+, X^-\}.$$

The identity $\text{Ad}_0 X^- = -X^+$ (c.f. proposition 2.6(b)) implies that

$$\text{Ad}_0^{-1} X^+ = 2\text{Ad}_0 X^+ - X^- \in \text{span} \{X^+, X^-\},$$

so the linear span of the set of all generators of $L^+$, $\{\text{Ad}_0^k X^+, k \geq 0\}$, coincides with the span of $X^+$ and $X^-$. Similarly, applying $\text{Ad}_0^{-1}$ to both sides of (10),

$$\text{Ad}_0^{-1} X^- = X^+ - 2\text{Ad}_0^{-1} X^+ = X^+ + 2X^-,$$

so the span of the $\{\text{Ad}_0^k X^-, k \leq 0\}$, the generators of $L^-$, is again the same. Finally,

$$[X^+, X^-] = 2(1 - z^2)(z^2 + 1)^{-2}Z,$$

from which it follows that $\{X^-, X^+, [X^+, X^-]\}$ and $\{X^-, Z\}$ span the same $C^\infty$ submodule of vector fields. The latter set is involutive, and we conclude that, for this example,

$$L^+ = L^- = L.$$

Thus the orbits have dimension 2 through each point except at those points with $x = y = 0$, where $Z$ vanishes, and there the dimension is 1. The tangent spaces are given by the vectors $\partial/\partial z$ and $2x\partial/\partial x + 3y\partial/\partial y$. The forward and backward accessible sets contain open subsets of each orbit, by the equality of these Lie algebras.

Of course, in this very simple example one can analyze the system directly. The initial states $(x_0, y_0, z_0)$ with $x_0 = y_0 = 0$ are such that the only possible directions of movement are those in which $z$ changes, as is clear from the equations (9), consistently with the above conclusion about tangent spaces. The points where exactly one of $x_0$ or $y_0$ is nonzero are also easy to analyze. Take now a point with both $x_0$ and $y_0$ nonzero. Consider the set $C$ consisting of all points $(x, y, z)$ with

$$y_0^2 x^3 = x_0^3 y^2.$$

This is the cross product of a cusp with a line. The forward accessible set consists of all $(x, y, z)$ in $C$ with $\text{sign } y = \text{sign } y_0$ for which $|z| \geq |x_0|$ and $|y| \geq |y_0|$. The backward accessible has both these inequalities reversed, and the orbit consists of the branch of $C$.
with just sign $y = \text{sign} y_0$. Note how each such set $C$, an algebraic variety, can be stratified into three submanifolds, which turn out to be its singular set (the orbit of $(0,0,0)$), the orbit of $(x_0, y_0, z_0)$, and the orbit of the 'conjugate' point $(x_0, -y_0, z_0)$.

Thus in this example both the forward-accessible set and the orbit from each point are open subsets of an irreducible algebraic variety. More generally, similar behavior may be expected when dealing with invertible polynomial systems and equilibrium initial states. We conjecture that the orbit is an open subset of the quasi-reachable set in the sense of [15] and [16]. This is an algebraic variety, and it can be computed explicitly, via Jacobians of the $n$-step transition map. Note that polynomial invertible systems may exhibit highly nonlinear behavior, such as in the case $x^+ = x^2 + x + u$, where the inverse of the transition mapping is not even rational. We plan to study such systems in greater depth in the future.

6 REFERENCES


