It is shown that every asymptotically controllable system can be stabilized by means of some (discontinuous) feedback law. One of the contributions of the paper is in defining precisely the meaning of stabilization when the feedback rule is not continuous.

1. Introduction

A longstanding open question in nonlinear control theory concerns the relationship between asymptotic controllability to the origin in \( \mathbb{R}^n \) of a nonlinear system

\[
\dot{x} = f(x, u)
\]

(1)

by an “open loop” control \( u : [0, +\infty) \to \mathbb{U} \) and the existence of a feedback control \( k : \mathbb{R}^n \to \mathbb{U} \) which stabilizes trajectories of the system

\[
\dot{x} = f(x, k(x))
\]

(2)

with respect to the origin.

For the special case of linear control systems \( \dot{x} = Ax + Bu \), this relationship is well understood: asymptotic controllability is equivalent to the existence of a continuous (even linear) stabilizing feedback law. But it is well-known that continuous feedback laws may fail to exist even for simple asymptotically controllable nonlinear systems. This is especially easy to see, as discussed in [18], for one-dimensional systems \( (U = \mathbb{R}, n = 1) \) systems (1): in that case asymptotic controllability is equivalent to the property “for each \( x \neq 0 \) there is some value \( u \) so that \( xf(x, u) < 0 \), but it is easy to construct examples of functions \( f \), even analytic, for which this property is satisfied but for which no possible continuous section \( k : \mathbb{R} \setminus \{0\} \to \mathbb{R} \) exists so that \( xf(x, k(x)) < 0 \)

for all nonzero \( x \). General results regarding the nonexistence of continuous feedback were presented in the paper [2], where techniques from topological degree theory were used (an exposition is given in the textbook [16]).

These negative results led to the search for feedback laws which are not necessarily of the form \( u = k(x) \), \( k \) a continuous function. One possible approach consists of looking for dynamical feedback laws, where additional “memory” variables are introduced into a controller, and as a very special case, time-varying (even periodic) continuous feedback \( u = k(t, x) \). Such time-varying laws were shown in [18] to be always possible in the case of one-dimensional systems, and in the major work [7] it was shown that they are also always possible when the original system is completely controllable and has “no drift”, meaning essentially that \( f(x, 0) = 0 \) for all states (see also [17] for numerical algorithms and an alternative proof of the time-varying result for analytic systems). However, for the general case of asymptotically controllable systems with drift, no dynamic or time-varying solutions are known. Thus it is natural to ask about the existence of discontinuous feedback laws \( u = k(x) \). Such feedbacks are often obtained when solving optimal-control problems, for example, so it is interesting to search for general theorems insuring their existence. Unfortunately, allowing nonregular feedback leads to an immediate difficulty: how should one define the meaning of solution \( x(\cdot) \) of the differential equation (2) with discontinuous right-hand side?

One of the best-known candidates for the concept of solution of (2) is that of a Filippov solution (cf. [9]), which is defined as the solution of a certain differential inclusion with multivalued right-hand side which is built from \( f(x, k(x)) \). However, it follows from the results in [13, 8] that the existence of a discontinuous stabilizing feedback in the Filippov sense implies the same Brockett necessary conditions as the existence of a continuous stabilizing feedback does. Moreover, it is shown in [8] that the existence of a stabilizing feedback in the Filippov sense is equivalent to the existence of a continuous stabilizing one, in the case of systems affine in controls. In conclusion, there is no hope of obtaining general results if one
In this paper, we develop a concept of solution of (2) for arbitrary feedback \( k(x) \) which has (a) a clear and reasonable physical meaning (perhaps even more so than the definitions derived from differential inclusions), and (b) allows proving the desired general theorem. Our notion is borrowed from the theory of positional differential games, and it was systematically studied in that context by Krasovskii and Subbotin in [12].

### 1.1. Definition of Feedback Solution

From now on, we assume that \( U \) is a locally compact metric space and that the mapping \( f : \mathbb{R}^n \times U \rightarrow \mathbb{R}^n : (x, u) \mapsto f(x, u) \) is continuous, and locally Lipschitz on \( x \) uniformly on compact subsets of \( \mathbb{R}^n \times U \). We use \( |x| \) to denote the usual Euclidean norm of \( x \in \mathbb{R}^n \), and \( \langle x, y \rangle \) for the inner product of two such vectors.

A locally bounded (for each compact, image is relatively compact) function \( k : \mathbb{R}^n \rightarrow U \) will be called a feedback.

Any infinite sequence \( \pi = \{t_i\}_{i \geq 0} \) consisting of numbers
\[
0 = t_0 < t_1 < t_2 < \ldots
\]
with \( \lim_{i \to \infty} t_i = \infty \) is called a partition (of \([0, +\infty)\)), and the number
\[
d(\pi) := \sup_{i \geq 0} (t_{i+1} - t_i)
\]
is its diameter.

We next define the trajectory associated to a feedback \( k(x) \) and any given partition \( \pi \) as the solution obtained by means of the following procedure: on each interval \([t_i, t_{i+1}]\), the initial state is measured, \( u_i = k(x(t_i)) \) is computed, and then the constant control \( u = u_i \) is applied until time \( t_{i+1} \), when a new measurement is taken. This notion of solution is an accurate model of the process used in computer control (“sampling”).

**Definition 1.1** Assume given a feedback \( k \), a partition \( \pi \), and an \( x_0 \in \mathbb{R}^n \). For each \( i, i = 0, 1, 2, \ldots \), recursively solve
\[
\dot{x}(t) = f(x(t), k(x(t_i))), \quad t \in [t_i, t_{i+1}]
\]
using as initial value \( x(t_i) \) the endpoint of the solution on the preceding interval (and starting with \( x(t_0) = x_0 \)). The \( \pi \)-trajectory of (2) starting from \( x_0 \) is the function \( x(\cdot) \) thus obtained.

Observe that this solution may fail to be defined on all of \([0, +\infty)\), because of possible finite escape times in one of the intervals, in which case we only have a trajectory defined on some maximal interval. In our results, however the construction will provide a feedback for which solutions are globally defined; we say in that case that the trajectory is well-defined.

### 1.2. Statement of Main Result

The main objective of this paper is to explore the relationship between the existence of stabilizing (discontinuous) feedback and asymptotic controllability of the open (ally) stabilizing feedback. This is a feedback law which, for fast enough sampling, drives all states asymptotically to the origin and with small overshoot. Of course, since sampling is involved, when near the origin it is impossible to guarantee arbitrarily small displacements unless a faster sampling rate is used, and, for technical reasons (for instance, due to the existence of possible explosion times), one might also need to sample faster for large states. Thus the sampling rate needed may depend on the accuracy desired when controlling to zero as well as on the rough size of the initial states, and this fact is captured in the following definition. (The “s” in “s-stabilizing” is for “sampling”.)

**Definition 1.2** The feedback \( k : \mathbb{R}^n \rightarrow U \) is said to s-stabilize the system (1) if for each pair \( 0 < r < R \) there exist \( M = M(R) > 0, \delta = \delta(r, R) > 0, \) and \( T = T(r, R) > 0 \) such that, for every partition \( \pi \) with \( d(\pi) < \delta \) and for any initial state \( x_0 \) such that \( |x_0| \leq R \), the \( \pi \)-trajectory \( x(\cdot) \) of (2) starting from \( x_0 \) is well-defined and it holds that:

1. (uniform attractiveness) \( |x(t)| \leq r \) for all \( t \geq T \);
2. (overshoot boundedness) \( |x(t)| \leq M(R) \) for all \( t \geq 0 \);
3. (Lyapunov stability) \( \lim_{t \to 0} M(R) = 0 \).

We next recall the definition of (global, null-) asymptotic controllability. By a control we mean a measurable function \( u : [0, +\infty) \rightarrow U \) which is locally essentially bounded (meaning that, for each \( T > 0 \) there is some compact subset \( U^T \subseteq U \) so that \( u(t) \in U^T \) for a.a. \( t \in [0, T] \)). In general, we use the notation \( x(t; x_0, u) \) to denote the solution of (1) at time \( t \geq 0 \), with initial condition \( x_0 \) and control \( u \). The expression \( x(t; x_0, u) \) is defined on some maximal interval \([0, t_{\text{max}}(x_0, u)]\).

**Definition 1.3** The system (1) is asymptotically controllable if:

1. (attractiveness) for each \( x_0 \in \mathbb{R}^n \) there exists some control \( u \) such that the trajectory \( x(t) = x(t; x_0, u) \) is defined for all \( t \geq 0 \) and \( x(t) \rightarrow 0 \) as \( t \rightarrow +\infty \);
2. (Lyapunov stability) for each \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for each \( x_0 \in \mathbb{R}^n \) with \( |x_0| < \delta \) there is a control \( u \) as in 1. such that \( |x(t)| < \varepsilon \) for all \( t \geq 0 \);
3. (bounded controls) there are compact subsets \( X_0 \) and \( U_0 \) of \( \mathbb{R}^n \) and \( U \) respectively such that, if the initial state \( x_0 \) in 2. satisfies also \( x_0 \in X_0 \), then the control in 2. can be chosen with \( u(t) \in U_0 \) for almost all \( t \).

This is a natural generalization to control systems of the concept of uniform asymptotic stability of solutions of differential equations. The last property – which is not part of the standard definition of asymptotic controllability given in textbooks, e.g. [16] – is introduced here for
Theorem 1 The system (1) is asymptotically controllable if and only if it admits an s-stabilizing feedback.

One implication is trivial: existence of an s-stabilizing feedback is easily seen to imply asymptotic controllability. Note that the bounded overshoot property, together with the fact that \( k \) is locally bounded, insures that the control applied (namely, a piecewise constant control which switches at the “sampling times” in the partition) is bounded. The attractiveness property holds by iteratively controlling to balls of small radius and using the overshoot and stability estimate to insure convergence to the origin. Finally, the Lyapunov stability property holds by construction.

The interesting implication is the converse, namely the construction of the feedback law. The main ingredients in this construction are: (a) the notion of control-Lyapunov function (called just “Lyapunov function” for a control system in [16]; see also [10, 11]), (b) methods of nonsmooth analysis, and (c) techniques from positional differential games. We review these ingredients in the next section, and then develop further technical results; the last section contains the proof.

2. Some Preliminaries

We start with a known characterization of asymptotic controllability in terms of control-Lyapunov functions. Given a function \( V : \mathbb{R}^n \to \mathbb{R} \) and a vector \( v \in \mathbb{R}^n \), the lower directional derivative of \( V \) in the direction of \( v \) is

\[
DV(x; v) := \liminf_{t \downarrow 0} \frac{1}{t} (V(x + tv) - V(x)).
\]

The function \( v \mapsto DV(x; v) \) is lower semicontinuous. For a set \( F \subseteq \mathbb{R}^n \), \( \text{co} F \) denotes its convex hull.

Definition 2.4 A control-Lyapunov pair for the system (1) consists of two continuous functions \( V, W : \mathbb{R}^n \to \mathbb{R}_{\geq 0} \) such that the following properties hold:

1. (positive definiteness) \( V(x) > 0 \) and \( W(x) > 0 \) for all \( x \neq 0 \), and \( V(0) = 0 \);
2. (properness) the set \( \{ x \mid V(x) \leq \beta \} \) is bounded for each \( \beta \);
3. (infinitesimal decrease) for each bounded subset \( G \subseteq \mathbb{R}^n \) there is some compact subset \( U_0 \subseteq U \) such that

\[
\min_{v \in \text{co}f(x; U_0)} DV(x; v) \leq -W(x) \quad (4)
\]

for every \( x \in G \).

If \( V \) is part of a control-Lyapunov pair \((V, W)\), it is a control-Lyapunov function (clf).

Observe that when the function \( V \) is smooth, condition (4) can be written in the more familiar form found in the literature, namely:

\[
\min_{u \in U_0} \langle \nabla V(x), f(x, u) \rangle \leq -W(x). \quad (5)
\]

In contrast to the situation with stability of (noncontrolled) differential equations, a system may be asymptotically controllable system and yet there may not exist any possible smooth clf \( V \). In other words, there is no analogue of the classical theorems due to Massera and Kurzweil. This issue is intimately related to that of existence of continuous feedback, via what is known as Artstein’s Theorem (cf. [1, 10, 11, 15]), which asserts that existence of a differentiable \( V \) is equivalent, for systems affine in controls, to there being a stabilizing regular feedback. Nevertheless, it is possible to reinterpret the condition (5) in such a manner that Equation (5) does hold in general, namely by using a suitable generalization of the gradient. Specifically, we will remark later that we may use the proximal subgradients of \( V \) at \( x \) instead of \( \nabla V(x) \), replacing (5) by:

\[
\min_{u \in U_0} \langle \zeta, f(x, u) \rangle \leq -W(x) \quad \text{for every } \zeta \in \partial_p V(x), \quad (6)
\]

where \( \zeta \) and \( \partial_p V(x) \) are the proximal subgradients and the subdifferential, respectively, of the function \( V \) at the point \( x \). The use of proximal subgradients as substitutes for the gradient for a nondifferentiable function plays a central role in our construction of feedback. The concept was originally developed in nonsmooth analysis for the study of optimization problems, see [3].

Finally, we rely on methods developed in the theory of positional differential games in [12]. These techniques were used together with nonsmooth analysis tools in the construction of discontinuous feedback for differential games of pursuit in [5] and games of fixed duration in [6], and these results are relevant to the construction of stabilizing feedback in our main result.

3. Proximal Subgradients and Inf-Convolutions

We recall the concept of proximal subgradient, one of the basic building blocks of nonsmooth analysis.

A vector \( \zeta \in \mathbb{R}^n \) is a proximal subgradient (respectively, supergradient) of the function \( V : \mathbb{R}^n \to (-\infty, +\infty] \) at \( x \) if there exists some \( \sigma > 0 \) such that, for all \( y \) in some neighborhood of \( x \),

\[
V(y) \geq V(x) + \langle \zeta, y - x \rangle - \sigma |y - x|^2 \quad (7)
\]
will be of special interest; it is simultaneously a proximal
for all regularization” of (convex)
known in classical convex analysis as the “Iosida-Moreau
convolution
This is called the inf-convolution of the function V (with
a multiple of the squared-norm). The function \( V_\alpha \) is well-
known in classical convex analysis as the “Iosida-Moreau
regularization” of (convex) \( V \). In the case of a lower semi-
continuous \( V \) which is bounded below (for instance, for a
clf \( V \), which is continuous and nonnegative), the function
\( V_\alpha \) is locally Lipschitz and is an approximation of \( V \)
in the sense that \( \lim_{\alpha \to 0} V_\alpha(x) = V(x) \); cf. [4]. In this case, the set of minimizing points \( y \) in (9) is nonempty. We
choose one of them and denote it by \( y_\alpha(x) \). (We are not
asserting the existence of a regular choice; any assignment
will do for our purposes.)

By definition, then, \( V_\alpha(x) = V(y_\alpha(x)) + \frac{1}{2\alpha^2} |y_\alpha(x) - x|^2 \leq V(y) + \frac{1}{2\alpha^2} |y - x|^2 \)
for all \( y \in \mathbb{R}^n \). The vector
\[ \zeta_\alpha(x) := \frac{x - y_\alpha(x)}{\alpha^2} \]
will be of special interest; it is simultaneously a proximal
subgradient of \( V \) at \( y_\alpha(x) \) and a proximal supergradient
of \( V_\alpha \) at \( x \), as we discuss next.

**Lemma 3.5** For any \( x \in \mathbb{R}^n \),
\[ \zeta_\alpha(x) \in \partial_p V(y_\alpha(x)) \, . \]

**Lemma 3.6** For any \( x \in \mathbb{R}^n \), \( \zeta_\alpha(x) \) is a proximal supergradient of \( V_\alpha \) at \( x \).

In fact, we also have:
\[ V_\alpha(y) \leq V_\alpha(x) + \langle \zeta_\alpha(x), y - x \rangle + \frac{1}{2\alpha^2} |y - x|^2 \, . \] (13)

We deduce from Equation (13) that, for any \( \tau \in \mathbb{R}^1 \)
and any \( v \in \mathbb{R}^n \),
\[ V_\alpha(x + \tau v) \leq V_\alpha(x) + \tau \langle \zeta_\alpha(x), v \rangle + \frac{\tau^2 |v|^2}{2\alpha^2} \, . \] (14)

This plays a role analogous to that of Taylor expansions
for evaluating increments of the function \( V_\alpha \).

4. Semi-Global Practical Stabilization

From now on, we assume that an asymptotically con-
trollable system (1) has been given. Applying Theorem 2,
we pick a Lyapunov pair \((V, W)\). Choose any \( 0 < r < R \).
We will construct a feedback that sends all states in the
ball of radius \( R \) (“semiglobal” stabilization) into the ball
of radius \( r \) (“practical” stability, as we do not yet ask
that states converge to zero). By the definition of Lyapunov
pair, there is some compact subset \( U_0 \subseteq U \) so that
property (4) holds for every \( x \in B_{R + \sqrt{2\beta(R)}} \). Because of
Theorem says that there is a feedback that steers every directional derivatives, we have that also condition (6) holds for the Lyapunov pair, for all \( x \in B_{R + \sqrt{2\beta(R)}} \).

Pick any \( \alpha \in (0, 1) \). In terms of the vectors \( \zeta_\alpha(x) \) introduced in Equation (11), we define a function \( \zeta_\alpha : B_R \to \mathbb{U}_0 \) by letting \( k_\nu(x) \) be a pointwise minimizer of \( \langle \zeta_\alpha(x), f(x, u) \rangle \):

\[
\langle \zeta_\alpha(x), f(x, k_\nu(x)) \rangle = \min_{u \in \mathbb{U}_0} \langle \zeta_\alpha(x), f(x, u) \rangle.
\] (22)

The choice \( x \to k_\nu(x) \) is not required to have any particular regularity properties. We use the subscript \( \nu = (\alpha, r, R) \) to emphasize the dependence of the function \( k \) on the particular parameters (which represent respectively the “degree of smoothing” of \( V \) that is used in its construction, the radius of the ball to which we are controlling, and the radius of the ball on which the feedback will be effective). The next theorem says that for any fixed \( r, R \), we can choose arbitrarily small \( \alpha > 0 \) and then \( \delta > 0 \) such that the set \( G^\alpha_R \) is invariant with respect to any \( \pi \)-trajectory of

\[
\dot{x} = f(x, k_\nu(x)),
\] (23)

and that \( x(t) \) enters and stays in \( B_r \) for all large \( t \), provided that the diameter of the partition \( \pi \) satisfies \( d(\pi) \leq \delta \).

**Theorem 3** Let \( V \) be a clf. Then, for any \( 0 < r < R \) there are \( \alpha_0 = \alpha_0(r, R) \) and \( T = T(r, R) \) such that, for any \( \alpha \in (0, \alpha_0) \) there exists \( \delta > 0 \) such that for any \( x_0 \in G^\alpha_R \) and any partition \( \pi \) with \( d(\pi) \leq \delta \), the \( \pi \)-trajectory \( x(\cdot) \) of (23) starting at \( x_0 \) must satisfy:

\[
x(t) \in G^\alpha_R, \quad \forall t \geq 0
\] (24)

and

\[
x(t) \in B_r, \quad \forall t \geq T.
\] (25)

Observe that, because of (17) and (19), for every \( R' > 0 \) there is some \( R > 0 \) such that \( B_{R'} \subseteq G^\alpha_R \). Thus the Theorem says that there is a feedback that steers every state of \( B_{R'} \) into the neighborhood \( B_r \); in this sense the result is semiglobal. The proof of the Theorem will take the rest of this section and will be based on a sequence of lemmas. The main idea behind the proof is to use the new function \( V_\alpha \) (for sufficiently small \( \alpha \)) as a Lyapunov function and to use the “Taylor expansion” formula (14) to estimate the variations of \( V_\alpha \) along \( \pi \)-trajectories.

By continuity of \( f(x, u) \) and the local Lipschitz property assumed, we know that there are some constants \( \ell, m \) such that

\[
|f(x, u) - f(x', u)| \leq \ell |x - x'|, \quad |f(x, u)| \leq m
\] (26)

for all \( x, x' \in B_R \) and all \( u \in \mathbb{U}_0 \). Let

\[
\Delta := \frac{1}{3} \min \left\{ W(y) \mid \frac{1}{2} \rho(r) \leq |y| \leq R + \sqrt{2\beta(R)} \right\}.
\]

Note that \( \Delta > 0 \), due to the positivity of \( W \) for \( x \neq 0 \).

\[
\sqrt{2\beta(R)} \alpha < \frac{1}{2} \rho(r), \quad 2\ell \omega_R \left( \sqrt{2\beta(R)} \alpha \right) < \Delta.
\] (27)

Then, for any \( x \in B_R \setminus B_{\rho(r)} \) it holds that

\[
\langle \zeta_\alpha(x), f(x, k_\nu(x)) \rangle \leq -2\Delta.
\] (28)

Now we consider any \( \pi \)-trajectory of (23) corresponding to a partition \( \pi = \{i_1\}_{i \geq 0} \) with \( d(\pi) \leq \delta \), where \( \delta \) satisfies the inequality

\[
\left( \frac{\ell m \sqrt{2\beta(R)}}{2\alpha} + \frac{m^2}{\alpha^2} \right) \delta \leq \Delta.
\] (29)

**Lemma 4.9** Let \( \alpha, \delta \) satisfy (20), (27), and (29), and assume that for some index \( i \) it is the case that \( x(t_i) \in G^\alpha_R \setminus B_{\rho(r)} \). Then

\[
V_\alpha(x(t_i)) - V_\alpha(x(t_i)) \leq -\Delta(t - t_i)
\] (30)

for all \( t \in [t_i, t_{i+1}] \). In particular, \( x(t) \in G^\alpha_R \) for all such \( t \).

Next we establish that every \( \pi \) trajectory enters the ball \( B_r \) at time \( t_N \), where \( N \) is the least integer such that \( x(t_N) \in G_r \), and stays inside thereafter. To do this, we first show that there is a uniform upper bound on such times \( t_N \).

**Lemma 4.10** Let \( \alpha \) satisfy (20) and (27), and pick any \( \delta \) so that (29) is valid. Then, for any \( \pi \)-trajectory \( x(\cdot) \) with \( d(\pi) \leq \delta \) and every \( x(0) \in G^\alpha_R \), it holds that

\[
t_N \leq T = \frac{\gamma(R)}{2\Delta}.
\] (31)

**Lemma 4.11** Assume that, in addition to the previously imposed constraints, \( \alpha \) and \( \delta \) also satisfy the following two conditions:

\[
\omega_R \left( \sqrt{2\beta(R)} \alpha \right) < \frac{1}{4} \gamma(r),
\] (32)

(which actually implies (20)) and

\[
\omega_R (m \delta) < \frac{1}{4} \gamma(r).
\] (33)

Then, \( x(t) \in B_r \) for all \( t \geq t_N \).

To conclude the proof of Theorem 3, let \( \alpha_0 \) be the supremum of the set of all \( \alpha \) which satisfy conditions (27) and (32). Then, for any \( \alpha \in (0, \alpha_0) \) we can choose \( \delta \) satisfying (29) and (33), so that, for each partition with \( d(\pi) \leq \delta \) and every \( \pi \)-trajectory starting from a state in \( G^\alpha_R \), the inclusions (24) and (25) hold.

5. **Proof of Global Result**

We now prove Theorem 1. The idea is to partition the state space \( \mathbb{R}^n \) into a number of “spherical shells” (more precisely, sets built out of sublevel sets of the functions \( \zeta_\alpha(x) \) and \( \zeta_\alpha(x) \)). For each such shell, we can find a function \( V_\alpha \) which is positive definite and has continuous derivatives, and which is strictly decreasing along \( \pi \)-trajectories.

By continuity of \( f(x, u) \) and the local Lipschitz property assumed, we know that there are some constants \( \ell, m \) such that

\[
|f(x, u) - f(x', u)| \leq \ell |x - x'|, \quad |f(x, u)| \leq m
\] (26)

for all \( x, x' \in B_R \) and all \( u \in \mathbb{U}_0 \). Let

\[
\Delta := \frac{1}{3} \min \left\{ W(y) \mid \frac{1}{2} \rho(r) \leq |y| \leq R + \sqrt{2\beta(R)} \right\}.
\]

Note that \( \Delta > 0 \), due to the positivity of \( W \) for \( x \neq 0 \).
for each integer $T$, we have:

$$2R_j \leq \rho(R_{j+1}), \quad j = 0, \pm 1, \pm 2, \ldots$$

(just define the $R_j$'s inductively for $j = 1, 2, \ldots$ and for $j = -1, -2, \ldots$; this is possible because of (17)). We also denote $r_j := \frac{1}{2}\rho(R_{j-1})$ for all $j$. We have that, for each integer $j$,

$$\rho(R_j) < R_j < 2R_j < \rho(R_{j+1}) \quad (34)$$

and

$$\lim_{j \to -\infty} R_j = 0, \quad \lim_{j \to +\infty} R_j = \infty. \quad (35)$$

Consider any integer $j$. For the pair $(r_j, R_j)$, Theorem 3 provides the existence of numbers $\alpha_j > 0$, $\delta_j > 0$, and $T_j > 0$, and a map $k_j : B_{R_j} \to U_j$, $k_j := k(\alpha_j, r_j, R_j)$, such that $G_{R_j}^{\alpha_j}$ is invariant with respect to all trajectories of (23) when $x = k_j$ and $d(\pi) \leq \delta_j$, and for each such trajectory it holds that

$$|x(t)| \leq r_j, \quad \forall t \geq T_j. \quad (36)$$

Recall that in the construction of $k_j$, we used the fact that there is some compact subset $\mathbb{U}_0 \subseteq \mathbb{U}$, to be called here $U_j$, to distinguish the sets used for the different indices $j$, so that condition (6) holds for the Lyapunov $R_j$ for all $x \in B_{R_j}$. Since the $R_j$ form an increasing sequence, and since if the min condition (6) also holds if we enlarge $U_0$ we may, and will, assume that the $U_j = \mathbb{U}_0$ for all $j \leq 0$ and that $U_j \subseteq U_{j+1}$ for all $j \geq 0$. In Equation (26) we picked a bound $n$ on the values of $|f(x, u)|$ for $x \in B_{R_j}$ and all $u \in U_j$; we call this bound $m_j$ to emphasize the dependence on $j$, and observe that $m_j \leq m_{j+1}$ for all $j$, because of the monotonicity of the sets $U_j$ and $B_{R_j}$.

Finally, we introduce the sets on which we will use the different feedbacks $k_j$:

$$H_j := G_{R_j}^{\alpha_{j+1}} \setminus G_{R_j}^{\alpha_j}. \quad (37)$$

Since the sets $H_j$ plus the origin constitute a partition of the state space, we may define a map $k : \mathbb{R} \to \mathbb{U}$ by means of the rule

$$k(x) := k_j(x), \quad \forall x \in H_j, \quad (37)$$

for each integer $j$, and $k(0) = u_0$, where $u_0$ is any fixed element of $U_0$. It is not difficult to show that this feedback is s-stabilizing. We must omit the proof due to space limitations; the full paper can be obtained by electronic mail from the authors.