A Characterization of Integral Input to State Stability

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Abstract

Just as input to state stability (ISS) generalizes the idea of finite gains with respect to supremum norms, the new notion of integral input to state stability (IISS) generalizes the concept of finite gain when using an integral norm on inputs. In this paper, we obtain a necessary and sufficient characterization of the IISS property, expressed in terms of dissipation inequalities.

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1 Introduction

One of the main issues in nonlinear control design concerns the study of the dependence of state trajectories on the size of inputs, a study which is especially relevant when the inputs in question represent disturbances acting on a system, or signals to be tracked. For nonlinear systems, there is no complete agreement as yet regarding what are the most useful formulations of system stability with respect to input perturbations. (For linear systems, similar considerations led to the development of gains and the operator-theoretic approach, including the formulation, when using $L^2$ norms, of $H^\infty$ control.) One candidate for such a formulation is the property called “input to state stability” (ISS), introduced in [17]. ISS differs fundamentally from the operator-theoretic notions, among others in two respects: (1) it takes account of initial states in a manner fully compatible with Lyapunov stability, and (2) it replaces finite linear gains, which represent far too strong a requirement for general nonlinear operators, with “nonlinear gains”. The ISS concept has already proven useful in many applications; see e.g. [2, 3, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 23]. Moreover, this concept has many equivalent versions, which indicates that it is mathematically natural: there are characterizations in terms of dissipation, robustness margins, and classical Lyapunov-like functions; see e.g. [19, 20].

Informally, the ISS property translates into the statement: “no matter what is the initial state, if the inputs are uniformly small, then the state must eventually be small”. It gives no useful bounds in the situation in which inputs $u(\cdot)$ are unbounded but still have in some sense “finite energy”. In [18], a new notion, of integral input to state stability, ISS for short, was introduced, to model the statement: “no matter what is the initial state, if integrals of the inputs are small, then the state must eventually be small”. That paper showed that ISS is, in general, strictly weaker than ISS, and provided a very conservative Lyapunov-type sufficient condition.

In this paper, we provide a complete, necessary and sufficient, Lyapunov-like characterization of the ISS property. Just as the equivalences for ISS, which have found wide applicability and serve to justify the ISS concept, are derived from its Lyapunov characterization, we expect that the current paper will be the first step in the understanding of which system properties are equivalent to ISS. In addition, the characterizations allow one to consider “LaSalle” types of dissipation inequalities (semidefinite derivatives), filling-in a theoretical gap in the ISS literature, and also leading to a characterization of the stability features of certain tracking designs, as we illustrate with a simple robotics example.

2 Main Results

Consider the system

$$\dot{x} = f(x, u)$$

(1)

with states $x(t)$ evolving in Euclidean space $\mathbb{R}^n$. Here, controls (or inputs) are measurable and locally essentially bounded functions $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$, and $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is assumed to be locally Lipschitz.

Given any control $u$ and any $\xi \in \mathbb{R}^n$, there is a unique maximal solution of the initial value problem $\dot{x} = f(x, u)$, $x(0) = \xi$. This solution is defined on some maximal open interval, and it is denoted by $x(\cdot, \xi, u)$.

Definition 2.1 ([18]) System (1) is integral input-to-state stable (IISS) if there exist functions$^*$ $\alpha \in \mathcal{K}_\infty$, $\beta \in \mathcal{KL}$, and $\gamma \in \mathcal{K}$, such that

$$\alpha(|x(t, \xi, u)|) \leq \beta(|\xi|, t) + \int_0^t \gamma(|u(s)|) \, ds$$

(2)

$^*$ We use standard terminology, cf. [4]: $\mathcal{K}$ is the class of functions $[0, \infty) \to [0, \infty)$ which are zero at zero, strictly increasing, and continuous, $\mathcal{K}_\infty$ is the subset of $\mathcal{K}$ functions that are unbounded, $\mathcal{L}$ is the set of functions $[0, +\infty) \to [0, +\infty)$ which are continuous, decreasing, and converging to 0 as their argument tends to $+\infty$, $\mathcal{KL}$ is the class of functions $[0, \infty)^2 \to [0, \infty)$ which are class $\mathcal{K}$ on the first argument and class $\mathcal{L}$ on the second argument. A positive definite function $[0, \infty) \to [0, \infty)$ is one that is zero at 0 and positive otherwise.
for all $t \geq 0$, all $\xi \in \mathbb{R}^n$, and all $u$, where $|\cdot|$ denotes the standard Euclidean norm.

Observe that a system is ISS if and only if there exist functions $\beta \in \mathcal{K}\mathcal{L}$ and $\gamma_1, \gamma_2 \in \mathcal{K}$ such that
\[
x(t, \xi, u) \leq \beta(|\xi|, t) + \gamma_1 \left( \int_0^t \gamma_2(|u(s)|) \, ds \right)
\]for all $t \geq 0$, all $\xi \in \mathbb{R}^n$, and all $u$.

Also note that if system (1) is ISS, then it is 0-GAS, that is, the 0-input system
\[
\dot{x} = f(x, 0)
\]
is globally asymptotically stable (GAS). (That is, the zero solution of this system is globally asymptotically stable.)

**Definition 2.2** A continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ is called an **ISS-Lyapunov function** for system (1) if there exist functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\sigma \in \mathcal{K}$, and a positive definite function $\alpha_3$, such that
\[
\alpha_1(|\xi|) \leq V(\xi) \leq \alpha_2(|\xi|)
\]for all $\xi \in \mathbb{R}^n$, and
\[
DV(\xi)f(\xi, \mu) \leq -\alpha_3(|\xi|) + \sigma(|\mu|)
\]for all $\xi \in \mathbb{R}^n$ and all $\mu \in \mathbb{R}^n$.

Note that the estimate (4) amounts to the requirement that $V$ must be positive definite (i.e., $V(x) > 0$ for all $x \neq 0$ and $V(0) = 0$), and proper (i.e., radially unbounded, namely, $V(x) \to \infty$ as $|x| \to \infty$).

Notice the difference between Definition 2.2 and the dissipation characterization of ISS (cf. [19, 20]): the ISS property is equivalent to the existence of a $V$ as here but for which $\alpha_3$ is required to be unbounded (in fact, class $\mathcal{K}_\infty$). As an example, consider the one dimensional system:
\[
\dot{x} = -\arctan x + u.
\]

Let $V(x) = x \arctan x$. Then
\[
DV(\xi)f(\xi, \mu) = \arctan \xi (-\arctan \xi + \mu) + \frac{\xi}{1 + \xi^2}(-\arctan \xi + \mu)
\]\[
\leq -(\arctan |\xi|)^2 + 2|\mu|.
\]

This shows that $V$ is an ISS-Lyapunov function for the system. But in the estimate (5) we have $\alpha_3(r) = (\arctan r)^2$, which is not of class $\mathcal{K}_\infty$, so one does not have an ISS-type estimate. Indeed, this system does not admit any ISS-Lyapunov function, since the system is not ISS (the trajectory with $x(0) = 1$ and $u(t) \equiv \pi/2$ is unbounded).

Our main result will establish that the existence of a smooth ISS-Lyapunov function is necessary as well as sufficient for the system to be ISS.

This fact will be stated in several essentially equivalent ways. One possibility is to relax the positive definiteness requirement on $\alpha_3$ to just nonnegativity, or simply omit it, but to assume explicitly that the system is 0-GAS.

Another possibility is to deduce the 0-GAS property from LaSalle’s invariance principle. This last variant is of considerable interest in applications such as the robotics example discussed in Section 5, and it may be stated using concepts of detectability, as is by now standard in the nonlinear dissipation literature (see, e.g., [24], section 3.2). Let us say that an output for the system (1) is a continuous map $h : \mathbb{R}^n \to \mathbb{R}^p$ (for some $p$), with $h(0) = 0$. For each initial state $\xi \in \mathbb{R}^n$, and each input $u$, we let $y(t, \xi, u)$ be the corresponding output function, i.e., $y(t, \xi, u) = h(x(t, \xi, u))$ (defined on some
The system (1) with output $h$ is said to be weakly zero-detectable if, for each $\xi$ such that $T_{\xi,0} = \infty$ and $y(t,\xi,0) \equiv 0$, it must be the case that $x(t,\xi,0) \rightarrow 0$ as $t \rightarrow \infty$. Finally, we will say for the purposes of this paper that the system (1) with output $h$ is dissipative if there exists a continuously differentiable, proper, and positive definite function $V$ (a storage function for the system), together with a $\sigma \in \mathcal{K}$ and a positive definite function $\alpha_4$, such that

$$DV(\xi)f(\xi, \mu) \leq -\alpha_4(|h(\xi)|) + \sigma(|\mu|)$$

for all $\xi \in \mathbb{R}^n$ and all $\mu \in \mathbb{R}^m$. If this property holds with a $V$ which is also smooth, we say that the system (1) with output $h$ is smoothly dissipative. Finally, if (6) holds with $h = 0$, i.e., if there exists a (smooth) proper and positive definite $V$, and a $\sigma \in \mathcal{K}$, so that

$$DV(\xi)f(\xi, \mu) \leq \sigma(|\mu|)$$

holds for all $\xi \in \mathbb{R}^n$ and all $\mu \in \mathbb{R}^m$, we say that the system (1) is zero-output (smoothly) dissipative.

We now are now able to state the main conclusions of this paper.

**Theorem 1** For any system (1), the following properties are equivalent:

1. The system is ISS.
2. The system admits a smooth ISS-Lyapunov function.
3. There is some output which makes the system smoothly dissipative and weakly zero-detectable.
4. The system is 0-GAS and zero-output smoothly dissipative.

The main step of the proof of Theorem 1 is given in Section 3, where we show 1$\Rightarrow$2 and also we prove Proposition 2.5 (see below), which characterizes the 0-GAS property. The implication 4$\Rightarrow$2 will be immediate from Proposition 2.5. The remaining implications are routine, so we can dispose of them immediately, as follows. First of all, notice that 2$\Rightarrow$3. To see this, take the ISS-Lyapunov function $V$ as a storage function, and consider the inequality in (5). We introduce the output function $h(x) := \alpha_4(|x|)$. The system is weakly zero-detectable (in fact, it is even “zero-observable”), because $h(x) = 0$ implies $x = 0$, since $\alpha_4$ is positive definite. Moreover, with $\alpha_4$ equal to the identity, we have that $\alpha_4(|h(\xi)|) = \alpha_4(|\xi|)$, so (6) is the same as (5). Finally, we show that 3$\Rightarrow$4. Suppose that (6) holds. With $\mu = 0$, we take $V$ as a Lyapunov function for the zero-input system $\dot{x} = f(x,0)$. The zero-detectability condition means that the LaSalle invariance principle, with the Lyapunov function $V_0$, can be applied, and we conclude 0-GAS. And since $-\alpha_4(h(\xi)) \leq 0$, also (7) holds.

**Remark 2.3** We stated Theorem 1 requiring that the corresponding functions $V$ (ISS-Lyapunov, storage) be smooth, that is, infinitely differentiable. This makes the existence of such $V$'s, which is the harder part to prove, more interesting. The sufficiency parts of the proofs do not require smoothness, however. In other words, system (1) is ISS if it admits an ISS-Lyapunov function, or if it has an output which makes the system dissipative and weakly zero-detectable or if it is 0-GAS and zero-output dissipative.

**Remark 2.4** We used the adjective “weak” when defining zero-detectability in order to distinguish this notion from true detectability, or “(zero-input) output to state stability”, cf. [21] and also Section 6 below, where one asks that “small output (when $u \equiv 0$) implies small state”, as opposed to merely asking that “zero output implies small state” here.

**A Characterization of 0-GAS Control Systems**

In the proof Theorem 1, we utilize the following characterization of 0-gas systems. It is in itself a result of some interest.

We call a positive definite function $W: \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ semi-proper if there exist a function $\pi(\cdot)$ of class $\mathcal{K}$ and a proper positive definite function $W_0$ such that $W = \pi \circ W_0$. (It is easy to see that a continuous positive definite $V: \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ is semi-proper if and only if, for each $r$ in the range of $V$, the sublevel set $\{x \mid V(x) \leq r\}$ is compact.)
Proposition 2.5  System (1) is 0-GAS if and only if there exist a smooth semi-proper function $W$, a $\sigma \in \mathcal{K}$, and a positive definite function $\rho : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$, such that

$$DW(\xi)f(\xi, \mu) \leq -\rho(|\xi|) + \sigma(|\mu|)$$

(8)

for all $\xi \in \mathbb{R}^n$ and all $\mu \in \mathbb{R}^m$.

The sufficiency part follows from the standard Lyapunov results for autonomous systems: if (8) holds with $W = \pi \circ W_0$, then $W_0$ is a Lyapunov function for the 0-input system. (This is because (8) implies that $DW_0(\xi)f(\xi, 0) < 0$ for all $\xi \neq 0$.) The necessity implication will be proved in Section 3.

Proof of $\Rightarrow$ in Theorem 1. Let the functions $V$ and $\sigma$ be so that (7) holds. Since the system is 0-GAS, by Proposition 2.5, there exists a semi-proper, positive definite function $V_0$ such that

$$DV_0(\xi)f(\xi, \mu) \leq -\rho_0(\xi) + \sigma_0(|\mu|), \quad \forall \xi \in \mathbb{R}^n, \forall \mu \in \mathbb{R}^m$$

for some positive definite function $\rho_0$ and some $\mathcal{K}$-function $\sigma_0$. Let $V_1(\xi) = V(\xi) + V_0(\xi)$. It then clear that $V_1$ is an ISS-Lyapunov function: it is proper because $V_0$ is, and

$$D(V + V_0)(\xi)f(\xi, \mu) \leq -\rho_0(\xi) + \sigma_0(|\mu|) + \sigma(|\mu|)$$

gives an estimate as in (5).

Motivation: Finite-Gain under Coordinate Changes

In closing this introduction, we wish to explain briefly how the notion of ISS arises in an extremely natural manner when generalizing linear $L^1$ to $L^\infty$ gains (sometimes called “$H_2$ gains”) to nonlinear systems. (See also [18], which explains why, when we apply the same reasoning to “$L^2$ to $L^2$ stability” or to “$L^\infty$ to $L^\infty$ stability,” we recover input to state stability.)

For linear systems, one defines finite-gain stability, with respect to square norm on inputs and sup norm on states, by requiring the existence of constants $c$ and $\lambda$, with $\lambda > 0$, so that, for each input $u(\cdot)$ and each initial state $\xi$, the solution $x(t)$ of $\dot{x} = Ax + Bu$, $x(0) = \xi$, satisfies the following estimate:

$$|x(t)| \leq ce^{-\lambda t} |\xi| + c \int_0^t |u(s)|^2 ds \quad \text{for all } t \geq 0.$$  

(9)

(Actually, most textbooks omit the initial state, but this is the appropriate estimate if nonzero initial states are taken into account.) In a nonlinear context, it is natural to require that notions of stability should be invariant under (nonlinear) changes of variables. Let us see what this leads us to. Suppose that we take an origin-preserving state change of coordinates $x = T(z)$ and an origin-preserving change of variables $u = S(v)$. That is, $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $S : \mathbb{R}^m \rightarrow \mathbb{R}^m$ are invertible, and they, as well as their inverses, are continuous; further, we suppose that $T(0) = 0$ and $S(0) = 0$. Then, there are two functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ so that

$$\alpha_1(|z|) \leq |T(z)| \leq \alpha_2(|z|)$$

for all $z \in \mathbb{R}^n$, and, similarly, we can write $|S(v)|^2 \leq \gamma(|v|)$ for each $v \in \mathbb{R}^m$, for some $\gamma \in \mathcal{K}_\infty$.

Therefore, the estimate (9) gives us, in terms of $z$ and $v$:

$$\alpha_1(|z(t)|) \leq ce^{-\lambda t} \alpha_2(|\xi|) + c \int_0^t \gamma(|u(s)|)^2 ds \quad \text{for all } t \geq 0,$$

when $z(t) = T(z(t))$ and $u(t) = S(v(t))$ for all $t$, and $\zeta = z(0) = T^{-1}(\xi)$. In other words,

$$|z(t)| \leq \beta(|\xi|, t) + \alpha_1^{-1} \left( \int_0^t 2c \gamma(|u(s)|)^2 ds \right) \quad \text{for all } t \geq 0,$$

where we let $\beta(r, t) := \alpha_1^{-1}(2\alpha_2(r)e^{-\lambda t})$. This is precisely as in the estimate (3), except that $\beta$ has what appears to be a very special form. Surprisingly, however, any $\mathcal{KL}$ function $\beta$ can be majorized by a function of this special form, see [18], so indeed one obtains the general notion of ISS with this reasoning.
3 Main Proofs

The following Lemma will be needed several times during the proofs.

**Lemma 3.1** Let $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a continuous positive definite function. Then there exist $\rho_1 \in \mathcal{K}_\infty$ and $\rho_2 \in \mathcal{L}$ such that:

$$\rho(r) \geq \rho_1(r) \rho_2(r). \quad (10)$$

The Lemma will be proved in the appendix; it is used in establishing the following comparison theorem.

**Lemma 3.2** Given any continuous positive definite function $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, there exists a $\mathcal{KL}$-function $\beta$ with the following property. Suppose that for some $0 < \tilde{t} \leq \infty$,

$$v : [0, \tilde{t}) \to \mathbb{R}_{\geq 0} \quad \text{and} \quad y : [0, \tilde{t}) \to \mathbb{R}$$

are, respectively, a continuous and a (locally) absolutely continuous function with $y(0) \geq 0$. Assume further that

$$\dot{y}(t) \leq -\rho \left( \max \{ y(t) + v(t), 0 \} \right) \quad (11)$$

holds for almost all $t \in [0, \tilde{t})$. Then, letting $||v_t||_{\infty}$ be the supremum of the restriction of $v$ to the interval $[0, t)$, the following estimate holds:

$$y(t) \leq \max \{ \beta(y(0), t), ||v_t||_{\infty} \} \text{ for all } t \in [0, \tilde{t}). \quad (12)$$

**Proof.** We start by picking $\rho_1 \in \mathcal{K}$, $\rho_2 \in \mathcal{L}$ as in Lemma 3.1, for the function $\rho$. Without loss of generality, we may assume that $\rho_1$ and $\rho_2$ are locally Lipschitz. Otherwise, we may always pick locally Lipschitz functions $\bar{\rho}_1 \in \mathcal{K}$ and $\bar{\rho}_2 \in \mathcal{L}$ that are majorized by $\rho_1$ and $\rho_2$ respectively to replace $\rho_1$ and $\rho_2$ respectively.

A standard comparison principle asserts the existence of a function $\beta \in \mathcal{KL}$ having the following property: if $q : [0, T] \to \mathbb{R}_{\geq 0}$ is any absolutely continuous function that satisfies the differential inequality $\dot{q} \leq -\rho_1(q) \rho_2(2q)$ almost everywhere, then it must be the case that $q(t) \leq \beta(q(0), t)$ for all $t \in [0, T]$. (See for instance Lemma 4.4 in [11]; the statement in that reference applies to $q$ defined on all of $[0, \infty)$, but exactly the same proof works for a finite interval. One choice for $\beta(s, t) = \beta(s, t) = z(t)$, where $z$ is the solution of the scalar initial value problem $\dot{z} = -\rho_1(z) \rho_2(2z)$, $z(0) = s$.)

Let now $v$ and $y$ be as in the statement of the Lemma, and define

$$t_0 := \min \{ t \geq 0 \mid y(t) \leq ||v_t||_{\infty} \}$$

(with $t_0 := \tilde{t}$ if $y(t) > ||v_t||_{\infty}$ for all $t \in [0, \tilde{t})$). For all $t \geq t_0$ (if $t_0 < \tilde{t}$), $y(t) \leq ||v_t||_{\infty}$ (because $y$ is nonincreasing, since $\dot{y}(t) \leq 0$ for all $t$, and $t \mapsto ||v_t||_{\infty}$ is nondecreasing), so (12) holds for all $t \geq t_0$.

Pick now any $t \in [0, t_0)$. We have then that $y(t) > ||v_t||_{\infty} \geq v(\tau)$ for all $\tau \in [0, t]$ (the last inequality by definition of $||v_t||_\infty$). Since $y$ is nonincreasing, this means that also $y(\tau) \geq y(t) > v(\tau)$ for all such $\tau$. Therefore

$$0 \leq y(\tau) \leq y(t) + v(\tau) \leq 2y(\tau)$$

for all $\tau \in [0, t]$. From (11) and the fact that $\rho_1$ is nondecreasing, we conclude that

$$\dot{y} \leq -\rho_1(y) \rho_2(2y) \quad (13)$$

almost everywhere on $[0, t]$. Since $t \in [0, t_0)$ was arbitrary, (13) holds on $[0, t_0)$ a.e. By the choice of $\beta$, it follows that $y(t) \leq \beta(y(0), t)$ for all $t \in [0, t_0)$. Thus (12) holds for all such $t$ as well. \[\square\]

The following is a consequence of Lemma 3.2.
Corollary 3.3 Given any continuous positive definite function $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, there exists a $\mathcal{KL}$-function $\beta$ with the following property. For any $0 < t \leq \infty$, and for any (locally) absolutely continuous function $y : [0, t] \to \mathbb{R}_{\geq 0}$ and any measurable, locally essentially bounded function $v : [0, t] \to \mathbb{R}_{\geq 0}$, if
\[
\dot{y}(t) \leq -\rho(y(t)) + v(t)
\]
holds for almost all $t \in [0, t]$, then the following estimate holds:
\[
y(t) \leq \beta(y(0), t) + \int_0^t 2v(s) \, ds \quad \text{for all } t \in [0, t].
\]

Proof. First observe that one may always assume that the function $\rho$ is locally Lipschitz, for otherwise one may replace $\rho$ by any such function majorized by $\rho$. We take a $\mathcal{KL}$-function $\beta$ as in Lemma 3.2, for this $\rho$. Take now any $y, v$ as in the statement, and consider the solution $w(t)$ to the following initial value problem:
\[
\dot{w}(t) = -\rho(|w(t)|) + v(t), \quad w(0) = y(0).
\]
It follows from the standard comparison principle that $0 \leq y(t) \leq w(t)$ for all $t \in [0, t]$. In particular, we can write $\rho(w(t))$ instead of $\rho(|w(t)|)$ in the above equation. Now define $v_1$ and $w_1$ as follows:
\[
v_1(t) = \int_0^t v(s) \, ds, \quad w_1(t) = w(t) - v_1(t).
\]
Taking the derivative of $w_1$ with respect to $t$ yields
\[
\dot{w}_1(t) = -\rho(w(t)) + v(t) - v(t) = -\rho(\max\{w_1(t) + v_1(t), 0\})
\]
for almost all $t \in [0, t]$, where the last equation holds because $w$ is nonnegative. By the choice of $\beta$, it follows that
\[
w_1(t) \leq \max\{\beta(w_1(0), t), \|v_1\|_{\infty}\} \quad \forall t \in [0, t],
\]
from which it follows that
\[
y(t) \leq w(t) \leq \beta(w(0), t) + \|v_1\|_{\infty} + \int_0^t v(s) \, ds = \beta(y(0), t) + \int_0^t 2v(s) \, ds
\]
for all $t \in [0, t]$. □

We also need the following result in our proofs.
Let $\mathcal{N}$ denote the class of all functions $k : \mathbb{R} \to \mathbb{R}$ that are:

1. nondecreasing,
2. continuous, and
3. unbounded below (i.e., $\inf_{x \in \mathbb{R}} k(x) = -\infty$).

We will prove:

Proposition 3.4 Suppose that $c : \mathbb{R}^2 \to \mathbb{R}$ is such that $c(\cdot, y) \in \mathcal{N}$ for each $y \in \mathbb{R}$ and $c(x, \cdot) \in \mathcal{N}$ for each $x \in \mathbb{R}$. Then, there exists some function $k \in \mathcal{N}$ such that
\[
c(x, y) \leq k(x) + k(y)
\]
for all $(x, y) \in \mathbb{R}^2$.

This result generalizes the one given in [18], which applied only to functions of the form $c(x, y) = g(x + y)$ with $g \in \mathcal{N}$. We will need the "exponential" form of this result, which is as follows:
Corollary 3.5 Suppose that $\gamma : \mathbb{R}^2_{\geq 0} \to \mathbb{R}$ is such that $\gamma(\cdot, s) \in \mathcal{K}$ for each $s \in \mathbb{R}_{\geq 0}$ and $\gamma(r, \cdot) \in \mathcal{K}$ for each $r \in \mathbb{R}_{\geq 0}$. Then, there exists some function $\sigma \in \mathcal{K}$ such that

\[ \gamma(r, s) \leq \sigma(r) \sigma(s) \]

for all $(x, y) \in (\mathbb{R}_{\geq 0})^2$.

Proof. Consider $c(x, y) := \ln \gamma(e^x, e^y)$; then $c$ is a class $\mathcal{N}$ function with respect to both arguments. Let $k \in \mathcal{N}$ be as in Proposition 3.4; without loss of generality, we may assume that $k$ is strictly increasing. Then $\sigma(r) := e^{k\ln r}$ (and $\sigma(0) = 0$) establishes the Corollary.

The proof of Proposition 3.4 will be given in the appendix.

Proof of $2 \Rightarrow 1$ in Theorem 1

We first prove that existence of a (just continuously differentiable, cf. Remark 2.3) ISS-Lyapunov function $V$ implies ISS. So pick $V$ so that (4)-(5) hold. Let $\rho_1 \in \mathcal{K}_{\infty}$ and $\rho_2 \in \mathcal{L}$ be functions as in Lemma 3.1 for $\alpha_3$. We let $\tilde{\rho}$ be any positive definite function which is locally Lipschitz and satisfies

\[ \tilde{\rho}(r) \leq \rho_1(\alpha_2^{-1}(r)) \rho_2(\alpha_1^{-1}(r)) \]

for all $r \geq 0$. By equation (4) we have:

\[ D V(\xi) f(\xi, \mu) \leq -\rho_1(|\xi|) \rho_2(|\xi|) + \sigma(|\mu|) \leq -\tilde{\rho}(V(\xi)) + \sigma(|\mu|) \quad (16) \]

for all $\xi$ and $\mu$. We let $\beta$ be associated to $\rho$ as in Corollary 3.3.

Now pick any trajectory $x(\cdot)$ corresponding to a control $u(\cdot)$. Equation (16) says that

\[ \dot{V}(x(t)) \leq -\tilde{\rho}(V(x(t))) + \sigma(|u(t)|) \]

for almost all $t$, so by Corollary 3.3, we know that

\[ V(x(t)) \leq \beta(V(x(0)), t) + \int_0^t 2\sigma(|u(s)|) \, ds \]

for all $t \geq 0$. Hence,

\[ \alpha_1(|x(t)|) \leq V(x(t)) \leq \beta(V(0), t) + \int_0^t 2\sigma(|u(s)|) \, ds \]

\[ \leq \beta(\alpha_2(|x(0)|), t) + \int_0^t 2\sigma(|u(s)|) \, ds \quad (17) \]

for all $t \geq 0$, and so the sufficiency proof is complete.

Proof of $1 \Rightarrow 2$ in Theorem 1

We first remark that the proof of Lemma 3.1 in [11] can be used to show the following:

Lemma 3.6 For each given $\mathcal{KL}$-function $\beta$, there exists a family of mappings $\{ T_r \}_{r \geq 0}$ with:

- for each fixed $r > 0$, $T_r : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is continuous and is strictly decreasing;
- for each fixed $\varepsilon > 0$, $T_r(\varepsilon)$ is (strictly) increasing as $r$ increases and $\lim_{r \to \infty} T_r(\varepsilon) = \infty$;

such that

\[ \beta(s, t) < \varepsilon \]

for all $s \leq r$, all $t \geq T_r(\varepsilon)$.  □
Assume now that system (1) is ISS with $\alpha, \beta, \gamma$ as in Definition 2.1. Let $\varphi$ be any smooth $K_\infty$-function such that $\gamma(\varphi(s)) \leq \alpha(s)$ for all $s \geq 0$. Consider the following system:

$$\dot{z}(t) = f(x(t), d(t)\varphi(|z(t)|)),$$

(18)

where we restrict the inputs $d$, thought of here as “disturbances”, to have values in the closed unit ball: $d(\cdot) : [0, \infty) \to \overline{B}$, where $\overline{B}$ denotes the closed unit ball $\{\mu \in \mathbb{R}^m : |\mu| \leq 1\}$ in $\mathbb{R}^m$. We let $M$ denote the set of all such inputs, and we let $x_\varphi(t, \xi, d)$ denote the trajectory of (18) corresponding to the initial state $\xi$ and the function $d$. This is defined on some maximal interval $[0, T_{\xi, d}^+]$ with $0 < T_{\xi, d}^+ \leq \infty$. It then follows from (2) that, for any given $\xi, d$, and each $t \in [0, T_{\xi, d}^+]$, and defining $\beta_0 := \beta(\cdot, 0)$:

$$\alpha(|x_\varphi(t, \xi, d)|) \leq \beta_0(|\xi|) + \int_0^t \gamma(|d(s)| \varphi(|x_\varphi(s, \xi, d)|)) ds$$

$$\leq \beta_0(|\xi|) + \int_0^t \gamma(\varphi(|x_\varphi(s, \xi, d)|)) ds$$

$$\leq \beta_0(|\xi|) + \int_0^t \alpha(|x_\varphi(s, \xi, d)|) ds.$$  

It thus follows, using Gronwall’s inequality, that

$$\alpha(|x_\varphi(t, \xi, d)|) \leq \beta_0(|\xi|) e^t$$

for all $0 \leq t < T_{\xi, d}^+$. Hence the maximal solution stays in a bounded set (the ball of radius $\beta_0(|\xi|) \exp(T_{\xi, d}^+)$), and thus $T_{\xi, d}^+ = +\infty$. In conclusion:

**Lemma 3.7** If system (1) is ISS, then there exists a smooth $K_\infty$-function $\varphi$ such that system (18) is forward complete, that is, $x_\varphi(t, \xi, d)$ is defined for all $t \geq 0$, all $\xi \in \mathbb{R}^n$, and all $d \in M$.  

Because of forward completeness, this follows from [11] (c.f. Propositions 5.1 and 5.5 of [11]).

**Lemma 3.8** Assume system (1) is ISS, and let $\varphi$ be given as in Lemma 3.7. For any fixed $T > 0$ and any compact $K \subset \mathbb{R}^n$, there is a compact $K_1 \subset \mathbb{R}^n$ such that $x_\varphi(t, \xi, d) \in K_1$ for all $t \in [0, T]$, all $\xi \in K$ and all $d \in M$. Furthermore, there is a constant $C > 0$ (which only depends on $T$ and the set $K$) such that

$$|x_\varphi(t, \xi, d) - x_\varphi(t, \eta, d)| \leq C |\xi - \eta|$$

for any $\xi, \eta \in K$, any $0 \leq t \leq T$, and any $d \in M$.  

We now continue with the proof of the necessity part of Theorem 1. Without loss of generality, one may assume that $\alpha$ is a smooth $K_\infty$-function. Otherwise, one can always replace $\alpha$ by a smooth $K_\infty$-function $\bar{\alpha}$ majorized by $\alpha$. We will first prove the result under the assumption that $\gamma$ is a smooth $K_\infty$-function, and then we show how to prove the result without this assumption.

Define $g : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ by

$$g(\xi) = \sup \{z(t, \xi, d) : t \geq 0, d \in M\},$$

(19)

where for each $\xi \in \mathbb{R}^n$ and $d \in M$, $z(\cdot, \xi, d)$ is defined by

$$z(t, \xi, d) = \alpha(|x_\varphi(t, \xi, u)|) - \int_0^t \gamma(|d(s)| \varphi(|x_\varphi(s, \xi, d)|)) ds.$$ 

Note that this function is well defined, and

$$\alpha(|\xi|) \leq g(\xi) \leq \beta_0(|\xi|)$$

for all $\xi \in \mathbb{R}^n$. In particular, $g(0) = 0$.

Let $T_\varphi(\epsilon)$ be defined as in Lemma 3.6. Then one sees that if $0 < r_1 < |\xi| < r_2$, then

$$g(\xi) = \sup \{z(t, \xi, d) : 0 \leq t \leq T_\varphi(\alpha(r_1)), d \in M\}.$$
Lemma 3.9 The function $g$ is locally Lipschitz on $\mathbb{R}^n \setminus \{0\}$ and continuous everywhere.

Proof. Fix any $\xi_0 \neq 0$, and let $s_0 = |\xi_0|$. Let $K_0 = \bar{B}(\xi_0, s_0/2)$, the closed ball centered at $\xi_0$ and with radius $s_0/2$. Let $T = T_{2s_0}(\alpha(s_0/2))$. Then

$$g(\xi) = \sup \{z(t, \xi, d) : t \in [0, T], d \in \mathcal{M}\}$$

for all $\xi \in K_0$. According to Lemma 3.8, one knows that there exists some $L > 0$ such that

$$|x_\varphi(t, \xi, d)| \leq L$$

for any $\xi \in K_0$, any $t \in [0, T]$, and any $d \in \mathcal{M}$. Since $\gamma$ is smooth, and in particular locally Lipschitz, there exists $C_1 > 0$ such that

$$|\gamma(r_1) - \gamma(r_2)| \leq C_1 |r_1 - r_2|$$

for all $r_1, r_2 \in [0, L]$. Consequently,

$$|\gamma(|d(s)| \varphi(|x_\varphi(s, \xi, d)|)) - \gamma(|d(s)| \varphi(|x_\varphi(s, \eta, d)|))| \leq C_1 |\varphi(|x_\varphi(s, \xi, d)|) - \varphi(|x_\varphi(s, \eta, d)|)|$$

for all $\xi, \eta \in K_0$, all $t \in [0, T]$, and all $d \in \mathcal{M}$. Since $\alpha$ and $\varphi$ are smooth, and $x_\varphi(s, \xi, d)$ is locally Lipschitz in $\xi$ uniformly in $t \in [0, T]$ and in $d \in \mathcal{M}$ (this is what is asserted by the last statement in Lemma 3.8), there exists some $C_2$ such that

$$|\alpha(|x_\varphi(t, \xi, d)|) - \alpha(|x_\varphi(t, \eta, d)|)| \leq C_2 |\xi - \eta|,$$

and

$$|\varphi(|x_\varphi(s, \xi, d)|) - \varphi(|x_\varphi(s, \eta, d)|)| \leq C_2 |\xi - \eta|$$

for all $\xi, \eta \in K_0$, all $t \in [0, T]$, and all $d \in \mathcal{M}$. It then follows that

$$\left|\int_0^t \gamma(|d(s)| \varphi(|x_\varphi(s, \xi, d)|)) ds - \int_0^t \gamma(|d(s)| \varphi(|x_\varphi(s, \eta, d)|)) ds\right| \leq C_3 |\xi - \eta|$$

for all $\xi, \eta \in K_0$, all $t \in [0, T]$, all $d \in \mathcal{M}$, where $C_3 = C_1 C_2 T$. This implies that

$$|z(t, \xi, d) - z(t, \eta, d)| \leq C_4 |\xi - \eta|$$

for all $\xi, \eta \in K_0$, all $t \in [0, T]$, all $d \in \mathcal{M}$, where $C_4 = C_2 + C_3$.

Pick any $\varepsilon > 0$. Then for each $\zeta \in K_0$, there is some $t_{\zeta, \varepsilon} \in [0, T]$ and $d_{\zeta, \varepsilon} \in \mathcal{M}$ such that

$$g(\zeta) \leq z(t_{\zeta, \varepsilon}, \zeta, d_{\zeta, \varepsilon}) + \varepsilon.$$

Let $\xi, \eta \in K_0$. Then

$$g(\xi) - g(\eta) \leq z(t_{\xi, \varepsilon}, \xi, d_{\xi, \varepsilon}) + \varepsilon - z(t_{\xi, \varepsilon}, \eta, d_{\xi, \varepsilon}) \leq C_4 |\xi - \eta| + \varepsilon. \quad (21)$$

Note that (21) holds for all $\varepsilon > 0$, so it follows that

$$g(\xi) - g(\eta) \leq C_4 |\xi - \eta|.$$

By symmetry, $g(\eta) - g(\xi) \leq C_4 |\xi - \eta|$. This proves that

$$|g(\xi) - g(\eta)| \leq C_4 |\xi - \eta|$$

for all $\xi, \eta \in K_0$. Thus, $g$ is locally Lipschitz on $\mathbb{R}^n \setminus \{0\}$.

To show that $g$ is continuous at $\xi = 0$, note that $g(0) = 0$, and $g(\xi) \leq \beta_0(\xi) \to 0$ as $\xi \to 0$. Thus $g$ is continuous everywhere.
We next show that \( g \) cannot increase too fast along trajectories. Pick any \( \xi \neq 0, h > 0 \), and \( |\mu| \leq 1 \). Let \( d_\mu \) denote the constant function \( d(t) \equiv \mu \). Then
\[
g(x, h, \xi, d_\mu) \\
= \sup_{t \geq 0, d \in M} \left\{ \alpha \left( |x(t, h, \xi, d_\mu)| \right) - \int_0^t \gamma(|d(s)| \varphi(|x(s, h, \xi, d_\mu)|)) \, ds \right\}
\]
for any \( \alpha \in \mathcal{K}_\infty \), and it holds that, for all \( |\mu| \leq 1 \),
\[
\gamma(|\xi|) \leq V_1(\xi) \leq \bar{\alpha}_2(|\xi|)
\]
for some \( \bar{\alpha}_2 \in \mathcal{K}_\infty \), and that for all \( |\mu| \leq 1 \),
\[
DV_1(\xi) f(\xi, \mu \varphi(|\xi|)) \leq -\rho(|\xi|) + \gamma(|\mu| \varphi(|\xi|))
\]
for all \( \xi \neq 0, \ |\mu| \leq 1 \). By Proposition 4.2 in [11], one sees that there exists a continuous function \( V_2 \), smooth on \( \mathbb{R}^n \setminus \{0\} \), such that
\[
\frac{\alpha(|\xi|)}{2} \leq V_2(\xi) \leq 2\bar{\alpha}_2(|\xi|), \quad \forall \xi \in \mathbb{R}^n,
\]
and
\[
DV_2(\xi) f(\xi, \mu \varphi(|\xi|)) \leq -\rho(|\xi|) + \gamma(|\mu| \varphi(|\xi|))
\]
for all \( \xi \neq 0, \ |\mu| \leq 1 \). By Proposition 4.2 in [11], one sees that there exists some smooth \( \mathcal{K}_\infty \)-function \( p \) such that \( p(s) > 0 \) for all \( s > 0 \) and \( p \circ V_2 \) is smooth everywhere. Without loss of
generality, one may assume that $p'(s) \leq 1$ for all $s > 0$. Otherwise, one may replace $p$ by any smooth $\mathcal{K}_\infty$-function $\tilde{p}$ with the property that $\tilde{p}(s) = p'(s)$ in a neighborhood of 0 where $p'(s) \leq 1$ and $\tilde{p}'(s) \leq 1$ everywhere else. Finally, we let $V = p \circ V_2$. Then $V$ satisfies (4) for some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, and

$$DV(\xi)f(\xi, \mu \varphi(\vert \xi \vert)) \leq -p'(V_2(\xi))\rho(\vert \xi \vert)/2 + p'(V_2(\xi))\gamma_1(\vert \mu \vert \varphi(\vert \xi \vert))$$

for all $\xi \in \mathbb{R}^n$, all $|\mu| \leq 1$, where $\alpha_3$ is any positive definite function with the property that $\alpha_3(\vert \xi \vert) \leq p'(V_2(\xi))\rho(\vert \xi \vert)/2$ (e.g., $\alpha_3(s) = p'(\alpha(s)/2)\rho(s)/2$). It then follows that

$$DV(\xi)f(\xi, \nu) \leq -\alpha_3(\vert \xi \vert) + \gamma_1(\vert \nu \vert)$$

for all $\xi \in \mathbb{R}^n$, all $|\nu| \leq \varphi(\vert \xi \vert)$. To show that $V$ satisfies an estimate of type (5), we let $\chi = \varphi^{-1}$, and let

$$\gamma(r) := \max_{|\xi| \leq \chi(|\nu|), |\nu| \leq 1}\{DV(\xi)f(\xi, \nu) + \alpha_3(\vert \nu \vert)\},$$

and

$$\gamma(r) = \max\{\gamma(r), \gamma_1(r)\}.$$ Then $\gamma \in \mathcal{K}$, and

$$DV(\xi)f(\xi, \nu) \leq -\alpha_3(\vert \xi \vert) + \gamma(\vert \nu \vert), \quad \forall \xi \in \mathbb{R}^n, \forall \nu \in \mathbb{R}^m$$

(consider the two cases: $|\xi| \leq \chi(|\nu|)$ and $|\xi| \geq \chi(|\nu|)$). This shows that $V$ is indeed an ISS-Lyapunov function for system (1).

Finally we show how to obtain the result without assuming that $\gamma$ is smooth. First of all, one may always assume that $\gamma(r) \geq r$ for all $r$. (Otherwise, replace $\gamma(r)$ by $\gamma(r) + r$.) Pick any $\mathcal{K}_\infty$-function $\theta$ such that $\theta(\sqrt{s})$ is smooth and $\theta(s) \leq \gamma^{-1}(s)$ for all $s \geq 0$. Consider the system

$$\dot{x}(t) = \tilde{f}(x(t), u(t)) := f(x(t), \sigma(u(t))\theta(|u(t)|)),$$  \hspace{1cm}  (22)

where $\sigma : \mathbb{R}^m \to \mathbb{R}$ is defined by $\sigma(\mu) = \frac{\mu}{|\mu|}$ if $|\mu| \neq 0$, and $\sigma(\mu) = 0$ if $|\mu| = 0$. Since $\theta$ is continuously differentiable and $\theta(0) = \theta'(0) = 0$, it follows that $\sigma(\nu)\theta(|\nu|)$ is $C^1$, and hence, $\tilde{f}$ is also a locally Lipschitz map. We let $x^u(t, \xi, u)$ denote the trajectory of this system corresponding to the initial state $\xi$ and the input $u$. It then holds that

$$\alpha\left(\left| z^u(t, \xi, u) \right| \right) \leq \beta(\vert \xi \vert, t) + \int_0^t \gamma(\vert \theta(u(s)) \vert) \, ds$$

for all $\xi$ and all $u$. Hence, the system $\dot{z} = \tilde{f}(x, u)$ is ISS with a smooth “gain”-function (which is the identity function). Applying the above proved result to this system, one sees that there exists a smooth ISS-Lyapunov function $V$ satisfying

$$DV(\xi)f(\xi, \sigma(\nu)\theta(|\nu|)) \leq -\alpha_3(\vert \xi \vert) + \gamma_2(|\nu|)$$

for some positive definite function $\alpha_3$ and some $\mathcal{K}$-function $\gamma_2$. Observe that

$$\nu = \sigma(\nu)\theta(\theta^{-1}(|\nu|))$$

for all $\nu \in \mathbb{R}^m$. Hence,

$$DV(\xi)f(\xi, \nu) \leq -\alpha_3(\vert \xi \vert) + \gamma_2(\theta^{-1}(|\nu|))$$

for all $\xi \in \mathbb{R}^n$, all $\nu \in \mathbb{R}^m$.  

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Proof of Proposition 2.5

To prove Proposition 2.5, we need the following result:

**Lemma 3.10** System (1) is 0-GAS if and only if there exist a smooth function \( V : \mathbb{R}^n \to \mathbb{R} \), \( \mathcal{K}_\infty \) functions \( \alpha_1, \alpha_2, \alpha_3 \), and \( \mathcal{K} \)-functions \( \lambda \) and \( \delta \), such that, for all \( \xi \in \mathbb{R}^n \) and \( \mu \in \mathbb{R}^n \),

\[
\alpha_1(|\xi|) \leq V(\xi) \leq \alpha_2(|\xi|),
\]

and

\[
DV(\xi)f(\xi, \mu) \leq -\alpha_3(|\xi|) + \lambda(|\xi|)\delta(|\mu|).
\]

**Proof.** Again one direction of the implication is easy to prove by the Lyapunov direct method applied to system (1) for \( u = 0 \). The reverse is more interesting. Assume (1) is 0-GAS. Then we have \( f(0, 0) = 0 \), and by a converse Lyapunov argument (see e.g. [11]), there exist a smooth \( V : \mathbb{R}^n \to \mathbb{R} \) and functions \( \alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty \) such that

\[
\alpha_1(|\xi|) \leq V(\xi) \leq \alpha_2(|\xi|)
\]

for all \( \xi \in \mathbb{R}^n \), and

\[
DV(\xi)f(\xi, 0) \leq -\alpha_3(|\xi|).
\]

Consider now the following function:

\[
\hat{\gamma}(r, s) := \max_{|\xi| \leq r, |\mu| \leq s} |f(\xi, \mu) - f(\xi, 0) - f(\xi, 0)|.
\]

Notice that \( \hat{\gamma}(r, s) \) is continuous, nondecreasing with respect to each argument, and vanishes for \( r = 0 \) or \( s = 0 \) (since \( f(0, 0) = 0 \)). Hence, it can be majorized by a function \( \gamma(r, s) \) separately of class \( \mathcal{K} \) (e.g., we can take \( \gamma(r, s) + r + s \)). We pick \( \sigma \) as in Corollary 3.5. Then, it follows from (26) and Corollary 3.5 that:

\[
DV(\xi)f(\xi, \mu) = DV(\xi)f(\xi, 0) + DV(\xi)[f(\xi, \mu) - f(\xi, 0)] \\
\leq -\alpha_3(|\xi|) + \gamma(|\xi|, |\mu|) + |DV(\xi)f(\xi, \mu)| \\
\leq -\alpha_3(|\xi|) + |DV(\xi)||\sigma(|\xi|)| + |DV(\xi)||f(0, \mu)|.
\]

It follows from (25) that \( \xi = 0 \) is a global minimum for \( V(\xi) \) and hence \( DV(0) = 0 \); so, since \( V \) is smooth, continuity of \( DV \) gives that

\[
\kappa(r) = r + \max_{|\xi| \leq r} |DV(\xi)|
\]

is a class \( \mathcal{K} \) function. By local Lipschitz continuity of \( f(\cdot, \cdot) \), also \( |f(0, \mu)| \leq \chi(|\mu|) \) for some \( \chi \in \mathcal{K} \).

Thus, recalling equations (28) and (29) we have,

\[
DV(\xi)f(\xi, \mu) \leq -\alpha_3(|\xi|) + \kappa(|\xi|)\sigma(|\xi|)|\mu| + \chi(|\xi|)|\mu| \leq -\alpha_3(|\xi|) + \lambda(|\xi|)\delta(|\mu|),
\]

with \( \lambda(r) = \kappa(r)\sigma(r) + \kappa(r) \) and \( \delta(r) = \sigma(r) + \chi(r) \).

**Remark 3.11** The same result can also be obtained along different lines, exploiting a result appeared in [16]. It is shown there that the 0-GAS property for system (1) implies the existence of an everywhere nonzero smooth function \( G(x) \) such that \( \dot{x} = f(x, G(x)v) \) is ISS with respect to \( v \). Then, the result follows from the Lyapunov characterization of input-to-state stability. □
We now can complete the proof of Proposition 2.5. Define \( \pi(\cdot) \) of class \( \mathcal{K} \) as follows:

\[
\pi(r) = \int_0^r \frac{ds}{1 + \chi(s)},
\]

with \( \chi \) a suitable class \( \mathcal{K} \) function to be defined later. It follows from 0-GAS that there exists a smooth \( V(\xi) \) as in Lemma 3.10. Composing \( \pi \) with \( V \) and taking derivatives yields:

\[
D[(\pi \circ V)(\xi)] f(\xi, \mu) = \frac{D V(\xi) f(\xi, \mu)}{1 + \chi(V(\xi))} \leq \frac{-\alpha_2(|\xi|)}{1 + \chi(V(\xi))} + \frac{\lambda(|\xi|)\delta(\mu)}{1 + \chi(V(\xi))}.
\]

Then, letting \( \chi(r) = \lambda \circ \alpha_1^{-1}, \ W = \pi \circ V, \) and recalling (23) we obtain:

\[
D W(\xi) f(\xi, \mu) \leq -\rho(|\xi|) + \delta(\mu),
\]

where \( \rho \) is the positive definite function defined as

\[
\frac{\alpha_2(r)}{1 + \lambda(\alpha_1^{-1}(\alpha_2(r)))}.
\]

4 A Counter-Example

As already remarked in section 2, \( \text{IASS} \) implies 0-GAS. The converse is easily seen to be false, taking any 0-GAS system that exhibits a finite escape time for some constant input signal \( u \neq 0 \). In fact, it follows by definition (2) that IASS implies forward completeness of the control system (1), viz., for any control \( u \) and any \( \xi \in \mathbb{R}^n \), the unique maximal solution of the initial value problem \( \dot{x} = f(x, u), \ x(0) = \xi \), is defined over the interval \([0, +\infty)\). It is reasonable to conjecture that IASS might be equivalent to simply forward completeness plus 0-GAS. This would make the IASS concept less interesting. In this section, we provide a counter-example to this conjecture, exhibiting a system that is forward complete and 0-GAS, but is not IASS. In other words, this example shows that, even when restricting attention to forward complete systems, IASS is a strictly stronger property than 0-GAS.

We begin the construction with a differential equation

\[
\dot{x} = f(x)
\]

which evolves in \( \mathbb{R}^2 \) and satisfies:

1. it is GAS;
2. \( |f(x)| \leq 1 \) for all \( x \in \mathbb{R}^2 \); and
3. there is a sequence of states

\[
x^0, x^1, x^2, x^3, \ldots
\]

so that \( x(T_k, x^k) = z^k \) for each \( k \), for some sequence of positive numbers \( \{T_k\} \), (where \( x(\cdot, p) \) denotes the trajectory of the system with initial value \( p \)) and

\[
x^k = \begin{pmatrix} 4k \\ * \end{pmatrix}, \quad z^k = \begin{pmatrix} 4k + 3 \\ * \end{pmatrix}
\]

for each \( k \), where \( "*" \) is arbitrary.

It is easy to construct such differential equations. For example, one may start with a linear system \( \dot{x} = Ax \), having an \( A \) matrix which is Hurwitz with non-real eigenvalues, and constructed so that its orbits are clockwise-turning converging spirals. One then scales the equation so that sequences of points as claimed exist, and finally one divides \( Ax \) by \( 1 + |Ax|^2 \) in order to guarantee that \( |f(x)| \leq 1 \) for all \( x \).

Let us denote \( \Delta_k := \max \{|z^{i+1} - z^i| : 0 \leq i \leq k\}, \ k = 0, 1, 2, \ldots \), and pick any scalar function \( \varphi : \mathbb{R} \to \mathbb{R} \) which satisfies the following properties:

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1. \( \varphi(r) = 0 \) when \( r \in [4k+1, 4k+2] \), for all \( k = 0, 1, 2, \ldots \).
2. \( \varphi(r) = 2^k \Delta_k + 1 \) when \( r \in [4k+3, 4(k+1)] \), for all \( k = 0, 1, 2, \ldots \).
3. \( 0 \leq \varphi(r) \leq 2^k \Delta_k + 1 \) if \( r \leq 4(k+1) \), for all \( k = 0, 1, 2, \ldots \).
4. \( \varphi(r) = 0 \) if \( r \leq 0 \).

The hypotheses imply that \( \varphi(r) \leq M(r) \) for all \( r \), where \( M \) is some increasing function which is zero for negative \( r \) (all we need, for \( r > 0 \), is \( M(r) \geq 2^k \Delta_k + 1 \), where \( k \) is the least positive integer so that \( r < 4(k+1) \)). See Fig. 1 for an illustration of the orbits of \( f \) and \( \varphi \).

Now we let \( G(x) = \varphi(x_1)I \) (that is, \( G \) depends only on the first coordinate) and consider the two-input system \( \dot{x} = f(x) + G(x)u \). We will show that this system is complete but is not ISS; it is 0-GAS by construction.

Claim: This system \( \dot{x} = f(x) + G(x)u \) is complete.

Proof: Let \( x(\cdot) \) be a maximal trajectory corresponding to a given control \( u \) and initial condition \( x(0) \), and suppose that \( x \) is defined on an interval \([0, T] \), with \( T < \infty \). Let \( K \) be an upper bound on the supremum norm of \( u \) (controls are locally essentially bounded, by definition). We will show that the trajectory is bounded, thus contradicting \( T < \infty \). We look at the first coordinates \( x_1 \) of the states along this trajectory. There are two cases to consider:

1. \( \{x_1(t), t \in [0, T] \} \) is bounded above.
   Suppose that \( x_1(t) \leq L \) for all \( t \). Then, since \( |G(x)| = \varphi(x_1) \leq M(x_1) \) for all \( x \), the velocities are bounded by \( 1 + M(L)K \) and so the trajectory stays in the ball about \( x(0) \) of radius \((1 + M(L)K)T \).
2. \( \{x_1(t), t \in [0, T] \} \) is not bounded above.
   There must be an infinite number of intervals of the form \([4k+1, 4k+2] \) which are traversed by \( x_1(t) \). That is, there are a countable set of disjoint closed intervals \( J_1, J_2, \ldots \) included in \([0, T] \), each \( J_i = [s_i, t_i] \), so that \( x_1(s_i) = 4k_i + 1 \) and \( x_1(t_i) = 4k_i + 2 \), for some \( k_i \)'s. We claim that every \( J_i \) has length at least one, which then contradicts \( T < \infty \). Indeed, take any interval \( J_i \) and suppose that \( t_i - s_i < 1 \). Observe that, on \( J_i \), \( \varphi \equiv 0 \), so the system equations are \( \dot{x} = f(x) \). By the Mean Value Theorem,
   \[
   1 \leq |x(t_i) - x(s_i)| \leq |f(x(t))|(t_i - s_i) < 1, 
   \]
a contradiction. This completes the proof of completeness.

Claim: This system is not ISS.

Proof: We will show that there is an initial state \( \xi \) and a control \( u \) so that \( |x(t, \xi, u)| \to \infty \) as \( t \to \infty \), where \( u \) has the following property: there is some increasing sequence \( t_k \to +\infty \) so that
   \[
   |u(t)| \leq 1 \text{ if } t \in [t_k, t_k + 2^{-k}] 
   \]
and \( u(t) = 0 \) otherwise; moreover, the sequence is also assumed to satisfy \( t_{k+1} - t_k > 2^{-k} \).

The existence of such \( \xi, u \) means that the system cannot be ISS. To see this, suppose that the system would be ISS. Then, for some \( \alpha, \gamma \in \mathcal{K} \), it holds that
   \[
   \limsup_{t \to \infty} \alpha(|x(t, \xi, u)|) \leq \int_0^\infty \gamma(|u(s)|) \, ds \leq \sum_{k=0}^{\infty} \int_{[t_k, t_k + 2^{-k}]} \gamma(1) \, ds = 2\gamma(1) < \infty,
   \]
but this contradicts the fact that \( |x(t, \xi, u)| \to \infty \) as \( t \to \infty \).

We start with \( \xi = x^1 \), and let \( u \equiv 0 \) on \([0, t_1] \), where \( t_1 := T_1 \). So, \( x(t_1, \xi, u) = x^1 \), and \( t_1 \geq 1 \) because of the assumption that \( |f(x)| \leq 1 \). Next we continue building the control \( u \) and the trajectory \( x(\cdot) = x(\cdot, \xi, u) \), inductively on the intervals
   \[
   [t_k, t_{k+1}].
   \]
We do the construction in such a fashion that
   \[
   x(t_k) = z_k 
   \]
and
\[ x(t_k + 2^{-k}) = x^{k+1} \]
for \( k = 1, 2, \ldots \). The idea is to switch between an uncontrolled motion on every interval of the form \([t_k + 2^{-k}, t_{k+1}]\) and appropriate control motions on the small intervals. To clarify the construction, we first do separately the case \( k = 1 \).

The control on the interval \([t_1, t_1 + 1/2]\) is defined as follows. We wish to force
\[ x(t) = x^1 + 2(t - t_1) (x^2 - x^1), \quad t \in [t_1, t_1 + 1/2] \]
so that we go from \( x(t_1) = x^1 \) to \( x(t_1 + 1/2) = x^2 \) along a straight line. The equation \( \dot{x} = f + Gu \) along this line is:
\[ 2(x^2 - x^1) - f(x(t)) = G(x(t))u(t) = (2\Delta_1 + 1) u(t) \]
so we may let
\[ u(t) := \frac{2(x^2 - x^1) - f(x^1 + 2(t - t_1)(x^2 - x^1))}{2\Delta_1 + 1}. \]
Since \( |x^2 - x^1| \leq \Delta_1 \) (by definition of the \( \Delta_k \)'s) and \( |f(x(t))| \leq 1 \) for all \( t \), we conclude that \( |u(t)| \leq 1 \) on \([t_1, t_1 + 1/2]\), as desired. Finally, we let \( u(t) = 0 \) for \( t \in [t_1 + 1/2, t_2] \), where \( t_2 := T_2 + (t_1 + 1/2) \), which makes \( x(t_2) = x^2 \).

Now we do the case of arbitrary \( k \). We pick the curve:
\[ x(t) = x^k + 2^k(t - t_k)(x^{k+1} - x^k), \quad t \in [t_k, t_k + 2^{-k}] \]
so that we go from \( x(t_k) = x^k \) to \( x(t_k + 2^{-k}) = x^{k+1} \) along a line, and the equation along this line becomes:
\[ 2^k (x^{k+1} - x^k) - f(x(t)) = G(x(t))u(t) = (2^k \Delta_k + 1) u(t) \]
so we may let
\[ u(t) := \frac{2^k (x^{k+1} - x^k) - f(x^k + 2^k(t - t_k)(x^{k+1} - x^k))}{(2^k \Delta_k + 1)}. \]
Since \( |x^{k+1} - x^k| \leq \Delta_k \) and \( |f(x(t))| \leq 1 \) for all \( t \), we have \( |u(t)| \leq 1 \) for all \( t \) in this interval of length \( 2^{-k} \). Finally, we let
\[ u(t) = 0 \quad \text{for} \quad t \in [t_k + 2^{-k}, t_{k+1}], \]
where \( t_{k+1} := T_{k+1} + (t_k + 1/2^n) \), so that we have \( x(t_{k+1}) = x^{k+1} \), as needed for the induction step. \( \blacksquare \)

5 An Example

In this section we provide an example of a tracking design for a robotic system which, when tracking signals are seen as inputs, is ISS. One interesting feature of this example, which in fact motivated much of the research reported here, is that it illustrates the use of the LaSalle-type condition that we obtained. Another noteworthy fact is that the system appears to not be ISS, in fact, bounded inputs may give rise to unbounded trajectories. The same example was used, for a different purpose (namely, to illustrate a different nonlinear tracking design which produces ISS, as opposed to merely ISS, behavior) in the paper \cite{1}.

Consider the manipulator shown in Fig. 2. A simple model is obtained considering the arm as a segment with mass \( M \) and length \( L \), and the hand as a material point with mass \( m \). If we denote with \( r \) the position of the hand and with \( \theta \) the angle of the arm, the equations for such a system are:
\[ (mr^2 + ML^2/3) \ddot{\theta} + 2mr\dot{r}\dot{\theta} = \tau \]
\[ mr^2 - m\dot{\theta}^2 = F, \]
where $F$ and $\tau$ indicate external torques. We now study the closed-loop system which is obtained by choosing $\tau$ and $F$ as:

$$
\tau = -k_d \dot{\theta} - k_p (\theta - \theta_d) \quad F = -k_d \dot{r} - k_p (r - r_d),
$$

with $k_p, k_d, k_{\theta}, k_r > 0$. This closed-loop system is given by:

$$
\begin{align*}
(m r^2 + M L^2/3) \ddot{\theta} + 2 m r \dot{r} \dot{\theta} &= -k_d \dot{\theta} - k_p (\theta - \theta_d) \\
(m r^2 - m r \dot{\theta})^2 &= -k_d \dot{r} - k_p (r - r_d).
\end{align*}
$$

(35)

For notational simplicity, we write $q = [\theta, r]^T$, and denote $\dot{q}$ by $z$.

This represents a typical passivity-based tracking design, when we think of $r_d$ and $\theta_d$ as signals to be followed by $r$ and $\theta$. Normally, one establishes tracking behavior, as well as the closed-loop stability of the system when the reference signal $u = (u_1, u_2) := q_d = (\theta_d, r_d)$ is constant (or, in particular, zero). For such signals, one obtains $q' \to 0$ and $q \to q_d$. In the spirit of input-to-state stability, however, it is natural to ask what happens with the full state when the reference signal is time-varying. We will now prove that this system is ISS, when $u$ is seen as the input, but it is not (or we think so, based on numerical simulations) ISS.

To fit the rest of the paper, we can of course represent the system as a 4-dimensional one in the following state space form:

$$
\begin{align*}
\dot{q}_1 &= z_1, \\
\dot{q}_2 &= z_2, \\
\dot{z}_1 &= \frac{-2 m q_2 z_2 - k_p q_1 - k_d z_1}{m r^2 + M L^2/3} + \frac{k_p u_1}{m r^2 + M L^2/3}, \\
\dot{z}_2 &= q_2 z_1^2 - \frac{k_p q_2}{m} - \frac{k_d z_2}{m} + \frac{k_p u_2}{m}.
\end{align*}
$$

(36)

(To be precise, to be able to apply the results in this paper, we must think of the state-space as $\mathbb{R}^4$, although a more natural state-space would consider the angle $\theta$ as an element of a unit circle.)

To prove the ISS property, we introduce, as usual for mechanical manipulators, the following matrix notation:

$$
H(q) = \begin{bmatrix} m r^2 + M L^2/3 & 0 \\ 0 & m \end{bmatrix}, \quad C(q, \dot{q}) = \begin{bmatrix} m r \dot{r} & m r \dot{\theta} \\ -m r \dot{\theta} & 0 \end{bmatrix},
$$

where $H(q)$ is the inertia matrix, and $C(q, \dot{q})$ expresses the Coriolis torques. Then (35) can be rewritten as

$$
H(q) \dot{q} + C(q, \dot{q}) \dot{q} = -K_p (q - q_d) - K_d q,
$$

where $K_p = \text{diag}(k_p, k_p)$, $K_d = \text{diag}(k_d, k_d)$, and $q_d = [\theta_d, r_d]^T$. We take the mechanical energy of the system as a candidate Lyapunov function:

$$
V(q, z) = \frac{1}{2} z^T H(q) z + \frac{1}{2} q^T K_p q = \frac{1}{2} q^T H(q) q + \frac{1}{2} q^T K_p q.
$$

(37)

Taking derivatives in (37) with respect to time along trajectories of (35) yields the following passivity-type estimate:

$$
\frac{d}{dt} V(q(t), z(t)) = \dot{q}(t)^T H(q(t)) \dot{q}(t) + \frac{1}{2} \dot{q}(t)^T \left[ \frac{d}{dt} H(q(t)) \right] \dot{q}(t) + q(t)^T K_p \dot{q}(t)
$$

$$
= -\dot{q}(t)^T K_d \dot{q}(t) + \dot{q}(t)^T K_p q_d(t)
$$

$$
\leq -c_1 |\dot{q}(t)|^2 + c_2 |q_d(t)|^2 = -c_1 |z(t)|^2 + c_2 |u(t)|^2,
$$

(38)

for some sufficiently small number $c_1 > 0$ and some sufficiently large number $c_2 > 0$. Inspection of the equations shows that, when $u \equiv 0$ and $z \equiv 0$, necessarily $q \equiv 0$ as well. Thus, thinking of $z$ as
an output, the system is weakly zero-detectable and dissipative; applying Theorem 1, one concludes that the system is ISS.

We believe that this system is not ISS. We have not proved this fact rigorously, but we have performed simulations which make that fact quite apparent. Specifically, we show in Fig. 3 the r component of a certain solution which corresponds to a certain bounded input and a certain initial state. This component is not bounded, contradicting the fact that an ISS system must have bounded-input bounded-state behavior. To obtain this trajectory, we did as follows. We started with the initial state \((0, 0, 1, 0, 0, 1)^T\), and took the feedback control \(u_1 = 3\tanh(z_1) (= 3\tanh(\theta))\) and \(u_2 = 0\). Since \(|\tanh(x)| \leq 1\) for all \(x \in \mathbb{R}\), the input signal, resulting from this destabilizing feedback, shown in Fig. 4, is bounded. Note that a sort of “non-linear resonant behavior” is obtained. (It is worth pointing out that a similar effect is met also for vanishing references, if the convergence of \(u_1\) to 0 is sufficiently slow.) The simulation used the parameters values shown in the next table, and were obtained using the ode23 MATLAB routine, with tolerance 0.001 and initial condition \([0, 0.1, 0, 0.1]^T\).

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(m)</td>
<td>1</td>
<td>(ML^2)</td>
<td>3</td>
</tr>
<tr>
<td>(k_{p_1})</td>
<td>2</td>
<td>(k_d)</td>
<td>2</td>
</tr>
<tr>
<td>(k_{p_2})</td>
<td>1</td>
<td>(k_{d_2})</td>
<td>1</td>
</tr>
</tbody>
</table>

6 Comments on Related Notions

The notion of ISS differs from ISS in its use of \(\int \gamma(|u|)\) instead of \(\sup \gamma(|u|)\). The same substitution may be used to define analogues of input/output stability and of detectability notions. We briefly discuss some of these now. Reasons of space preclude a detailed discussion, but proofs of the various claims are not difficult to obtain by following steps like those used in the rest of the paper. In this section, we deal with systems with outputs

\[
\dot{x} = f(x, u), \quad y = h(x),
\]

where, as earlier, the output map \(h : \mathbb{R}^n \to \mathbb{R}^p\) is assumed to be continuous and \(h(0) = 0\). For each \(\xi \in \mathbb{R}^n\) and each input \(u\), we let \(y(t, \xi, u)\) be the output function of the system, i.e., \(y(t, \xi, u) = h(x(t, \xi, u))\) (defined on some maximal interval \([0, T_{\xi, u})\).

Consider the following type of estimation:

\[
\alpha(|x(t, \xi, u)|) \leq \beta(|\xi|, t) + \int_0^t \gamma_1(|y(s, \xi, u)|) ds + \int_0^t \gamma_2(|u(s)|) ds
\]

for all \(t \in [0, T_{\xi, u})\), where \(\beta \in \mathcal{KL}, \alpha \in \mathcal{K}_{\infty}\), and \(\gamma_1, \gamma_2 \in \mathcal{K}\). If (40) holds for every trajectory of (39), then we say that system (39) is integral input-output-to-state stable (iISS). This is a notion of detectability: inputs and outputs are “small” implies that states are also small; see [21] for the corresponding notion of (sup-norm) iISS. We say that a smooth function \(V\) is an iISS-Lyapunov function for system (39) if there exist functions \(\alpha_1, \alpha_2 \in \mathcal{K}_{\infty}, \sigma_1, \sigma_2 \in \mathcal{K}\), and a positive definite function \(\alpha_3\), such that

\[
\alpha_1(|\xi|) \leq V(\xi) \leq \alpha_2(|\xi|)
\]

for all \(\xi \in \mathbb{R}^n\), and

\[
DV(\xi)f(\xi, \mu) \leq -\alpha_3(|\xi|) + \sigma_1(|h(\xi)|) + \sigma_2(|\mu|)
\]

for all \(\xi \in \mathbb{R}^n\) and all \(\mu \in \mathbb{R}^m\). A proof analogous to that of Theorem 1, by virtue of Corollary 3.3, shows the following: If a system admits an iISS-Lyapunov function, then the system is iISS.

The area of input-to-output (as opposed to input-to-state) stability deals with properties which may be described, informally, as “small inputs produce small outputs.” Such properties appear naturally in regulation problems. In particular, one may define a concept of iOS (input-to-output stability), see [17] and [22]. This is yet another obvious candidate for the replacement of sup norms
by integrals. So let us call system (39) integral input-to-output stable (I2OS) if there exist \( \alpha \in \mathcal{K}_\infty \), \( \beta \in \mathcal{K} \), and \( \gamma \in \mathcal{K} \) such that

\[
\alpha(|y(t, \xi, u)|) \leq \beta(|\xi|, t) + \int_0^t \gamma(|u(s)|) \, ds
\]

(43)

holds on the maximal interval \([0, T_{\xi, u}]\) for every trajectory of the system. Correspondingly, we may define a Lyapunov function as follows. A smooth function \( V \) is called an I2OS-Lyapunov function for system (39) if there exist functions \( \alpha_1, \alpha_2 \in \mathcal{K}_\infty \), \( \sigma \in \mathcal{K} \), and a positive definite function \( \alpha_3 \), such that

\[
\alpha_3(|h(\xi)|) \leq V(\xi) \leq \alpha_2(|\xi|)
\]

(44)

for all \( \xi \in \mathbb{R}^n \), and

\[
DV(\xi)f(\xi, \mu) \leq -\alpha_3(V(\xi)) + \sigma(|\mu|)
\]

(45)

for all \( \xi \in \mathbb{R}^n \) and all \( \mu \in \mathbb{R}^m \). Again, by virtue of Corollary 3.3, one can prove: If a system admits an I2OS-Lyapunov function, then the system is I2OS.

Note that the difference between (4) and (44) is that the function \( V \) in (4) is proper (i.e., radially unbounded), while in (44) the function \( V \) only majorizes a \( \mathcal{K}_\infty \)-function of \(|y|\). It is also interesting to consider the following type of condition on \( V \):

\[
\alpha_1(|h(\xi)|) \leq V(\xi) \leq \alpha_2(|h(\xi)|)
\]

(46)

for some \( \mathcal{K}_\infty \)-function \( \alpha_1, \alpha_2 \). It then follows, again appealing to Corollary 3.3, that if system (39) admits a \( V \) satisfying (46) and (45), then the following holds for all trajectories of the system:

\[
\alpha(|y(t, \xi, u)|) \leq \beta(|y_0|, t) + \int_0^t \gamma(|u(s)|) \, ds, \quad \forall t \in [0, T_{\xi, u}),
\]

for some \( \alpha \in \mathcal{K}_\infty, \beta \in \mathcal{K} \) and \( \gamma \in \mathcal{K} \), where \( y_0 = y(0, \xi, u) \).

7 Appendix

7.1 Proof of Lemma 3.1

Assume without loss of generality that \( \rho(r) \to 0 \) as \( r \to +\infty \) (otherwise one can always consider \( \bar{\rho}(r) = \min\{\rho(r), 1/(1 + r)\} \)). Then the function \( \rho \) admits a global maximum over the interval \([0, +\infty)\). Let \( M = \max_{r \geq 0} \rho(r) \), and define \( \bar{\rho}(r) = \rho(r)/M \). Pick now \( r_M > 0 \) such that \( \bar{\rho}(r_M) = 1 \). Then, we can define the following functions:

\[
\beta_1(r) = \begin{cases} 
\min_{r \leq s \leq r_M} \bar{\rho}(s) & \text{for } r \leq r_M, \\
1 & \text{for } r > r_M,
\end{cases}
\]

\[
\beta_2(r) = \begin{cases} 
\min_{r_M \leq s \leq r} \bar{\rho}(s) & \text{for } r < r_M, \\
1 & \text{for } r \geq r_M.
\end{cases}
\]

(47)

Notice that \( \beta_1, (\beta_2) \) is a non-decreasing (non-increasing) function, and, by equation (47), considering separately the cases \( r > r_M \) and \( r \leq r_M \),

\[
\beta(r) \geq \beta_1(r) \beta_2(r).
\]

(48)

Then, we can choose \( \rho_1 \) and \( \rho_2 \) according to:

\[
\rho_1(r) = M \beta_1(r) r \quad \rho_2(r) = \bar{\rho}(r)/(1 + r).
\]

Notice that \( \rho_1 \in \mathcal{K}_\infty \) and \( \rho_2 \in \mathcal{L} \). Taking into account (48) we have:

\[
\rho(r) = M \beta(r) \geq M \rho_1(r) \rho_2(r) \geq \rho_1(r) \rho_2(r)
\]

(49)

for all \( r \geq 0 \).
7.2 Proof of Proposition 3.4

Proposition 3.4 will follow from a sequence of Lemmas.

**Lemma 7.1** Let $c$ be as in the statement of Proposition 3.4. Then, there exists a function $g \in \mathcal{N}$ such that, for each $y \in \mathbb{R}$,

$$c(x, y) - g(x) \to -\infty \text{ as } |x| \to \infty. \quad (50)$$

**Proof.** We will assume without loss of generality that $c(0, 0) > 0$. (If $c(0, 0) < 0$, we simply pick any constant $a$ so that $a < c(0, 0)$, apply the Lemma to $c' := c - a$ to obtain some function $g'$, and then let $g := g' + a$.) Let

$$\tilde{x} := \sup \left\{ x \mid c(x, 0) = 0 \right\}.$$

(Note that, since $c(\cdot, 0)$ is continuous, unbounded below, and achieves the some positive value, there is indeed at least one $x$ so that $c(x, 0) = 0$; and by continuity, $c(\tilde{x}, 0) = 0$). Now introduce the following set:

$$\mathcal{G} := \left\{ (x, y) \in \mathbb{R}^2 \mid y \geq 0, c(x, y) + y = 0 \right\}.$$

**Claim:** $\mathcal{G}$ is the graph of a continuous, nonincreasing, onto function

$$g_0 : (-\infty, \tilde{x}] \to [0, \infty).$$

To establish this claim, we first prove that, if $x_2 \leq x_1$ and $(x_1, y_1) \in \mathcal{G}$, then there is a $y_2$ so that $(x_2, y_2) \in \mathcal{G}$, and any such $y_2$ must satisfy $y_1 \leq y_2$. Consider the function $C(y) := c(x_2, y) + y$. As

$$C(y_1) = c(x_2, y_1) + y_1 \leq c(x_1, y_1) + y_1 = 0$$

and $C(y) \to +\infty$ as $y \to +\infty$ (because $c(x_2, \cdot)$ is nondecreasing), we conclude, using continuity of $C$, that there is some $y_2$ so that $C(y_2) = 0$, as required. And, given any $y_2$ so that $(x_2, y_2) \in \mathcal{G}$, if it were the case that $y_1 > y_2$ then it would hold that

$$0 = c(x_1, y_1) + y_1 > c(x_2, y_2) + y_2 = 0,$$

a contradiction. Thus, as stated, $y_1 \leq y_2$.

In particular, it follows that if $(x, y_1)$ and $(x, y_2)$ are both in $\mathcal{G}$ then necessarily $y_1 = y_2$ (apply with $x_1 = x_2 = x$), so $\mathcal{G}$ is the graph of some function $g_0$, and $g_0$ is nonincreasing.

Next, we note that the projection of $\mathcal{G}$ on the $x$ coordinate (that is, the domain of the function $g_0$) is $(-\infty, \tilde{x}]$. Pick any $(x, y) \in \mathcal{G}$. Suppose that $x > \tilde{x}$. Then, $c(\tilde{x}, 0) \leq c(x, 0) \leq c(x, y) + y = 0 = c(\tilde{x}, 0)$ (using that $c$ is nondecreasing in each variable, and $y \geq 0$). Then, $c(x, 0) = 0$, so $x \leq \tilde{x}$ by maximality of $\tilde{x}$, a contradiction. Thus, $x \leq \tilde{x}$. Conversely, given any $x \leq \tilde{x}$, we may again apply the argument given earlier (now with $x_2 = x$ and $x_1 = \tilde{x}$) to obtain that there is some $y$ so that $(x, y) \in \mathcal{G}$.

The projection of $\mathcal{G}$ on the $y$ coordinate is $[0, \infty)$. Indeed, pick any $y \geq 0$; as $c(\tilde{x}, y) + y \geq c(\tilde{x}, 0) = 0$ and $c(\cdot, y)$ is continuous and unbounded below, there is some $x$ so that $c(x, y) + y = 0$.

To complete the proof of the claim, we need to see that $g_0$ is continuous. But this is an immediate consequence of the fact that $g_0$ is monotonic and onto an interval.

Finally, we define

$$g(x) := \begin{cases} \frac{-1}{2} g_0(x) & \text{if } x \leq \tilde{x} \\ c(x, x) - c(\tilde{x}, \tilde{x}) + x - \tilde{x} & \text{if } x > \tilde{x}. \end{cases}$$

By construction, $g \in \mathcal{N}$. We show the desired limit property. Pick any $y \in \mathbb{R}$.

For $x \geq \max\{\tilde{x}, y\}$,

$$c(x, y) - g(x) = c(x, y) - c(x, x) - x + c \leq c - x$$

20
where $c := \bar{c} + c(\bar{x}, \bar{x})$, so $c(x, y) - g(x) \to -\infty$ as $x \to +\infty$.

On the other hand, for any $x \leq \bar{x}$ such that $g_0(x) > y$,
\[
c(x, y) - g(x) = c(x, y) + \frac{1}{2} g_0(x) \leq c(x, g_0(x)) + g_0(x) - \frac{1}{2} g_0(x) = -\frac{1}{2} g_0(x)
\]
and $-\frac{1}{2} g_0(x) \to -\infty$ as $x \to -\infty$.

We now complete the proof of Proposition 3.4. Let $g$ be as in Lemma 7.1, and define the following function $h : \mathbb{R} \to \mathbb{R}$:
\[
h(y) := \sup_{x \in \mathbb{R}} [c(x, y) - g(x)].
\]
(As $c(\cdot, y) - g$ is continuous, and is negative for large $|x|$, the supremum is indeed finite.)

Since $h$ is the sup of a family $\{c(x, \cdot) - g(x), x \in \mathbb{R}\}$ of continuous functions, $h$ is itself continuous, and since each member of this family is nondecreasing, $h$ is also nondecreasing. We prove now that $h(y) \to -\infty$ when $y \to -\infty$, which will then allow us to conclude that $h \in \mathcal{N}$.

Pick any $K \in \mathbb{R}$. For this $K$, we pick a $\rho > 0$ so that $c(x, 0) - g(x) < K$ whenever $|x| > \rho$. Next we pick an $L \leq 0$ so that $c(\rho, L) < K + g(-\rho)$. (Such an $L$ exists because $c(\rho, \cdot)$ is unbounded below.) We claim that
\[
y < L \Rightarrow h(y) < K.
\]
Take one such $y$, and any $x \in \mathbb{R}$; we need to see that $c(x, y) - g(x) < K$. Consider first the case $|x| \leq \rho$; then
\[
c(x, y) - g(x) \leq c(\rho, L) - g(-\rho) < K.
\]
If instead $|x| > \rho$, then also
\[
c(x, y) - g(x) \leq c(x, 0) - g(x) < K
\]
(assuming that $y < L \leq 0$).

So we have constructed $g$ and $h$ in $\mathcal{N}$ such that
\[
c(x + y) \leq g(x) + h(y) \leq \max\{g(x), h(x)\} + \max\{g(y), h(y)\}
\]
for all $x$ and $y$. Thus, $k := \max\{g, h\}$ is as wanted for Proposition 3.4.

References


Figure 1: Flow $f$ and function $\varphi$ for example of forward complete, 0-GAS but not ISS system
Figure 2: Example of a non-IS stable robot

Figure 3: Non-linear resonance
Figure 4: Input signal