A new notion of input-to-state stability involving infinity norms of input derivatives up to a finite order $k$ is introduced and characterized. We show by examples that this notion of stability is indeed weaker than the usual iss. Applications to the study of global asymptotic stability of cascaded nonlinear systems are discussed.

1 Introduction

The study of stability robustness with respect to exogenous input disturbances is a fundamental issue in control theory. For nonlinear systems, input to state stability (iss) and integral iss, as well as $\mathcal{H}_\infty$ theory, have proven to be powerful tools in order to tackle problems of robustness analysis and control synthesis, see for instance [2,4,14,20].

A “disturbance” is in the iss-related literature a measurable function (locally essentially bounded). Such a choice represents, of course, an extremely rich set of possible input perturbations, and is well suited to model gaussian and random noises, as well as constant or periodic signals, slow parameter drift, and so on. While, on the one hand, this makes the notion of iss extremely powerful, on the other hand it is known that iss might sometimes be too strong a requirement to be satisfied for feedback control systems, as illustrated by the problem discussed in [1]. In the output regulation literature [6], by contrast, the focus is on often “deterministic” disturbances, i.e., signals that can be generated by a finite dimensional nonlinear systems whose the state evolves in a neighborhood of a neutrally stable equilibrium. This is an extremely interesting class of persistent disturbances for which, roughly

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Research supported in part by US Air Force Grant F49620-98-1-0242
Research supported in part by NSF Grant DMS-9457826
speaking, the following is true:

\[ \|d\|_\infty \text{ small } \Rightarrow \|\dot{d}\|_\infty \text{ and derivatives of arbitrary order are also small.} \]

The classical definition of input-to-state stability completely disregards such additional information. Tracking of output references, see [11], is another area where “derivative” knowledge is usually disregarded (the analysis is often performed only taking into account constant setpoints), whereas such an information could be exploited to get tighter estimates for the steady-state tracking error due to time-varying, smooth reference signals.

Another role of “disturbances” in feedback system analysis arises when controlling systems which depend on parameters, as in adaptive control. If the design has been made under the assumption of constant parameters, it is the information contained on derivatives of parameters, not the values themselves, which may lead to instability under even moderate parameter drift. Indeed, for this reason the study of systems with slowly varying parameters has long been an important subject, see e.g. [8] and [9]. The analysis of such a system is usually carried out by first considering the family of systems corresponding to “frozen” parameters. If, for all the fixed parameters, the corresponding frozen systems posses certain stability properties uniformly, then it is reasonable to expect that the system with slowly varying parameters will posses the same properties. See, for instance, [9] for a result of this type. A more general question is how the magnitudes of the time derivatives of the time varying parameters affect the behavior of the systems.

The main purpose of this note is to remark how, in the context of ISS, stability notions can be adjusted in order to take into account of robustness with respect to disturbances and their time derivatives. A new, but totally natural, notion of $D^k$ISS is defined through an ISS-like estimate which involves the magnitudes of the inputs and of their derivatives up to the $k$-th order. It is worth mentioning that in a seemingly different context [12,13], namely in the study of ISS for time-varying systems by means of averaging techniques, similar notions of stability were already introduced. The basic idea in [12] is to investigate stability properties of a time-varying systems by addressing the simpler problem of ISS for a time-invariant system that can be seen as the “average” in time of the original vector field. Unlike the case of autonomous systems, where a unique notion of average is usually taken into account, for systems with inputs it is useful to define averages in a weak or strong sense. It turns out that, if only a weak average is avaible for a given nonlinear system, and the averaged system is ISS, then the original system enjoys, for sufficiently fast time scalings, a semiglobal (and practical) version of the $DISS$ property. In other words, for each neighborhood of the origin $N$, each set of initial conditions and each bound on $\|u\|$ and $\|\dot{u}\|$ it is possible to choose a sufficiently fast time-scale, so that a practical ISS estimate holds, viz. $|x(t,t_0,\xi,u)| \leq \max\{\beta(|\xi|,t-t_0),\gamma(\|u\|)\} + c$ where $c = \sup_{s \in N} |s|$.

We also propose several variations of the $D^k$ISS notion, which all serve to formalize the idea of “stable” dependence upon the inputs and their time derivatives. These properties differ in the formulation of the decay estimates which specify how the magnitudes of input derivatives affect state behavior. We illustrate by a set of interesting examples how these properties differ from each other and from the standard ISS property.
We provide equivalent Lyapunov characterizations for these properties. Interestingly enough, one of our Lyapunov formulations already appeared in [8] (see formula (5) in that reference). In this work, we provide a stability property that is equivalent to the existence of such a type of Lyapunov function. Finally, we present an application of the newly introduced notion to the analysis of cascaded nonlinear systems. The well known result that cascading preserves the iss property is generalized to the $D^k\text{iss}$ property.

The paper is organized as follows: Section 2 provides the basic definitions. Section 3 contains the main results and Lyapunov characterizations of the $D^k\text{iss}$ property and some other related properties. Sections 4 and 5 are devoted to the study of cascaded systems. Sections 6 and 7 provide examples of systems which illustrate the newly introduced stability notions.

2 Basic Definitions

Consider finite-dimensional nonlinear systems of the following form

$$\dot{x}(t) = f(x(t), u(t)), \quad (1)$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ for each $t \geq 0$. The function $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is locally Lipschitz continuous. Thus, for any measurable, locally essentially bounded function $u(t) : \mathbb{R} \rightarrow \mathbb{R}^m$, and any initial condition $\xi \in \mathbb{R}^n$, there exists a unique solution $x(t, \xi, u)$ of (1) satisfying the initial condition $x(0, \xi, u) = \xi$, defined on some maximal interval $(T^-_{\xi, u}, T^+_{\xi, u})$.

Recall that the system (1) is input-to-state stable (iss for short) if there exist $\gamma \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$ so that the following holds:

$$|x(t, \xi, u)| \leq \beta(|\xi|, t) + \gamma(\|u_{[0,t]}\|_\infty) \quad (2)$$

for all $t \geq 0$, $\xi \in \mathbb{R}^n$ and all input signals $u$, where for any interval $I$, $u_I$ denotes the restriction of $u$ to $I$, and where $\|v\|$ denotes the usual $L^m_{\infty}$-norm (possibly infinite) of $v$. Usually one can think of $u$ as an exogenous disturbance entering the system. Note that if (2) holds for any trajectory on any interval where the trajectory is defined, then the system is automatically forward complete.

We denote by $W^{k,\infty}(J)$, for any integer $k \geq 1$ and any interval $J$, the space of all functions $u : J \rightarrow \mathbb{R}^m$ for which the $(k-1)$st derivative $u^{(k-1)}$ exists and is locally Lipschitz. For $k = 0$, we define $W^{0,\infty}(J)$ as the set of locally essentially bounded $u : J \rightarrow \mathbb{R}^m$. When $J = [0, +\infty)$, $\mathcal{K}_\infty$ functions which will often be used in the following sections is the so-called “weak triangular inequality” $\gamma(a + b) \leq \gamma(2a) + \gamma(2b)$ for all $a, b \geq 0$.
we omit \( J \) and write simply \( W^{k,\infty} \). (Since absolutely continuous functions have essentially bounded derivatives if and only if they are Lipschitz, the definition of \( W^{k,\infty}(J) \), for positive \( k \), amounts to asking that the \((k-1)\)st derivative \( u^{(k-1)} \) exists and is absolutely continuous, and its derivative is locally essentially bounded; thus \( W^{k,\infty} \) is a standard Sobolev space, justifying our notation.)

**Definition 2.1** System (1) is said to be \( k \)th derivative input-to-state stable (\( D^k \mathrm{iss} \)) if there exist some \( \mathcal{KL} \)-function \( \beta \), and some \( \mathcal{K} \)-functions \( \gamma_0, \gamma_1, \ldots, \gamma_k \) such that, for every input \( u \in W^{k,\infty} \) the following holds:

\[
|x(t, \xi, u)| \leq \beta(|\xi|, t) + \gamma_0(\|u\|) + \gamma_1(\|\dot{u}\|) + \cdots + \gamma_k(\|u^{(k)}\|)
\]  

(3)

for all \( t \geq 0 \).

As with \( \mathrm{iss} \), we remark that if the estimate (3) is only required to hold along the maximal interval of definition of the corresponding solution, then, as the right-hand side is bounded independently of \( t \) on any bounded interval, the solution must be in fact globally defined if \( u \in W^{k,\infty} \). Note, however, a \( D^k \mathrm{iss} \) system may still fail to be complete with respect to arbitrary measurable, locally essentially bounded input functions, see for instance, Example 6.2.

We simply say that the system is \( D \mathrm{iss} \) if the system is \( D^1 \mathrm{iss} \); of course, \( \mathrm{iss} \) is the same as \( D^k \mathrm{iss} \) for \( k = 0 \).

**Remark 2.2** Note that a system is \( D^k \mathrm{iss} \) if and only if there exist some \( \beta \in \mathcal{KL} \) and some \( \gamma \in \mathcal{K} \) such that

\[
|x(t, \xi, u)| \leq \beta(|\xi|, t) + \gamma(\|u\|^{[k]})
\]  

(4)

for all \( t \geq 0 \), where \( \|u\|^{[k]} = \max_{1 \leq i \leq k} \{\|u^{(i)}\|\} \).  

\( \square \)

**Smooth inputs**

It is easy to see that the same definition results if we employ smooth inputs instead of just \( u \in W^{k,\infty} \) in our definitions:

**Lemma 2.3** The system (1) is \( D^k \mathrm{iss} \) if and only if property (4) holds for all smooth input functions (with the same \( \beta, \gamma \)).

**Proof.** It is enough to show that if (4) holds for all smooth input functions, then (4) also holds for all \( u \in W^{k,\infty} \).
Assume for some $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$, (4) holds for all smooth input functions. By causality, it follows that

$$|x(t, \xi, u)| \leq \beta(|\xi|, t) + \gamma \left( \|u_{[0,t]}\|^{[k]} \right)$$

(5)

for all $u \in C^\infty$.

Let $u \in W^{k,\infty}$. Fix $T > 0$ such that $x(t, \xi, u)$ is defined on $[0, T]$. Note that $u^{(k)}$ is essentially bounded on $[0, T]$. It is a routine approximation fact (reviewed in Corollary B.2 in the appendix) that there exists an equibounded sequence of $C^\infty$ functions $\{u_j\}$ such that

- $u_j \to u$ pointwise on $[0, T]$; and
- $\|u_j(0,T)\|^{[k]} \leq \|u_{[0,T]}\|^{[k]}$.

By (5), we have

$$|x(t, \xi, u_j)| \leq \beta(|\xi|, t) + \gamma \left( \|u_j_{[0,T]}\|^{[k]} \right)$$

(6)

for all $t \in [0, T]$. On the other hand, by Theorem 1 in [15], one knows that $x(t, \xi, u_j) \to x(t, \xi, u)$ for all $t \in [0, T]$. Letting $j \to \infty$ in (6), we get

$$|x(t, \xi, u)| \leq \beta(|\xi|, t) + \gamma \left( \|u_{[0,T]}\|^{[k]} \right)$$

(7)

for all $t \in [0, T]$. In particular, $|x(t, \xi, u)| \leq \beta(|\xi|, 0) + \gamma \left( \|u\|^{[k]}_{[0,\tau]} \right)$ for all $t \in [0, T]$. This implies that on the maximal interval $[0, T^{+}_{\xi, u}]$ of existence of $x(t, \xi, u)$, it holds that

$$|x(t, \xi, u)| \leq \beta(|\xi|, 0) + \gamma \left( \|u\|^{[k]}_{[0,\tau_{\xi, u}]} \right).$$

Hence, $T^{+}_{\xi, u} = \infty$, that is, $x(t, \xi, u)$ is defined on $[0, \infty)$. Thus $T$ can be picked arbitrarily. It then follows from (7) that

$$|x(t, \xi, u)| \leq \beta(|\xi|, t) + \gamma \left( \|u\|^{[k]} \right)$$

for all $t \geq 0$.

3 A Lyapunov Characterization of $D^k$ISS

Fix $k \geq 1$. For system (1), consider the auxiliary system

$$\dot{x} = f(x, z_0), \quad \dot{z}_0 = z_1, \ldots, \dot{z}_{k-1} = v.$$  

(8)
Let
\[
\hat{x}(t, \xi, \eta, v) := \begin{pmatrix}
x(t, \xi, \eta, v) \\
z(t, \eta, v)
\end{pmatrix}
\]
denote the trajectory of (8) with the initial state \(x(0) = \xi, z(0) = \eta\), (note that the \(z\)-component of the solution is independent of the choice of \(\xi\)).

Observe that, if property (4) is known to hold for all inputs in \(W^{k,\infty}\), then, for the trajectories \((x(t, \xi, z_0, v), z(t, \xi, z_0, v))\) of the auxiliary system, the following property holds:
\[
|x(t, \xi, \eta, v)| \leq \beta(|\xi|, t) + \bar{\gamma}_0 \left(\|z\|_{[0,t]}\right) + \bar{\gamma}_1(\|v\|) \tag{9}
\]
for all measurable, locally essentially bounded inputs \(v\). Given the fact that \(|z(t)| \leq \|z\|_{[0,t]}\) is always true, we get
\[
|x(t, \xi, \eta, v)| \leq \beta(|\xi| + |\eta|, t) + \bar{\gamma}_0 \left(\|z\|_{[0,t]}\right) + \bar{\gamma}_1(\|v\|)
\]
for some \(\bar{\gamma}_0, \bar{\gamma}_1 \in \mathcal{K}\). This shows that if (1) is \(D^{k}\)iss, then (8) is input-output-to-state stable, i.e., \(I\)oss, with \(v\) as input and \(z = (z_0, z_1, \ldots, z_{k-1})\) as outputs (cf. [10], or the appendix).

On the other hand, if the auxiliary system (8) is \(I\)oss, then there exist some \(\beta \in \mathcal{KL}\) and \(\gamma_0, \gamma \in \mathcal{K}\) such that
\[
|x(t, \xi, \eta, v)| \leq \beta(|\xi|, t) + \gamma_0 \left(\|z\|_{[0,t]}\right) + \gamma(\|v\|)
\]
for all locally essentially bounded inputs \(v\). Observe that
\[
\beta(|\xi| + |\eta|, t) \leq \beta(2|\xi|, t) + \beta(2|\eta|, 0) \leq \beta(2|\xi|, t) + \beta(2\|z\|_{[0,t]}, 0).
\]

It follows that
\[
|x(t, \xi, \eta, v)| \leq \beta(|\xi|, t) + \gamma_0 \left(\|z\|_{[0,t]}\right) + \gamma(\|v\|)
\]
holds for all locally essentially bounded \(v\), where \(\bar{\beta}(s, t) = \beta(2s, t)\), and \(\bar{\gamma}_0(s) = \beta(2s, 0) + \gamma_0(s)\). In particular,
\[
|x(t, \xi, \eta, v)| \leq \bar{\beta}(|\xi|, t) + \bar{\gamma}_0 \left(\|z\|_{[0,t]}\right) + \gamma(\|v\|).
\]

This implies that for any \(u \in W^{k,\infty}\), the trajectory of system (1) with initial state \(\xi\) satisfies the estimate:
\[
|x(t, \xi, u)| \leq \beta(|\xi|, t) + \gamma_1 \left(\|u\|^{[k]}\right),
\]
where $\gamma_1(s) = \bar{\gamma}_0(s) + \gamma(s)$.

**Lemma 3.1** Let $k \geq 1$. The system (1) is $D^k\text{iss}$ if and only the associated auxiliary system (8) is iOSS with $v$ as input and $z = (z_0, z_1, \ldots, z_{k-1})$ as output. \hfill $\square$

By the main result in [10], the system (8) is iOSS if and only if it admits an iOSS-Lyapunov function, that is, a smooth function $V : \mathbb{R}^n \times \mathbb{R}^{km} \to \mathbb{R}_{\geq 0}$ such that

- for some $\alpha, \overline{\alpha} \in \mathcal{K}_\infty$, it holds that
  \[ \alpha(||(x, z)||) \leq V(x, z) \leq \overline{\alpha}(||z||) \]
  for all $(x, z)$; and
- for some $\alpha, \rho \in \mathcal{K}_\infty$,\n  \[ \frac{\partial V}{\partial x}(x, z)f(x, z) + \frac{\partial V}{\partial z_0}(x, z)z_1 + \cdots + \frac{\partial V}{\partial z_{k-1}}(x, z)v \leq -\alpha(V(x, z)) + \rho(||z, v||) \]
  for all $(x, v)$ and $z$.

Interpreting $z$ as the input and its derivatives for system (1), we get the following:

**Theorem 1** Let $k \geq 1$. The system (1) is $D^k\text{iss}$ if and only if there exists a smooth function $V : \mathbb{R}^n \times \mathbb{R}^{km} \to \mathbb{R}_{\geq 0}$ such that

- there exist some $\alpha, \overline{\alpha} \in \mathcal{K}_\infty$ such that for all $(x, \mu^{[k-1]}) \in \mathbb{R}^n \times \mathbb{R}^{mk}$ with $\mu^{[k-1]} = (\mu_0, \mu_1, \ldots, \mu_{k-1})$, it holds that
  \[ \alpha(||(x, \mu^{[k-1]})||) \leq V(x, \mu^{[k-1]}) \leq \overline{\alpha}(||x, \mu^{[k-1]}||) \]; \hfill (10)
- there exist some $\alpha \in \mathcal{K}_\infty, \rho \in \mathcal{K}_\infty$ such that for all $x \in \mathbb{R}^n$ and all $\mu^{[k]} \in \mathbb{R}^{m(k+1)}$ with $\mu^{[k]} = (\mu_0, \mu_1, \ldots, \mu_k)$, it holds that
  \[ \frac{\partial V}{\partial x}(x, \mu^{[k-1]})f(x, \mu_0) + \frac{\partial V}{\partial \mu_0}(x, \mu^{[k-1]})\mu_1 + \frac{\partial V}{\partial \mu_1}(x, \mu^{[k-1]})\mu_2 + \cdots + \frac{\partial V}{\partial \mu_{k-1}}(x, \mu^{[k-1]})\mu_k \leq -\alpha(V(x, \mu^{[k-1]})) + \rho(||\mu^{[k]}||). \hfill (11) \]

**Remark 3.2** Note that inequality (10) implies that

\[ \alpha(||x||) \leq V(x, \mu^{[k-1]}) \leq \overline{\alpha}(||x, \mu^{[k-1]}||) \]. \hfill (12)

On the other hand, if a system admits a Lyapunov function $V$ satisfying (12) and (11), then the system is $D^k\text{iss}$. Hence, an equivalent Lyapunov characterization of $D^k\text{iss}$ is the existence of a smooth function $V$ satisfying (12) and (11) for some $\alpha, \overline{\alpha}, \alpha \in \mathcal{K}_\infty$ and some $\rho \in \mathcal{K}$. \hfill $\square$
3.1 Related Notions

In this section, we consider two properties related to \( \text{Diss} \).

We say that system (1) is \( \text{ISS in } \dot{u} \) if, for some \( \beta \in \mathcal{K}\mathcal{L} \) and some \( \gamma \in \mathcal{K} \), the following estimate holds for all trajectories with inputs in \( W^{1,\infty} \):

\[
|x(t, \xi, u)| \leq \beta(|\xi|, t) + \gamma(\|\dot{u}\|) \quad \forall t \geq 0.
\]  

(13)

We say that system (1) is \( \text{ISS in constant inputs} \) if, for some \( \beta \in \mathcal{K}\mathcal{L} \) and \( \gamma \in \mathcal{K} \), the following estimate holds for all trajectories corresponding to constant inputs \( u \):

\[
|x(t, \xi, u)| \leq \beta(|\xi|, t) + \gamma(\|u\|) \quad \forall t \geq 0.
\]  

(14)

It is not hard to see that if a system is \( \text{ISS in } \dot{u} \), then it is \( \text{GAS uniformly in all constant inputs} \), that is, for some \( \beta \in \mathcal{K}\mathcal{L} \), the following holds for all trajectories with constant inputs:

\[
|x(t, \xi, u)| \leq \beta(|\xi|, t) \quad \forall t \geq 0.
\]

Also note that

\[
(\text{ISS in } \dot{u}) \Rightarrow (\text{Diss}).
\]

The converse is in general false. This can be seen through the following argument. Suppose \( \text{Diss} \) implies \( \text{ISS in } \dot{u} \). Then we would have \( \text{ISS} \Rightarrow \text{Diss} \Rightarrow \text{ISS in } \dot{u} \), and hence, \( \text{ISS} \Rightarrow \text{ISS in } \dot{u} \). But this is false, as shown by the simple linear example \( \dot{x} = -x + u \), where the system is \( \text{ISS} \) but not \( \text{ISS in } \dot{u} \).

It is also not hard to see that, for any \( k \geq 0 \),

\[
(\text{D}^k\text{ISS}) \Rightarrow (\text{ISS in constant } u).
\]

Again, the inverse implication is in general false as shown by examples in Section 7.

Using similar arguments as in the proof of Lemma 3.1, we get the following:

**Lemma 3.3** System (1) is \( \text{ISS in } \dot{u} \) if and only if the auxiliary system

\[
\dot{x} = f(x, z), \quad \dot{z} = v
\]  

(15)

is state-independent-input-to-output stable, i.e., \( \text{SIOS} \) (see [18] or the appendix) with \( v \) as inputs and \( x \) as outputs. \( \square \)
Lemma 3.4 System (1) is ISS in constant inputs if and only if the auxiliary system

\[ \dot{x} = f(x, z), \quad \dot{z} = 0 \]  

is output-to-state-stable, i.e., OSS (see [10] or the appendix) with \( z \) as outputs. \qed

Applying Theorem 1.2 of [17] in conjunction with Remark 4.1 in [17] to the SIOS property for system (15), we get the following:

**Proposition 3.5** System (1) is ISS with respect to \( \dot{u} \) if and only if there exists a smooth Lyapunov function \( V : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_{\geq 0} \) satisfying the following:

- for some \( \alpha, \overline{\alpha} \in \mathcal{K}_\infty \),
  
  \[ \alpha(|x|) \leq V(x, \mu_0) \leq \overline{\alpha}(|x|) \quad \forall x \in \mathbb{R}^n \forall \mu_0 \in \mathbb{R}^m; \]  

- for some \( \chi \in \mathcal{K}_\infty \) and some continuous, positive definite function \( \alpha \),
  
  \[ V(x, \mu_0) \geq \chi(|\mu_1|) \Rightarrow D_x V(x, \mu_0) f(x, \mu_0) + D_{\mu_0} V(x, \mu_0) \mu_1 \leq -\alpha(V(x, \mu_0)) \]  
  for all \( x \in \mathbb{R}^n \) and all \( \mu_0, \mu_1 \in \mathbb{R}^m \). \( \blacksquare \)

Observe that if one restricts the set where the input functions take values to be a bounded set \( \mathcal{U} \) (as in the case of [8]), then the Lyapunov characterization in Proposition 3.5 is equivalent to the existence of a smooth Lyapunov function \( V \) satisfying (17) for some \( \alpha, \overline{\alpha} \in \mathcal{K}_\infty \) such that for some \( \alpha \in \mathcal{K}_\infty \) and \( \sigma \in \mathcal{K} \),

\[ D_x V(x, \mu_0) f(x, \mu_0) + D_{\mu_0} V(x, \mu_0) \mu_1 \leq -\alpha(V(x, \mu_0)) + \sigma(|\mu_1|) \]

for all \( x \in \mathbb{R}^n \), all \( \mu_0 \in \mathcal{U} \), and all \( \mu_1 \in \mathbb{R}^m \). Such an Lyapunov estimate was used in [8] to analyze the asymptotic behavior of systems with slowly varying parameters.

Applying Theorem 2 of [10] to the OSS property for system (16), we have the following:

**Proposition 3.6** System (1) is ISS with respect to constant inputs if and only if it admits a smooth Lyapunov function \( V : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_{\geq 0} \) such that

- for some \( \alpha, \overline{\alpha} \in \mathcal{K}_\infty \),
  
  \[ \alpha(|(x, \mu)|) \leq V(x, \mu) \leq \overline{\alpha}(|(x, \mu)|) \quad \forall x \in \mathbb{R}^n, \forall \mu \in \mathbb{R}^m; \]  

- for some \( \alpha \in \mathcal{K}_\infty \), \( \sigma \in \mathcal{K} \),
  
  \[ D_x V(x, \mu) f(x, \mu) \leq -\alpha(V(x, \mu)) + \sigma(|\mu|) \]  
  for all \( x \in \mathbb{R}^n, \mu \in \mathbb{R}^m \). \( \blacksquare \)
Remark 3.7 It is also worth introducing a $D$iss property with respect to different indexes on different components of the inputs. For instance, for a system
\[ \dot{x} = f(x, u, v) \] (21)
with $(u, v)$ as inputs, one may consider the property that for some $\beta \in K\mathcal{L}, \gamma_u \in K$ and $\gamma_v \in K$, it holds that
\[ |x(t, \xi, u, v)| \leq \beta(|\xi|, t) + \gamma_u(\|u\|) + \gamma_v(\|v\|) + \gamma_v(\|\dot{v}\|). \] (22)

One can also get a Lyapunov characterization for such a property by using the same argument as in the proof of Theorem 1 with the ioss results. For instance, a system as in (21) satisfies property (22) if and only if there exists a smooth Lyapunov function $V$ such that

- for some $\alpha, \overline{\alpha} \in K_\infty$, 
  \[ \alpha(|(\xi, \nu_0)|) \leq V(\xi, \nu_0) \leq \overline{\alpha}(|(\xi, \nu_0)|); \]
- for some $\alpha \in K_\infty$, some $\rho_u, \rho_v \in K$, it holds that
  \[ \frac{\partial V}{\partial x}(x, \nu_0)f(x, \mu_0, \nu_0) + \frac{\partial V}{\partial \nu_0}(x, \nu_0)\nu_1 \leq -\alpha(V(x, \nu_0)) + \rho_u(\|\mu_0\|) + \rho_v(\|\nu_0\|) + \rho_v(\|\nu_1\|) \]
  for all $x, \mu_0, \nu_0$ and $\nu_1$. \[

3.2 ISS with respect to Inputs Generated by an Exogenous System

The ioss approach provides a very convenient tool for studying the $D^k$iss and related properties. The idea is to treat the inputs and their derivatives as part of the state variables of an augmented system. The same approach also allows one to treat the notion of iss with respect to inputs generated by a given exogenous system
\[ \dot{u} = S(u) \] (23)
for some locally Lipshitz map $S$. By considering the augmented system
\[ \dot{x} = f(x, z), \quad \dot{z} = S(z), \]
the problem of iss with respect to the inputs generated by (23) becomes the problem of oss with $z$ as the outputs. In fact, this was one of the main motivations for the work [10] and [19].

A more general problem is to consider the inputs generated by an exogenous system
\[ \dot{q} = S(q), \quad u = \psi(q) \] (24)
where, for the sake of simplicity, we assume that $\psi$ is smooth.

We say that a system (1) is $D^k\text{ISS}$ with respect to the input generated by (24) if an estimate as in (3) holds for all the trajectories of (1) with $u(\cdot)$ generated by (24).

Observe that if a system is $D^k\text{ISS}$, then it is $D^k\text{ISS}$ with respect to inputs generated by any exogenous system. On the other hand, $D^k\text{ISS}$ with respect to inputs generated by a exogenous system is much weaker than $D^k\text{ISS}$. For instance, consider the scalar system

$$\dot{x} = -\text{sat}(x) + u,$$

where $\text{sat}(\cdot)$ is the saturation function defined by $\text{sat}(s) = s$ if $|s| \leq 1$, and $\text{sat}(s) = s/|s|$ for all $|s| > 1$. This system is not $D^k\text{ISS}$ for any $k$. In fact, even with the constant input $u \equiv 1$, the asymptotic gain property fails, that is, there is no $\gamma \in \mathcal{K}$ such that $\lim_{t \to \infty} |x(t)| \leq \gamma(\|u\|) = \gamma(1)$.

This is because every $|x_0| \geq 1$ is an equilibrium of the system. On the other hand, we will show at the end of this section that this system is $D\text{ISS}$ with respect to sinusoid signals with any frequency. That is, for any $\omega \neq 0$, the system is $D\text{ISS}$ with respect to the inputs generated by the exogenous system

$$\dot{q}_1 = \omega q_2, \quad \dot{q}_2 = -\omega q_1, \quad u = q_1.$$

To study the property of $\text{ISS}$ for (1) subject to an exogenous system (24), we consider the following “measurement-to-output” stability. Given a system with no input $u$ (for simplicity):

$$\dot{x} = f(x), \quad z = \psi(x), \quad y = h(x),$$

where $z$ is considered as the measurements, and $y$ the outputs, we say that the system is measurement-to-output stable if there exist some $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that

$$|y(t)| \leq \beta(|y(0)|, t) + \gamma \left( \|z\|_{[0,t]} \right) \quad \forall t \geq 0,$$

holds for every trajectory of the system. (A more general setup is to replace $y(0)$ by a function in $x(0)$ in the $\beta$ term.) This is one of the topics investigated in [5].

It then can be seen that the property of $\text{ISS}$ with respect to the inputs generated by (24) is equivalent to the measurement-to-output stability for the augmented system

$$\dot{x} = f(x, z), \quad \dot{q} = S(q), \quad z = \psi(q),$$

with $z$ as the measurements and $x$ as the inputs. Similarly, the property of $D^k\text{ISS}$ with respect to the inputs generated by (24) is equivalent to the measurement-to-output stability for the augmented system (27) with $(z_0, \ldots, z_k)$ as the measurements, and with $x$ as the outputs, where $z_i = \psi_i(q)$ ($i = 0, 1, \ldots, k$), and where $\psi_i$ is defined inductively by

$$\psi_0(q) = \psi(q), \quad \psi_{i+1}(q) = D\psi_i(q)S(q), \quad i \geq 0.$$
We expect that the results to be developed in [5] will make substantial contributions on the topics related to measurement-to-output stability which, in turn, will lead to progress in the understanding of the properties $D^kiss$ with respect to inputs generated by exogenous systems.

Finally, we return to show that system (25) is $Diss$ with respect to sinusoid signals of any frequency. For this purpose, we consider the Lyapunov function

$$V(x,q) = \left(x + \frac{q_2}{\omega}\right)^2 + q_1^2 + q_2^2.$$  

Notice that $V$ is a positive definite quadratic form of the extended state $\chi \doteq [x, q_1, q_2]$. With $S$ defined by (26), we have

$$DV_x(x,q)f(x,q_1) + DV_q S(q) = 2\left(x + \frac{q_2}{\omega}\right)(-\text{sat}(x))$$

$$\leq -2\text{sat}(x)x + \frac{2|q_2|}{\omega} \leq -\alpha(|\chi|) + \frac{2|q_2|}{\omega} + q_1^2 + q_2^2$$

for some $\alpha \in \mathcal{K}_\infty$. Hence, system (25), augmented with equation (26), is output-to-state stable with respect to the output $y = [q_1, q_2]$ and therefore, the following estimate holds along its trajectories

$$|x(t)| \leq |\chi(t)| \leq \beta(|\chi(0)|, t) + \gamma_1(\|q_1\|) + \gamma_2(\|q_2\|), \quad \forall t \geq 0 \quad (28)$$

where $\beta \in \mathcal{KL}$ and $\gamma_1, \gamma_2 \in \mathcal{K}$. Since $|\chi(0)| \leq |x(0)| + |q_1(0)| + |q_2(0)|$ and $|q_i(0)| \leq \|q_i\|$, $i = 1, 2$, equation (28) implies $Diss$ of (25) with respect to sinusoidal signals by a standard application of the weak triangular inequality to the term $\beta(|\chi(0)|, t)$. We would also like to remark that, though system (25) is $Diss$ for all sinusoid signals of a given frequency, the $Diss$ property does not hold for the system uniformly in the frequencies.

4 Application of $Diss$ to the Analysis of Cascade Systems

An interesting feature of $iss$, which makes it particularly useful in feedback design, is that the property is preserved under cascades. Unfortunately, this is not the case for the weaker notion of integral $iss$, as remarked in [2]. Interestingly, however, although $Diss$ is also a weaker property than $iss$, it is preserved under cascades, as shown in this section.

For a system

$$\dot{x} = f(x,v,u)$$

with $(v,u)$ as inputs, we say that the system is $D^kiss$ in $v$ and $D^liss$ in $u$ if there exist
\[ \beta \in KL \text{ and } \gamma \in K \text{ such that the following holds along any trajectory } x(t, \xi, v, u) \text{ with initial state } \xi, \text{ any input } (v, u) \text{ for which } v \in W^{k,\infty} \text{ and } u \in W^{l,\infty}: \]

\[ |x(t, \xi, v, u)| \leq \beta(|\xi|, t) + \gamma \left( \|v\|^{[k]} \right) + \gamma \left( \|u\|^{[l]} \right) \quad \forall t \geq 0. \]

**Lemma 4.1** Consider a cascade system

\[
\begin{align*}
\dot{x} &= f(x, z, u), \\
\dot{z} &= g(z, u),
\end{align*}
\]

where \( x(\cdot) \) and \( z(\cdot) \) evolve on \( \mathbb{R}^{n_1} \) and \( \mathbb{R}^{n_2} \) respectively, the input \( u \) takes values in \( \mathbb{R}^m \), and where \( f \) is locally Lipschitz and \( g \) is smooth. Let \( k \geq 0 \). Suppose that the \( y \)-subsystem is \( D^kiss \) with \( u \) as input, and that the \( x \)-subsystem \( D^{k+1}iss \) in \( z \) and \( D^kiss \) in \( u \). Then the cascade system (29) is \( D^kiss \).

**Proof.** By assumption, there exist \( \beta_z \in KL \) and \( \gamma_z \in K \) such that, along any trajectory \( z(t) \) of the \( z \)-subsystem with input \( u \), it holds that

\[ |z(t)| \leq \beta_z(|z(0)|, t) + \gamma_z \left( \|u\|^{[k]} \right) \quad \forall t \geq 0; \quad (30) \]

and there exist some \( \beta_x \in KL \) and \( \gamma_x \in K \) such that, for any trajectory \( x(t, v, u) \) of the system \( \dot{x} = f(x, v, u) \),

\[ |x(t, \xi, v, u)| \leq \beta_x(|x(0)|, t) + \gamma_x \left( \|v\|^{[k+1]} \right) + \gamma_x \left( \|u\|^{[l]} \right) \quad \forall t \geq 0. \quad (31) \]

To prove Lemma 4.1, we need to find a suitable estimate for the \( x \)-component of solutions of (29). For this purpose, we define by induction for \( 1 \leq i \leq k + 1 \):

\[ g_i(a, b_0, b_1, \ldots, b_{i-1}) = \frac{\partial g_{i-1}}{\partial a} g(a, b_0) + \sum_{j=0}^{i-2} \frac{\partial g_{i-1}}{\partial b_j} b_{j+1}, \]

where \( g_1(a, b_0) = g(a, b_0) \). It can be seen that \( g_i(0, 0, \ldots, 0) = 0 \) for all \( 0 \leq i \leq k \), hence, there exists some \( \sigma_i \in K \) such that

\[ g_i(a, b_0, \ldots, b_{i-1}) \leq \sigma_i(|a|) + \sigma_i \left( |b_{i-1}| \right). \]

Again, by induction, one can show that, along any trajectory \( z(t) \) of the \( z \)-subsystem of (29) with an input \( u \in W^{k,\infty} \), it holds that

\[ \frac{d^i}{dt^i} z(t) = g_i \left( \eta(t), d(t), \dot{d}(t), \ldots, d^{i-1}(t) \right) \]
for all $1 \leq i \leq k + 1$. It then follows that
\[
\|z^{[k+1]}\| \leq \sigma(\|z\|) + \sigma(\|u^{[k]}\|)
\] (32)

for some $\sigma \in \mathcal{K}$. It then follows from (31) and (32) that, for some $\rho \in \mathcal{K}$, it holds that along any trajectory $(x(t), z(t))$ of (29),
\[
|x(t)| \leq \beta_x(|x(0)|, t) + \rho(\|z\|) + \rho(\|u^{[k]}\|) \quad \forall t \geq 0.
\] (33)

Applying a standard argument to (33) and (30) as in the proof of the result that a cascade of ISS systems is again ISS, one can show that system (29) is $D^k$-ISS. To be more precise, (33) implies that
\[
|x(t)| \leq \beta_x\left(|x(t/2)|, \frac{t}{2}\right) + \rho(\|z\|_{[0,t/2]}) + \rho(\|u^{[k]}\|_{[0,t/2]}) \quad \forall t \geq 0
\] (34)

along any trajectory of (29). Fix an input $u$ and pick any trajectory $(x(t), z(t))$ of (29) with the input $u$. Let $x_1 = x(t/2)$. We also have:
\[
|x_1| \leq \beta_x\left(|\xi|, \frac{t}{2}\right) + \rho(\|z\|_{[0,t/2]}) + \rho(\|u^{[k]}\|_{[0,t/2]}) \quad \forall t \geq 0.
\]

Hence, there exist some $\tilde{\beta}_x, \tilde{\beta}_z$ and some $\tilde{\rho} \in \mathcal{K}$ (which depend only on $\beta_x, \rho$) such that
\[
\beta_x\left(|x_1|, \frac{t}{2}\right) \leq \tilde{\beta}_x(|\xi|, t) + \tilde{\beta}_z(\|z\|_{[0,t/2]}, t) + \tilde{\rho}(\|u^{[k]}\|_{[0,t/2]})
\] (35)

for all $t \geq 0$. By (30), $|z(\tau)| \leq \beta_z(|z(0)|, \tau) + \gamma_z(\|u^{[k]}\|)$ for all $\tau \geq 0$, hence,
\[
\tilde{\beta}_z(\|z\|_{[0,t/2]}, t) \leq \tilde{\beta}_z(|z(0)|, t) + \tilde{\gamma}_z(\|u^{[k]}\|) \quad \forall t \geq 0
\] (36)

for some $\tilde{\beta}_z \in \mathcal{K} \mathcal{L}$ and some $\tilde{\gamma}_z \in \mathcal{K}$. With (36), one sees from (35) that for some $\tilde{\beta} \in \mathcal{K} \mathcal{L}$ and some $\tilde{\gamma}_z \in \mathcal{K}$, it holds that
\[
\beta_x\left(|x_1|, \frac{t}{2}\right) \leq \tilde{\beta}_x(|\xi|, t) + \tilde{\beta}_z(|z(0)|, t) + \tilde{\gamma}_u(\|u^{[k]}\|)
\] (37)

for all $t \geq 0$. Since
\[
|z(\tau)| \leq \beta_z(|z(0)|, \tau/2) + \gamma_z(\|u^{[k]}\|) \quad \forall \tau \geq t/2,
\]
it follows that for some $\tilde{\beta}_z \in \mathcal{K} \mathcal{L}$ and some $\tilde{\gamma}_z \in \mathcal{K}$, it holds that
\[
\rho(\|z\|_{[t/2, t]}) \leq \tilde{\beta}_z(|z(0)|, t) + \tilde{\gamma}_z(\|u^{[k]}\|) \quad \forall t \geq 0.
\] (38)
Combining (33), (37) and (38), one sees that there exist some $\beta \in KL$ and some $\gamma \in K$ such that

$$|x(t)| \leq \beta(|x(0)| + |z(0)|, t) + \gamma \left(\|u\|^{[k]}\right) \quad \forall t \geq 0.$$ 

Note that the choice of $\beta$ and $\gamma$ was made independent of the trajectory of (29). Together with (30) this shows that (29) is $D^k\text{iss}$. \hfill \blacksquare

**Remark 4.2** Observe from the above proof that to show that system (29) is $D^k\text{iss}$, the assumption that requires $g$ be smooth can be relaxed to requiring that $g$ be $C^k$ if $k \geq 1$, or to requiring that $g$ be locally Lipschitz in the case when $k = 0$.

Applying Lemma 4.1 to the special case of $k = 1$, one gets the following:

**Corollary 4.3** Consider a cascade system as in Lemma 4.1, where $f$ and $g$ are $C^1$ maps. Suppose that the $x$-subsystem is $\text{Disss}$ with $(z,u)$ as inputs that the $z$-subsystem is $\text{Disss}$ with $u$ as inputs, then the cascade system (29) is $\text{Disss}$ with $u$ as inputs. \hfill \Box

Applying Lemma 4.1 to the following autonomous system

$$\begin{align*}
\dot{x} &= f(x, z), \\
\dot{z} &= g(z),
\end{align*}$$

(39)

where $f$ is locally Lipschitz, and $g$ is smooth, one sees that the system is GAS provided that the $z$-subsystem is GAS and the $x$-subsystem is $D^k\text{iss}$ with $z$ as inputs for some $k \geq 0$.

It is by now a standard result that a system (39) is GAS if the $x$-subsystem is ISS and the $z$-subsystem is GAS. Now one sees that the ISS property of the $x$-subsystem can be relaxed to $D^k\text{iss}$. This result can be further improved as in the following.

For $\delta > 0$, we define the saturation function $\text{sat}_\delta$ by

$$\text{sat}_\delta(r) = \begin{cases} 
  r & \text{if } |r| < \delta, \\
  \text{sign}(r) \delta & \text{otherwise.}
\end{cases}$$

(40)

For $z = (z_1, z_2, \ldots, z_m) \in \mathbb{R}^m$, we define $\text{sat}_\delta(z) := (\text{sat}_\delta(z_1), \text{sat}_\delta(z_2), \ldots, \text{sat}_\delta(z_m))$.

**Proposition 4.4** A forward complete system as in (39) is GAS provided that for some $\delta > 0$ and for some $k \geq 0$, the system

$$\dot{x} = f(x, \text{sat}_\delta(z))$$

(41)

is $D^k\text{iss}$ and that the $z$-subsystem is GAS.
Proof. The local asymptotic stability property of (39) follows directly from the local asymptotic stability property of the \(x\) and \(z\) subsystems. Thus we only need to show the global attraction property, in particular, the convergence property of \(x(t)\) for any trajectory \((x(t), z(t))\) of (39).

First of all, the forward completeness assumption guarantees that \(x(t, \xi, z)\) is defined on \([0, \infty)\) for any trajectory of the \(x\)-subsystem with initial state \(\xi\) and external signal \(z\).

Pick any trajectory \((x(t), z(t))\) of (39). Since the \(z\)-subsystem is GAS, there is some \(T > 0\) such that \(|z(t)| \leq \delta\) for all \(t \geq T\). Consequently, \((x(t), z(t))\) is also a trajectory of (41) with the \(z\)-subsystem for all \(t \geq T\). Since system (41) cascaded with the \(z\)-subsystem is GAS, it follows that \(x(t)\) converges to 0. \(\blacksquare\)

5 An ISS Related Interpretation of \(D^k\text{iss}\)

Definition 5.1 A smoothly invertible ISS filter is an ISS system

\[
\dot{w} = g(w, d) \quad (42)
\]

with \(w(t), d(t) \in \mathbb{R}^m\), where \(g: \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m\) is a smooth map for which there exists a smooth map \(G: \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m\) such that \(G(\nu_0, g(\nu_0, d)) = d\) and \(g(\nu_0, G(\nu_0, \nu_1)) = \nu_1\) for all \(\nu_0, \nu_1\).

The main result in this section is the following characterization of \(D^k\text{iss}\):

Theorem 2 Let \(k\) be a positive integer. The following facts are equivalent:

(i) System (1) is \(D^k\text{iss}\);
(ii) There exists a smoothly invertible ISS filter

\[
\dot{\eta} = g(\eta, \mu) \quad (43)
\]

such that the system

\[
\begin{align*}
\dot{x} &= f(x, \eta) \\
\dot{\eta} &= g(\eta, \mu)
\end{align*} \quad (44)
\]

is \(D^{k-1}\text{iss}\).
(iii) For each smoothly invertible ISS filter as in (43) the cascade system (44) is \(D^{k-1}\text{iss}\).

Proof. The implication (iii) \(\Rightarrow\) (ii) is obvious. Let us consider (ii) \(\Rightarrow\) (i). Since system (44) is \(D^{k-1}\text{iss}\), there exist some \(\beta \in \mathcal{KL}\) and some \(\gamma \in \mathcal{K}\) such that along any trajectory \((x(t), \eta(t))\) of (44), it holds that
\[ |x(t)| \leq \beta(|x(0)| + |\eta(0)|, t) + \gamma \left( \|d\|^{k-1} \right) \]
\[ \leq \beta(2|x(0)|, t) + \beta_0(2\|\eta\|)\gamma \left( \|d\|^{k-1} \right), \quad (45) \]
where \( \beta_0(s) = \beta(s, 0) \). Let \( G \) be the smooth function as in Definition 5.1 for the function \( g \) in system (44).

Observe that any trajectory \( x(t) \) of (1) with an input \( \eta \in W^{k,\infty} \), \( (x(t), \eta(t)) \) is a trajectory of (44) with the input \( d \in W^{k,\infty} \) defined by \( d(t) = G(\eta(t), \dot{\eta}(t)) \).

Notice that \( g(0, 0) = 0 \) implies \( G(0, 0) = 0 \). Thus, by continuity of \( G \), there exists \( \gamma_0 \in \mathcal{K}_\infty \) such that \( G(a, b) \leq \gamma_0(|a|) + \gamma_0(|b|) \). Take any trajectory \( x(t) \) of (1) with an input \( \eta \in W^{k,\infty} \), and let \( d(t) = G(\eta(t), \dot{\eta}(t)) \). Then
\[ \|d\| \leq \gamma_0(\|\eta\|) + \gamma_0(\|\dot{\eta}\|). \quad (46) \]

Hence, in the case when \( k = 1 \), that is, when system (44) is ISS, (45) combined with (46) implies that
\[ |x(t)| \leq \beta(2|x(0)|, t) + \beta_0(2\|\eta\|) + \gamma_0(\|\eta\|) + \gamma_0(\|\dot{\eta}\|). \quad (47) \]

This shows that system (1) is DISS. To consider the more general case for \( k \geq 2 \), we consider inductively the following functions defined by:
\[ G_i(a_0, a_1, \ldots, a_i, a_{i+1}) = \sum_{j=0}^{i} \frac{\partial G_{i-1}}{\partial a_j}(a_0, a_1, \ldots, a_{i})a_{j+1} \]
with \( G_0(a_0, a_1) := G(a_0, a_1) \). Observe that for each \( i \), \( G_i(0, \ldots, 0) = 0 \), and hence, there exists some \( \gamma_i \in \mathcal{K} \) such that
\[ |G_i(a_0, a_1, \ldots, a_{i+1})| \leq \gamma_i \left( |a|^{i+1} \right), \]
where \( a^{[i]} = (a_0, a_1, \ldots, a_i) \).

By induction, one can show that for any trajectory \( \eta(t) \) of the \( \eta \)-subsystem of (44) with the input \( d \in W^{k-1,\infty} \), it holds that, for any \( 0 \leq i \leq k - 1 \),
\[ d^{(i)}(t) = G_i(\eta(t), \dot{\eta}(t), \ldots, \eta^{(i+1)}(t)). \]

Consequently, one has
\[ \|d\|^{k-1} \leq \gamma_k(\|\eta\|^{k}). \]

Thus, for any trajectory \( x(t, \xi, \eta) \) of (1), it holds that, with \( d = G(\eta, \dot{\eta}) \),
\[ |x(t, \xi, \eta)| \leq \beta(2 |\xi|, t) + \beta_0(2 \|\eta\|) + \gamma \left( \|d\|^{k-1} \right) \]
\[ \leq \beta(2 |\xi|, t) + \beta_0(2 \|\eta\|) + \tilde{\gamma}_k \left( \|\eta\|^k \right), \]

where \( \tilde{\gamma}_k = \gamma \circ \gamma_k \). Hence, system (1) is \( D^k \text{iss} \).

To complete the proof of Theorem 2, it only remains to show the implication (i)\( \Rightarrow \) (iii). But this implication is an immediate consequence of Lemma 4.1.

6 Are \( D \text{iss} \) Systems Always \( \text{iss} \)?

In this section we will discuss an example of a \( D \text{iss} \) system which is not \( \text{iss} \). This shows that \( D \text{iss} \) is indeed strictly weaker than \( \text{iss} \). First, however, we show that \( D \text{iss} \) and \( \text{iss} \) are equivalent for scalar systems.

**Proposition 6.1** A one-dimensional system in the form of (1) is \( \text{iss} \) if and only if it is \( D \text{iss} \).

**Proof.** Clearly, we only need to show one direction of the implication. Let a one-dimensional system (1) be \( D \text{iss} \). Then, there exists a smooth function \( V : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}_{\geq 0} \) such that

\[ \alpha(|x| + |\mu_0|) \leq V(x, \mu_0) \leq \sigma(|x| + |\mu_0|) \quad \forall (x, \mu_0) \in \mathbb{R} \times \mathbb{R}^m \]

(48)

for some \( \alpha, \sigma \in \mathcal{K}_\infty \), and

\[ \frac{\partial V}{\partial x}(x, \mu_0) f(x, \mu_0) + \frac{\partial V}{\partial \mu_0} \mu_1 \leq -\alpha(|x|) + \gamma(|\mu_0| + |\mu_1|), \quad \forall x, \forall (\mu_0, \mu_1) \]

(49)

for some \( \alpha, \gamma \) of class \( \mathcal{K}_\infty \). In particular, with \( \mu_1 = 0 \), (49) yields

\[ \frac{\partial V}{\partial x}(x, \mu_0) f(x, \mu_0) \leq -\alpha(|x|) + \gamma(|\mu_0|). \]

(50)

This implies that there exists a \( \mathcal{K}_\infty \) gain margin \( \chi \) (for instance \( \chi = \alpha^{-1} \circ 2\gamma \)) such that

\[ |x| \geq \chi(|\mu_0|) \implies \frac{\partial V}{\partial x}(x, \mu_0) f(x, \mu_0) \leq -\tilde{\alpha}(|x|) \]

(51)

where \( \tilde{\alpha} \) is of class \( \mathcal{K}_\infty \). If \( V \) is independent of \( \mu_0 \), this would already provide an \( \text{iss} \)-Lyapunov function for the system, and the \( \text{iss} \) property would follow. For the general case, let \( V_0(x) = V(x, 0) \). From (48), one sees that

\[ \alpha(|x|) \leq V_0(x) \leq \sigma(|x|) \quad \forall x; \]

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and from (51), one sees that

\[ DV_0(x)f(x,0) < -\bar{\alpha}(|x|) \quad \forall x. \]  

(52)

Since both \( DV_0 \) and \( f(x,0) \) are scalar functions, it follows that \( DV_0(x) \neq 0 \) for all \( x \neq 0 \). Since the 0-input system \( \dot{x} = f(x,0) \) is GAS, it follows that \( xf(x,0) < 0 \) for all \( x \neq 0 \). This together with (52) implies that \( xDV_0(x) < 0 \) for all \( x < 0 \). We will complete the proof by showing the following:

\[ |x| \geq \chi(|\mu_0|) \implies DV_0(x)f(x,\mu_0) < 0 \]  

(53)

for all \( x \neq 0 \), from which it follows that \( V_0 \) is an ISS-Lyapunov function for the system.

Suppose (53) fails for some \( x_0 \neq 0 \). Applying the intermediate value theorem to the continuous function \( DV_0(x)f(x,\mu_0) \) with the property that \( DV_0(x_0)f(x_0,0) < 0 \), one sees that there exists some \( \bar{\mu}_0 \) for which \( \chi(\bar{\mu}_0) \leq |x_0| \) such that \( DV_0(x_0)f(x_0,\bar{\mu}_0) = 0 \). It then follows from the fact that \( DV_0(x_0) \neq 0 \) that \( f(x_0,\bar{\mu}_0) = 0 \). This is impossible since it contradicts (51). Hence, (53) holds for all \( x \).

In what follows we show by example that Proposition 6.1 in general fails in higher dimensions.

**Example 6.2** Take any 2 by 2 matrix \( \Phi \) with the property that \( \Phi \) is Hurwitz but \( \Phi^T + \Phi \) has at least one positive eigenvalue (where \( A^T \) denotes the transpose of \( A \)). An example of such a matrix is

\[ \Phi = \begin{bmatrix} -1 & 4 \\ -1 & -1 \end{bmatrix}. \]

Let \( \bar{\lambda} \) be such an eigenvalue of \( (\Phi^T + \Phi) \), and let \( v_1 \) be a unit eigenvector of \( \Phi \) corresponding to \( \bar{\lambda} \). For \( \theta \in \mathbb{R} \), let \( U(\theta) \) be defined by

\[ U(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}. \]  

(54)

Notice that \( U(\theta)^T U(\theta) = I \). Consider now the system:

\[ \dot{x} = (x^T x) U(\theta)^T \Phi U(\theta) x, \]  

(55)

where \( \theta(\cdot) \) is taken to be the input to the system.

To see that this system is not ISS, we will show that there is some input which is bounded and for which the solution of (55) with \( x(0) = (0,1)' \) is not defined for all \( t > 0 \).
To define this input, we proceed as follows. We start by writing the eigenvector $v_1$ in polar form: $v_1 = (\cos \phi_0, \sin \phi_0)$, with $0 \leq \phi_0 < 2\pi$. Viewing the system away from zero as a system on $\mathbb{R}^2 \setminus \{0\}$, we consider the feedback law $\theta(x) := \phi - \phi_0$, where, using polar coordinates, $x_1 = r \cos \phi$ and $x_2 = r \sin \phi$. In defining the feedback, we may assume that arguments are taken in the following range: $0 \leq \phi < 2\pi$. However, the choice is irrelevant, since only trigonometric functions of $\theta$ appear in the system description.

In principle, there is no reason for a solution to exist for (55), under this feedback law, since the feedback law is discontinuous. However, again from periodicity of the equations, substitution into the right-hand side of (55) results in a smooth differential equation. Thus there is a unique solution, defined on some maximal interval $[0, T_{\text{max}})$, starting from the initial state $x(0) = (0, 1)'$. We consider the input $u$ which coincides with $\theta(x(t))$ on the maximal interval $[0, T_{\text{max}})$, and equals some arbitrary value, let us say zero, for $t > T_{\text{max}}$. This input is bounded (by $4\pi$). We now show that $r = |x(t)| \to \infty$ as $t \to T_{\text{max}}$.

Transforming to polar coordinates, we have that along trajectories of (55):

$$2r\dot{r} = 2(x^T x)x^T U(\theta)^T \Phi U(\theta)x = 2r^4 \begin{pmatrix} \cos \phi & \sin \phi \end{pmatrix} U(\theta)^T \Phi U(\theta) \begin{pmatrix} \cos \phi \\ \sin \phi \end{pmatrix},$$

where $\Psi = \Phi + \Phi^T$, and

$$\dot{\phi} = \frac{x_1 \dot{x}_2 - x_2 \dot{x}_1}{x_1^2 + x_2^2} = r^2 \begin{pmatrix} - \sin \phi & \cos \phi \end{pmatrix} U(\theta) \Phi U(\theta) \begin{pmatrix} \cos \phi \\ \sin \phi \end{pmatrix},$$

Thus, away from the equilibrium $x = 0$, we have that the system system (55) on $\mathbb{R}^2 \setminus \{0\}$ is, up to a coordinate change, the same as the following system which evolves on $\mathbb{R}_{>0} \times S^1$:

$$\dot{r} = \frac{1}{2} r^3 \begin{pmatrix} \cos(\phi - \theta) & \sin(\phi - \theta) \end{pmatrix} \Psi \begin{pmatrix} \cos(\phi - \theta) \\ \sin(\phi - \theta) \end{pmatrix},$$

$$\dot{\phi} = r^2 \begin{pmatrix} \sin(\theta - \phi) & \cos(\theta - \phi) \end{pmatrix} \Phi \begin{pmatrix} \cos(\phi - \theta) \\ \sin(\phi - \theta) \end{pmatrix}. \tag{56}$$

$$\dot{\phi} = r^2 \begin{pmatrix} \sin(\theta - \phi) & \cos(\theta - \phi) \end{pmatrix} \Phi \begin{pmatrix} \cos(\phi - \theta) \\ \sin(\phi - \theta) \end{pmatrix}. \tag{57}$$
With the feedback law $\theta = \phi - \phi_0$, Equation (56) becomes
\[ \dot{r} = \frac{1}{2} r^3 v_1^T \Psi v_1 = \frac{\bar{\lambda}}{2} r^3. \] (58)

Thus $r$ diverges monotonically to infinity in finite time, as claimed.

Nevertheless we claim that (55) is $\text{DiSS}$. For this purpose, consider the system
\[
\begin{cases}
\dot{x} = (x' x) U(\theta)' \Phi U(\theta) x \\
\dot{\theta} = -\theta + d.
\end{cases}
\] (59)

Here $d$ takes value in $\mathbb{R}$ and plays the role of an exogenous input, whereas $\theta$ is a component of the extended state $[x', \theta]'$. By virtue of the main result in Section 5, the $\text{DiSS}$ property for (55) is equivalent to the $\text{iss}$ property for (59). Pick as a candidate Lyapunov function:
\[ W(x, \theta) = x^T U(\theta)^T P U(\theta) x + k \theta^2, \] (60)

where $P = P' > 0$ is the solution of the Lyapunov equation
\[ \Phi^T P + P \Phi = -I_2. \] (61)

Notice that
\[ \lambda_{\min}(P)|x|^2 + k \theta^2 \leq W(x, \theta) \leq \lambda_{\max}(P)|x|^2 + k \theta^2, \] (62)

where $\lambda_{\min}$ and $\lambda_{\max}$ denote the largest and smallest eigenvectors of $P$ respectively. Thus $W$ is proper. Taking derivatives of $W$ along any trajectory $(x(t), \theta(t))$ of (59) yields

\[
\frac{d}{dt} W(x(t), \theta(t)) = |x(t)|^2 \left[ (U(\theta(t))^T \Phi U(\theta(t)) x(t))^T U(\theta(t))^T P U(\theta(t)) x(t) \\
+ x(t)^T U(\theta(t))^T P U(\theta(t))(U(\theta(t))^T \Phi U(\theta(t)) x(t)) \right] \\
+ \left[ 2x(t)^T U(\theta(t))^T P \frac{\partial}{\partial \theta} U(\theta(t)) x(t) + 2k \theta(t) \right] (-\theta(t) + d(t)).
\] (63)

Since $U(\omega)$ is orthonormal for all $\omega \in \mathbb{R}$, it follows from (61) that
\[ U(\omega)^T P U(\omega) U(\omega)^T \Phi U(\omega) + U(\omega)^T \Phi^T U(\omega) U(\omega)^T P U(\omega) = -I_2 \] (64)

for all $\omega$. Let $c > 0$ be such that $\left| U(\omega) P \frac{\partial}{\partial \omega} U(\omega) \right| \leq c$ for all $\omega$, where $|A|$ denotes the operator norm of $A \in \mathbb{R}^{2 \times 2}$. Then
\[ 2x(t)^T U(\theta(t))^T P \frac{\partial}{\partial \theta} U(\theta(t)) x(t) + 2k\theta(t) \leq 2c|x(t)|^2 (|\theta(t)| + |d(t)|) - 2k\theta(t)^2 + 2k\theta(t)d(t) \]
\[ \leq \frac{1}{4} |x(t)|^4 + 4c^2 (|\theta(t)| + |d(t)|)^2 - k\theta(t)^2 + k d(t)^2 \]
\[ \leq \frac{1}{4} |x(t)|^4 + 8c^2 \theta(t)^2 + 8c^2 d(t)^2 - k\theta(t)^2 + k d(t)^2. \] (65)

Combining (63), (64), and (65), one sees that

\[ \frac{d}{dt} W(x(t), \theta(t)) \leq -\frac{3}{4} |x(t)|^4 - (k - 8c^2)\theta(t)^2 + (8c^2 + k)d(t) \]

along any trajectory of (59). It then can be seen that if \( k > 8c \), \( W \) is an ISS-Lyapunov function for system (59). Consequently, system (59) is ISS as we wanted to show. \( \square \)

7 Examples

It is clear that one has the following implications for each \( k \geq 1 \):

ISS \( \Rightarrow \) \( D^k \)ISS \( \Rightarrow \) ISS in constant \( u \).

Below we show by examples how the converse implications may fail. For this purpose, we first show the following.

**Lemma 7.1** Consider a locally Lipschitz map \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \).

(i) The system \( \dot{x} = f(x + u) \) is ISS in constant \( u \) if and only if the 0-input system \( \dot{x} = f(x) \) is GAS.

(ii) If the system \( \dot{x} = f(x) + u \) is ISS, then the system \( \dot{x} = f(x + u) \) is \( D^k \)ISS.

**Proof.** (i) It is trivial that if \( \dot{x} = f(x + u) \) is ISS in constant \( u \), then the system \( \dot{x} = f(x) \) is GAS (just take \( u \equiv 0 \)). Conversely, suppose that the system \( \dot{z} = f(z) \) is GAS. Then, there is some \( \beta_0 \in KL \) such that

\[ |z(t, \xi)| \leq \beta_0(\xi, t) \quad \forall t \geq 0, \]

holds for every trajectory \( z(t, \xi) \) of the system with initial state \( z(0) = \xi \). Let \( u(t) \equiv \mu \) be a constant input, and consider a trajectory \( x(t, \xi, u) \) of the system \( \dot{x} = f(x + u) \). Let \( z(t) = x(t, \xi, u) + u(t) = x(t, \xi, u) + \mu \). Then \( z(t) \) is a trajectory of \( \dot{z} = f(z) \). Hence

\[ |z(t)| \leq \beta_0(|z(0)|, t) \leq \beta_0(2|\xi|, t) + \beta_0(2|u(0)|, t) \leq \beta_0(2|\xi|, t) + \beta_0(2|\mu|, 0). \]

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Combining this with the fact that \( |x(t, \xi, u)| \leq |z(t)| + \|u\| \), we get
\[
|x(t, \xi, u)| \leq \beta(2|\xi|, t) + \gamma(\|u\|),
\]
where \( \gamma(s) = \beta_0(2s, 0) + s \). This shows that the system is ISS in constant inputs.

[(ii)] Suppose that system \( \dot{z} = f(z) + u \) is ISS. Then, for some \( \beta_0 \in \mathcal{KL} \) and some \( \gamma_0 \in \mathcal{K} \), it holds that
\[
|z(t, \xi, u)| \leq \beta_0(|\xi|, t) + \gamma_0(\|u\|)
\]
for the trajectory \( z(t, \xi, u) \) of the system with initial state \( z(0) = \xi \) and input \( u \). Take a trajectory \( x(t, \xi, u) \) of the system \( \dot{x} = f(x + u) \) for some input \( u \in W^{1,\infty} \). Let \( z(t) = x(t, \xi, u) + u(t) \). Obviously, \( z(\cdot) \) is a solution of \( \dot{z} = f(z) + \dot{u} \). Hence,
\[
|z(t, \xi, u)| \leq \beta_0(|z(0)|, t) + \gamma_0(\|\dot{u}\|).
\]
Arguing as in the proof of (i), it can be seen that
\[
|x(t, \xi, u)| \leq \beta_0(2|\xi|, t) + \gamma_0(\|\dot{u}\|) + \gamma(\|u\|),
\]
where \( \gamma(s) = \beta_0(2s, 0) + s \). Hence, the system is Diss.

To show that (ISS in constant \( u \) \( \not\Rightarrow \) Diss), we first show the following.

**Lemma 7.2** There exists a smooth system \( \dot{x} = f(x) \) in \( \mathbb{R}^2 \) with the following properties:

(i) The origin is globally asymptotically stable for \( \dot{x} = f(x) \).

(ii) For each \( a > 2 \) there exists an input \( u^a \) such that:
   - \( u \) is smooth, periodic, and \( u \) as well as all its derivatives are bounded in norm by 1, and
   - the solution of
     \[
     \dot{x} = f(x + u^a(t)) , \quad x(0) = \begin{pmatrix} a \\ 0 \end{pmatrix}
     \]
     is \( x(t) = (a \cos t, a \sin t)' \).

**Proof:** We fix a smooth nonincreasing function \( \gamma : \mathbb{R} \to \mathbb{R} \) with the following properties:
\[
\gamma(r) < 0 \quad \forall r > 0,
\]
\[
0 \leq r < \frac{1}{2} \Rightarrow \gamma(r) = -r,
\]
\[
r \geq 1 \Rightarrow \gamma(r) = -1.
\]
In terms of this $\gamma$, we define the following system:

\[
\begin{align*}
\dot{x}_1 &= \frac{\gamma(\sqrt{x_1^2 + x_2^2})}{\sqrt{x_1^2 + x_2^2}} x_1 - x_2 \\
\dot{x}_1 &= \frac{\gamma(\sqrt{x_1^2 + x_2^2})}{\sqrt{x_1^2 + x_2^2}} x_2 + x_1 .
\end{align*}
\]

Note that this is a smooth system on $\mathbb{R}^2$, since $\gamma(\sqrt{x_1^2 + x_2^2})/\sqrt{x_1^2 + x_2^2} \equiv -1$ for $x \approx 0$ (the dynamics are, in fact, linear near 0). In polar coordinates, we have

\[
\dot{r} = \gamma(r) , \quad \dot{\phi} = -1
\]

so the origin is indeed globally asymptotically stable. Observe that near the origin we have $\dot{r} = -r$, but for $|x| \geq 1$ we have $\dot{r} = -1$. This “slowing down” will allow us to obtain the desired result.

For each $a > 2$, we define the input $u_a$ as follows:

\[
u_a(t) = \begin{pmatrix} u_1^a(t) \\ u_2^a(t) \end{pmatrix} := \begin{pmatrix} \sqrt{a^2 - 1} \sin t - \cos t \\ \frac{a}{a^2 - 1} \cos t \end{pmatrix}
\]

and observe that $du_1^a/dt = u_2^a$, $du_2^a/dt = -u_1^a$, and $(u_1^a)^2 + (u_2^a)^2 \equiv 1$. These facts imply that $u$ and all its derivatives are bounded by 1.

The form $x^a(t) = (a \cos t, a \sin t)'$ for the solutions of (66) may be verified by substitution into the equation: one needs only to check that

\[
\begin{align*}
\frac{\gamma(\sqrt{(a \cos t + u_1^a(t))^2 + (a \sin t + u_2^a(t))^2})}{\sqrt{(a \cos t + u_1^a(t))^2 + (a \sin t + u_2^a(t))^2}} (a \cos t + u_1^a(t)) - (a \sin t + u_2^a(t)) \\
\frac{\gamma(\sqrt{(a \cos t + u_1^a(t))^2 + (a \sin t + u_2^a(t))^2})}{\sqrt{(a \cos t + u_1^a(t))^2 + (a \sin t + u_2^a(t))^2}} (a \sin t + u_2^a(t)) + (a \cos t + u_1^a(t))
\end{align*}
\]

equal $-a \sin t$ and $a \cos t$ respectively. As $(a \cos t, a \sin t)'$ has constant norm $a > 2$ and $u$ has unit norm, the vector $x + u$ has norm always bigger than one, so the two multipliers of the form $\gamma(r)/r$ reduce to $-1/r$. In summary, it suffices to verify that

\[
\begin{align*}
-a \sin t &= -\frac{(a \cos t + u_1^a(t))}{\sqrt{(a \cos t + u_1^a(t))^2 + (a \sin t + u_2^a(t))^2}} - (a \sin t + u_2^a(t)) \\
a \cos t &= -\frac{(a \sin t + u_2^a(t))}{\sqrt{(a \cos t + u_1^a(t))^2 + (a \sin t + u_2^a(t))^2}} + (a \cos t + u_1^a(t))
\end{align*}
\]

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with the above choice of \( u^a \). It can be checked that this is indeed the case.

Note that in the above example, \( \|u^a\|[k] = 1 \) for all \( k \geq 0 \). Hence, the system \( \dot{x} = f(x + u) \) is not \( D^k\text{iss} \) for any \( k \geq 0 \). To see this, suppose that the system is \( D^k\text{iss} \) for some \( k \geq 0 \). Then there exist some \( \beta \in KL \) and \( \sigma \in K \) such that

\[
|x^a(t)| \leq \beta(|x^a(0)|, t) + \sigma \left( \|u^a\|[k] \right) = \beta(a, t) + \sigma(1)
\]

for any \( a > 2 \). Consequently,

\[
\limsup_{t \to \infty} |x^a(t)| \leq \sigma(1)
\]

for any \( a \geq 2 \). This is a contradiction since \( |x^a(t)| \equiv a \) for all \( a > 2 \).

Also note that since the system \( \dot{x} = f(x) \) as in the example is GAS, the system \( \dot{x} = f(x + u) \) is ISS in constant inputs. Thus, the lemma provides an example where a system is ISS in constant inputs but fails to be \( D^k\text{iss} \) for any \( k \geq 0 \).

Below we modify the system to get a system that is \( D\text{iss} \) but not ISS. Thus we obtain an alternative to the counterexample in Section 7.

Let \( f \) be as defined in Lemma 7.2 and consider the system

\[
\dot{z} = \varphi(|z|)f(z) + u,
\]

where \( \varphi(r) = \sqrt{1 + r^2} \geq |r| \). Let \( V(z) = \frac{z_1^2 + z_2^2}{2} \). One has

\[
DV(z)(\varphi(|z|)f(z) + u) = \varphi(|z|) \frac{\gamma \left( \sqrt{z_1^2 + z_2^2} \right)}{|z|} |z|^2 + z \cdot u \leq \gamma \left( \sqrt{z_1^2 + z_2^2} \right) |z|^2 + z \cdot u.
\]

It follows that

\[
|u| \leq \frac{\gamma(|z|)|z|}{2} \Rightarrow DV(z)(\varphi(|z|)f(z) + u) \leq \frac{\gamma(|z|)|z|^2}{2}.
\]

Consequently, system (67) is ISS. According to Lemma 7.2, the system

\[
\dot{x} = \varphi(|x + u|)f(x + u)
\]

is \( D\text{iss} \). Below we show that system (68) is not ISS. To see this, consider, for each \( a > 2 \), the input \( \tilde{u}^a \) defined by \( \tilde{u}^a(t) = u(at) \). Let \( \tilde{x}^a(t) = (a \cos at, a \sin at) \). One has:

- \( |\tilde{u}^a(t)| \equiv 1 \) and \( |\tilde{x}^a(t)| \equiv a \).
• For any $a > 2$, $\tilde{x}^a(t) \cdot \tilde{u}^a(t) = -1$ and 
\[
|\tilde{x}^a(t) + \tilde{u}^a(t)|^2 = |\tilde{x}^a(t)|^2 + 2\tilde{x}^a(t) \cdot \tilde{u}^a(t) + |\tilde{u}^a(t)|^2 = -1.
\]

Hence, $\varphi(|\tilde{x}^a(t) + \tilde{u}^a(t)|) = a$.

Since $\tilde{x}^a(t) = x^a(at)$ and $\tilde{u}^a(t) = u^a(at)$, and since $x^a(t)$ is a solution of (66), it follows that $\tilde{x}^a(t)$ is a solution of the equation
\[
\dot{x}^a(t) = af(\tilde{x}^a(t), \tilde{u}^a(t)).
\]

Combining this with the fact that $\varphi(|\tilde{x}^a(t) + \tilde{u}^a(t)|) = a$, one sees that $\tilde{x}^a(t)$ is a solution of (68) with the input $\tilde{u}^a$. It follows from the fact that $|\tilde{u}^a(t)| \equiv 1$ and $|\tilde{x}^a(t)| \equiv a$ that it is impossible for system (68) to be iISS.

Appendix

A Stability Properties regarding Inputs and Outputs

In this section we briefly discuss some work on some stability properties regarding inputs and outputs. The details can be found in [10], [18] and [17].

Consider finite-dimensional nonlinear systems with output of the following form
\[
\dot{x}(t) = f(x(t), u(t)), \quad y = h(x), \quad \tag{A.1}
\]
where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}^p$ are locally Lipschitz continuous.

We say that the system is input-output-to-state stable (iOSS for short) if there exist $\beta \in \mathcal{K}\mathcal{L}$, $\sigma \in \mathcal{K}$ and $\gamma \in \mathcal{K}$ such that for every trajectory of the system, it holds that
\[
|x(t)| \leq \beta(|x(0)|, t) + \sigma(\|y_{[0,t]}\|) + \gamma(\|u\|) \tag{A.2}
\]
for all $t$ in the maximal interval $[0,T_{\xi,u})$. In the autonomous case when there is no input $u$ acting on the system, the system is called output-to-state stable (oSS for short) if (A.2) holds with $\gamma \equiv 0$.

An iOSS-Lyapunov function $V$ for system $\dot{x} = f(x, u), y = h(x)$ is a smooth function defined on $\mathbb{R}^n$ such that

- there exist $\underline{\sigma}, \overline{\sigma} \in \mathcal{K}_\infty$ such that
  \[
  \underline{\sigma}(|x|) \leq V(x) \leq \overline{\sigma}(|x|) \quad \forall \xi \in \mathbb{R}^n; \tag{A.3}
  \]
there exist $\alpha \in \mathcal{K}_\infty$ and $\rho \in \mathcal{K}$ such that
\[
\frac{\partial V}{\partial x} f(x, u) \leq -\alpha(V(x)) + \rho(||y, u||).
\]

An oss-Lyapunov function $V$ for system $\dot{x} = f(x), y = h(x)$ is a smooth function defined on $\mathbb{R}^n$ such that

- for some $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$, (A.3) holds; and
- there exists some $\alpha \in \mathcal{K}$ such that
\[
\frac{\partial V}{\partial x} f(x) \leq -\alpha(V(x)).
\]

The following is one of the main results in [10]:

**A system is IOSS if and only if it admits an IOSS-Lyapunov function.**

An autonomous system is OSS if and only if it admits an OSS-Lyapunov function.

A forward complete system as in (A.1) is input-to-output stable (IOS for short) if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for every trajectory $x(t)$ and the corresponding output function $y(t) = h(x(t))$ of the system it holds that
\[
|y(t)| \leq \beta(|x(0)|, t) + \gamma(||u||) \quad \forall t \geq 0.
\]

A forward complete system as in (A.1) is state-independent-input-to-output stable (SIOS for short) if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for every trajectory $x(t)$ and the corresponding output function $y(t) = h(x(t))$ of the system it holds that
\[
|y(t)| \leq \beta(|y(0)|, t) + \gamma(||u||) \quad \forall t \geq 0.
\]

The following is one of the main results in [17]:

A forward complete system (A.1) is SIOS if and only if there exists a smooth Lyapunov function $V$ defined on $\mathbb{R}^n$ such that

- for some $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$, it holds that
\[
\underline{\alpha}(|h(x)|) \leq V(x) \leq \overline{\alpha}(|h(x)|) \quad \forall \xi \in \mathbb{R}^n;
\]
- there exist $\alpha \in \mathcal{K}_\infty$ and $\chi \in \mathcal{K}_\infty$ such that
\[
V(x) \geq \chi(|u|) \Rightarrow \frac{\partial V}{\partial x} f(x, u) \leq -\alpha(V(x)).
\]

**B Smooth approximation to measurable, essentially bounded functions**

We need in the text several routine smooth approximation results; for ease of reference, we provide proofs here.
Let \( \varphi \) be measurable, essentially bounded on \([a, b] \). Then there exists a sequence of measurable simple functions \( \{ \varphi_j \} \) that converges to \( \varphi \) almost everywhere on \([a, b] \) such that \( \| \varphi \|_j \leq \| \varphi \| \) (c.f. [21, Theorem 4.13]), (where \( \| \cdot \| \) stands for the \( L_\infty \) norm on \([a, b] \)). Furthermore, it is also easy to see that for every measurable simple function \( \rho \), there exist a sequence of measurable piecewise constant functions \( \{ \rho_j \} \) that converges to \( \rho \) almost everywhere on \([a, b] \), and the \( \{ \rho_j \} \) can be choosen so that \( \| \rho_j \| \leq \| \rho \| \) for all \( j \) (see, for instance, Remark C.1.2 in [15]). In turn, for each piecewise constant function \( \psi : [a, b] \to \mathbb{R} \), one can find a sequence of continuous functions \( \{ \psi_j \} \) that approaches \( \psi \) almost everywhere with the property that \( \| \psi_j \| \leq \| \psi \| \). Finally, by the Weierstrass theorem, each continuous function \( \sigma : [a, b] \to \mathbb{R} \) can be approximated by a sequence of polynomial functions \( \{ \sigma_j \} \) uniformly on \([a, b] \). Since the convergence is uniform, one sees that \( \lim_{j \to \infty} \| \sigma_j \| \leq \| \sigma \| \). Combining the above arguments together, we have the following small variation of Remarks C.1.1 and C.1.2 in [15]:

**Lemma B.1** Let \( \varphi : [a, b] \to \mathbb{R} \) be measurable, essentially bounded. Then there exists an equibounded sequence \( \{ \varphi_j \} \) of smooth functions such that

- \( \varphi_j \to \varphi \) a.e. on \([a, b] \);
- \( \limsup_{j \to \infty} \| \varphi_j \| \leq \| \varphi \| \); and
- by Lebesgue dominated convergence theorem, \( \lim_{j \to \infty} \| \varphi_j - \varphi \|_1 = 0 \), where \( \| \cdot \|_1 \) is the \( L_1 \) norm on \([a, b] \).

Observe that the above approximation result also holds for functions from \([a, b] \) to \( \mathbb{R}^m \).

Let \( k \geq 1 \). Suppose that \( u \in W^{k, \infty}(a, b) \). Let \( \{ \varphi_j \} \) be a sequence of smooth functions that approaches \( \varphi := u^{(k)} \) as in Lemma B.1. Define inductively, for \( i = 1, 2, \ldots, k \)

\[
\varphi_j^i(t) = u^{(k-i)}(a) + \int_a^t \varphi_j^{i-1}(s), ds,
\]

where \( \varphi_j^0(t) = \varphi_j(t) \). Let \( u_j(t) = \varphi_j^k(t) \). It can be seen that, for \( i = 1, \ldots, k \), \( u_j^{(i)} = \varphi_j^{k-i} \).

Since

\[
\left| u_j^{(k-1)}(t) - u^{(k-1)}(t) \right| \leq \int_a^t \left| \varphi_j(s) - u^{(k)}(s) \right| ds \leq \| \varphi_j - \varphi \|_1,
\]

it follows that \( u_j^{(k-1)} \to u^{(k-1)} \) uniformly on \([a, b] \). Processing inductively, one shows that, for \( i = 0, 1, \ldots, k-1 \), \( \{ u_j^{(i)} \} \) converges to \( u^{(i)} \) uniformly on \([a, b] \). It then follows from the uniform convergence that \( \lim_{j \to \infty} \| u_j^{(i)} \| = \lim_{j \to \infty} \| u^{(i)} \| \) for all \( i = 0, 1, \ldots, k-1 \). Hence, we have shown that, for \( k \geq 1 \), if \( u \in W^{k, \infty}(a, b) \), there there exists a sequence of smooth functions \( \{ u_j \} \) that converges to \( u \) uniformly with the property that \( \limsup_{j \to \infty} \| u_j \|^{[k]} \leq \| u \|^{[k]} \). Combining with the case of \( k = 0 \) as stated in Lemma B.1, we get the following:

**Corollary B.2** Let \( k \geq 0 \). Suppose that \( u \in W^{k, \infty}(a, b) \). Then there exists an equibounded
sequence of smooth functions \( \{u_j\} \) that converges to \( u \) pointwise on \([a, b]\) with the property that \( \limsup_{j \to \infty} \|u_j\|_k \leq \|u\|_k \). \( \Box \)

**Acknowledgement** The authors are grateful to Prof. A. Teel for useful discussions.

**References**


