On First-Order Equations for Multidimensional Filters

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Abstract—A construction is given to obtain first-order equation representations of a multidimensional filter, whose dimension is of the order of the degree of the transfer function.

I. INTRODUCTION

Let \( k \) be any field, \( m, t, \) integers. Let \( k[z_1, z_2] \) (respectively, \( k(z_1, z_2) \)) denote the ring of polynomials (respectively, field of rational functions) in two indeterminates \( z_1, z_2 \). Let the degree of a polynomial \( p = p(z_1, z_2) \) in \( z_1 \) be denoted by \( d_1 p \).

The degree of \( p \) is then \( dp = (d_1 p, d_2 p) \). A rational function \( w(z_1, z_2) = p(z_1, z_2)q(z_1, z_2)^{-1} \) is causal iff \( dp \leq dq, i = 1, 2 \) and \( q = x_1^{d_1}x_2^{d_2}x_1^{d_1}x_2^{d_2} + \cdots \). Denote then \( d_w = (d_1 w, d_2 w) \). A transfer function is a matrix

\[
W(z_1, z_2) = (w_{ij}(z_1, z_2)) \in k(z_1, z_2)^{m \times n}
\]

where each \( w_{ij} \) is causal.

A system \( \Sigma \) of dimension \( (n_1, n_2) \) is given by four matrices \( (F, G, H, J) \) where \( F \in k^{n \times n}, G \in k^{n \times n}, H \in k^{n \times t}, J \in k^{t \times t} \) \((n = n_1 + n_2)\).

Introduce the polynomial matrix

\[
A_{n_1, n_2} = \begin{bmatrix}
z_1I_{n_1} & 0 \\
0 & z_2I_{n_2}
\end{bmatrix}
\]

where \( I_r \) is the identity matrix of dimension \( r \). Then, for each \( \Sigma \) of dimension \( (n_1, n_2) \), let

\[
W_{\Sigma}(z_1, z_2) = H(A_{n_1, n_2} - F)^{-1}G.
\]

It is easy to verify that \( W_{\Sigma} \) is a transfer function. When \( m = t = 1, d_w W_{\Sigma} \leq n_1 \). The realization problem that we wish to consider is: "given a transfer function \( W \), find \( \Sigma \) with \( W_{\Sigma} = W \)." The motivation for this problem is the following. A transfer function describes a recursive "northeast causal" two-dimensional filter, and a system realization \( \Sigma \) corresponds to a set of first-order equations realizing the corresponding filter; specifically, denoting

\[
F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}, \quad G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}, \quad H = (H_1, H_2),
\]

then \( \Sigma \) corresponds to a system of equations

\[
x_1(h + 1, k) = F_{11}x_1(h, k) + F_{12}x_2(h, k) + G_1u(h, k)
\]

\[
x_2(h, k + 1) = F_{21}x_1(h, k) + F_{22}x_2(h, k) + G_2u(h, k)
\]

\[
y(h, k) = H_1x_1(h, k) + H_2x_2(h, k) + Ju(h, k)
\]

where \( x_1(\cdot, \cdot) : Z \times Z \rightarrow k^n \).

Generalizations to transfer functions in three or more variables \( z_1, \ldots, z_r \) are straightforward and will not be discussed here.

Models of the above type were independently suggested by Roesser [8] and Sontag [9]; the latter reference also explains how the same mathematical problem appears in modeling neutral delay-differential systems.

In the present paper we expand in detail the realization method outlined in [9, discussion (5.6)]. The results were presented at the Amherst Workshop on Algebraic System Theory, held at the University of Massachusetts, June 14-19, 1976.

II. MAIN RESULTS

Theorem (2.1): Every \( W(z_1, z_2) \) has a realization.

Theorem (2.2): Let \( m = t = 1, d_w = (n_1, n_2) \).

Then

a) there is a realization \( \Sigma \) of dimension \( (n_1, 2n_2) \);

b) if \( q(z_1, z_2) = q_1(z_1) q_2(z_2) \), there is a realization \( \Sigma \) of dimension \( (n_1, n_2) \).

Analogous conclusions hold, of course, by reversing the roles of \( z_1 \) and \( z_2 \).

Note that b) deals with what are usually called "separable" transfer functions. We shall prove both of the above theorems as corollaries of a general construction, outlined in the rest of this section. It is this construction, rather than the theorems themselves, which we consider to be the main contribution of this work. In the next section we shall present a conjecture for the case when \( k \) is algebraically closed, and in the last
section we indicate the connection between certain realizations of "separable" transfer functions and the theory of "recognizable" power series as developed by Fliess [3].

Let \( k((z_2)) \) denote the set of causal transfer functions in the variable \( z_2 \) only, i.e., the set of all \( p(z_2) q(z_2)^{-1} \) with \( \deg q \geq \deg p \). We introduce the notion of a 

**system \( R \) over \( k((z_2)) \) of dimension \( r \):**

this is just a 4-tuple of matrices

\[
R = (A(z_2), B(z_2), C(z_2), D(z_2)),
\]

where \( A, B, C, \) and \( D \) are matrices over \( k((z_2)) \) of dimensions \( r \) by \( r \), \( m \) by \( r \), \( t \) by \( r \), and \( t \) by \( m \), respectively. (This is a particular case of a *system over a commutative ring; the general concept is studied in [10].)

A system \( R \) as above induces a transfer function

\[
W_R(z_1, z_2) = C(z_2)(z_1 I_r - A(z_2))^{-1}B(z_2) + D(z_2). 
\]

Conversely, each \( W(z_1, z_2) \) has an \( R \)-realization, i.e., is of the form \( W_R \) for some \( R \). Indeed, given \( W \) there is some \( r \) such that we may write

\[
W(z_1, z_2) = (P_r(z_2) - P_0(z_2))^{-1}(q_0(z_2) + q_1(z_2)z_2^{-1} + \cdots + q_0(z_2)z_2^{-r})^{-1} 
\]

\[
= P_r(z_2)q_0(z_2) + (T_{r-1}(z_2)z_2^{-r} + \cdots + T_0(z_2)) 
\]

\[
\times (z_1 I_r + u_{r-1}(z_2)z_2^{-r-1} + \cdots + u_0(z_2))^{-1}
\]

where the \( P_r \) are polynomial matrices, \( q_j \) are polynomials, and

\[
u_j(z_2) = q_j(z_2)q_r(z_2)^{-1} \quad J_r(z_2) = (P_r(z_2) - P_0(z_2)u_j(z_2))q_r(z_2)^{-1}.
\]

Then \( W = W_R \) where \( R = (A, B, C, D) \) is the \( r \)-dimensional system over \( k((z_2)) \) given by

\[
A = \begin{bmatrix} 0 & 0 & \cdots & 0 & -u_0 I_r \\ I_t & 0 & \cdots & 0 & -u_1 I_r \\ & \ddots & \ddots & \ddots & \vdots \\ & & 0 & 0 & -u_{r-1} I_r \\ 0 & 0 & \cdots & I_t & -u_r - I_r \end{bmatrix}, \quad B = \begin{bmatrix} T_0 \\ \vdots \\ T_{r-1} \end{bmatrix},
\]

\[
C = [0 \cdots 0 I_t], \quad D = P_rq_r^{-1}. 
\]

The proof of Theorems (2.1) and (2.2) will be completed upon finding, for each \( R \), a suitable \( \Sigma \) with \( W_\Sigma = W_R \). We study then the possible passages from \( W_R \) to \( W_\Sigma \). To study this, we note that for any \( \Sigma = (F, G, H) \), \( W_\Sigma = W_R \) when \( R \) is defined by (using the notations (1.1))

\[
A(z_2) = F_{11} + F_{12}(z_2 I_n - F_{22})^{-1}F_{21},
\]

\[
B(z_2) = G_1 + F_{12}(z_2 I_n - F_{22})^{-1}G_2
\]

\[
C(z_2) = H_1 + H_2(z_2 I_n - F_{22})^{-1}F_{21}
\]

\[
D(z_2) = J + H_2(z_2 I_n - F_{22})^{-1}G_2.
\]

Equivalently, denoting

\[
\tilde{W}_R(z_2) = \begin{bmatrix} F_{12} \\ H_2 \end{bmatrix} (z_2 I_n - F_{22})^{-1}(F_{21}, G_2),
\]

we see that

\[
\begin{bmatrix} A(z_2) & B(z_2) \\ C(z_2) & D(z_2) \end{bmatrix} = \begin{bmatrix} F_{11} & G_1 \\ H_1 & J \end{bmatrix} + \tilde{W}_R(z_2).
\]

Therefore, if we begin with \( R = (A, B, C, D) \), finding \( \Sigma \) is equivalent to obtaining a decomposition (2.4). In other words, we must solve a "minimal realization problem" for the transfer matrix (in \( z_2 \)) in the left-hand side of (2.4). It is well known from linear system theory (see, for instance, [2, p. 219]) that the minimal possible dimension \( n_2 \) is given by

\[
n_2 = \text{M.d.} \ \tilde{W}_R(z_2) = \text{McMillan degree of} \ \tilde{W}_R(z_2)
\]

where

\[
\text{M.d.} \ \tilde{W}_R(z_2) = \text{degree of the least common denominator of all minors of} \ \tilde{W}_R(z_2).
\]

The problem of finding \( \Sigma \) can thus be decomposed into two parts: first find \( R \), for instance (but not necessarily!), via (2.3), obtaining \( n_1 \); then construct \( \Sigma \) with \( n_2 = \text{M.d.} \ \tilde{W}_R(z_2) \). There are well-known algorithms for such a construction [2, p. 235 ff.]. It must be noted that different \( R \) can lead to different \( \tilde{W}_R(z_2) \), so both \( n_1 \) and \( n_2 \) depend on the construction. Moreover, one may instead first "extract" \( z_1 \) and then \( z_2 \), obtaining a different result. Although the method can be expected to give realizations of rather low dimensions, it is clear that much more research is needed before the situation becomes well understood.

In any case, the bounds of Theorem (2.2) are easily proved from the above construction via (2.3) and (2.4). Indeed,

\[
\tilde{W}_R(z_2) = \begin{bmatrix} \text{constants} \\ -v_0 & T_0 \\ \vdots & \vdots \\ -v_{r-1} & T_{r-1} \end{bmatrix};
\]

since \( -v_0 T_0 - T_1(-v_0) = P_rq_q^{-1} \), it follows that \( q_q^{2} \) is a common denominator for all minors, so

\[
\text{M.d.} \ \tilde{W}_R(z_1) \leq 2 \deg q_q = 2n_2,
\]

proving a). And b) is clear since it means that all \( v_j \) are constant.

**Example (2.5):** Let

\[
W(z_1, z_2) = \frac{z_1 + z_2}{z_1 z_2 - 1}
\]

\( k \) is real numbers. Then an \( R \)-realization is

\[
x_1 x = z_1 x + (1 + z^2) u 
\]

\[
y = x + z_1 u
\]

and a \((1, 2)\)-dimensional \( \Sigma \)-realization is

\[
x_1(h + 1, k) = x_1(h, k) + u(h, k)
\]

\[
x_2(h, k + 1) = x_2(h, k) + x_3(h, k)
\]

\[
x_3(h, k + 1) = u(h, k)
\]

\[
y = x_1 + x_3.
\]

III. A CONJECTURE

**Conjecture (3.1):** Let \( k \) be algebraically closed (for instance, \( k = \text{complex numbers} \)). Let \( W(z_1, z_2) \) be a scalar (i.e., \( m = t = 1 \)) transfer function of degree \((n_1, n_2)\). Then there exists a realization \( \Sigma \) of \( W \) such that \( \dim \Sigma = (n_1, n_2) \).

When \( n_1 \) (or \( n_2 \)) = 1, this conjecture is true, since one may always factor \( T_0 = (P_0 q_0^{-1}) \cdot (q_0 q_0^{-1}) \) as a product of causal \( z_2 \)-transfer functions, so \( R = (-q_0 q_0^{-1}, P_0 q_0^{-1}, q_0 q_0^{-1}, P_r q_r^{-1}) \) gives rise to a \( W_R(z_2) \) with McMillan degree \( n_2 \).

**Example (3.2):** Consider again the \( W \) in (2.5), but this time let \( k = \text{complex numbers} \). A direct calculation shows that a \( \Sigma \) of dimension \((1, 1)\) realizes \( W \) if and only if

\[
x_1(h + 1, k) = ax_2(h, k) + bu(h, k)
\]

\[
x_2(h, k + 1) = a^{-1} u_2(h, k) + cu(h, k)
\]

\[
y = b^{-1} x_1 + c^{-1} x_2
\]

and \((abc)^2 + 1 = 0\).
Remarks (3.3):
- There can be no rational procedure for finding realizations of minimal dimensions. Indeed, such a procedure, involving only additions and multiplications, would give the same result independently of the field of realization. But Examples (2.5) and (3.2) show that minimality is field-dependent. This situation is completely at variance with the ordinary linear-system case.
- If the conjecture is true, one may expect also, by an algebraic-geometric argument, generically finitely many realizations under the group $GL(n_1) \times GL(n_2)$ acting in the obvious way.

IV. RECOGNIZABLE TRANSFER FUNCTIONS

A transfer function is recognizable iff every entry has the form $q(z_1)^{-1} q(z_2)^{-1}$. We now show how some results of [4] can be translated into the context of two-dimensional filters by looking at another type of first-order difference equations.

We consider in this section systems $\Gamma = (F_1, F_2, G, H)$ of dimension $n$ given by equations of type

$$x(h, k) = F_1 x(h - 1, k) + F_2 x(h, k - 1) + Gw(h, k)$$

$$y = H x$$

where $x(h, k)$ is in $k^n$ and $F_1 F_2 = F_2 F_1$.

Theorem (4.1): $W(z_1, z_2)$ has a $\Gamma$-realization if and only if $W(z_1, z_2)$ is recognizable. Furthermore, expanding

$$W(z_1, z_2) = \sum_{i,j \geq 0} A_{ij} z_1^i z_2^j,$$

the dimension of a minimal $\Gamma$-realization is equal to the rank $n$ of the block Hankel matrix $H(W)$ which has rows and columns indexed by the pairs $(i, j)$ and the $(i, j)$th block is

$$\begin{pmatrix} i+j \end{pmatrix}^{-1} A_{ij}.$$

Any two realizations of this dimension $n$ are equal except for a change of basis in $k^n$.

Proof: Easy consequence of [4]. The proof by Fliess provides an explicit $\Gamma$-realization of dimension $n = \text{rank } H(W)$.

Remarks (4.2):
- Representations of “separable,” i.e., recognizable, transfer functions by systems somewhat similar to our $\Gamma$ were given by Attasi [11] and Fornasini and Marchesini [5], [16], adding a term of the form $-F_1 F_2 x(h - 1, k - 1)$ to the right side. This minor variation of the above can be treated also via the theory of recognizable series.
- In general, there are $\Sigma$-realizations of dimension $(n_1 + n_2)$, much less than the dimensions of possible $\Gamma$-realizations. For instance, a straightforward modification of an example of [6] is $W(z_1, z_2) = \sum (z_1^{-1} z_2^{-1})$, which has a $(1, 1)$-dimensional $\Sigma$-realization [by Theorem (2.2)], but rank $H(W) = 4$.

V. FINAL REMARKS

The bound given in Theorem (2.2a) improves considerably that given by [5], [6], in which the dimension of $\Sigma$ is of the order of $n_1 + n_2$, rather than $n_1 + n_2$.

In an interesting recent paper, Kung et al. [7] have independently arrived at Theorem (2.2).

ACKNOWLEDGMENT

The author wishes to express his gratitude to Dr. E. W. Kamen for introducing him to the operator study of delay-differential systems, and hence, indirectly, to the present topic, and to Prof. M. A. Arbib and Dr. E. G. Manes for the opportunity of presenting his results at the Amherst Workshop.

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