

The geometry of photon surfaces

Clarissa-Marie Claudel^{a)}

Mathematical Institute, University of Umeå, 901 87 Umeå, Sweden

K. S. Virbhadra^{b)} and G. F. R. Ellis^{c)}

*Department of Applied Mathematics, University of Cape Town,
Rondebosch 7701, South Africa*

(Received 23 May 2000; accepted for publication 11 July 2000)

The photon sphere concept in Schwarzschild space–time is generalized to a definition of a photon surface in an arbitrary space–time. A photon sphere is then defined as an $SO(3) \times \mathbb{R}$ -invariant photon surface in a static spherically symmetric space–time. It is proved, subject to an energy condition, that a black hole in any such space–time must be surrounded by a photon sphere. Conversely, subject to an energy condition, any photon sphere must surround a black hole, a naked singularity or more than a certain amount of matter. A second order evolution equation is obtained for the area of an $SO(3)$ -invariant photon surface in a general nonstatic spherically symmetric space–time. Many examples are provided. © 2001 American Institute of Physics. [DOI: 10.1063/1.1308507]

I. INTRODUCTION

The exterior region of the maximally extended Schwarzschild space–time is described by the metric

$$\mathbf{g} = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad r > 2m. \quad (1)$$

For any null geodesic in this exterior region the null geodesic equations give

$$\frac{d^2 r}{d\lambda^2} = (r - 3m) \left\{ \left(\frac{d\theta}{d\lambda}\right)^2 + \sin^2\theta \left(\frac{d\phi}{d\lambda}\right)^2 \right\}, \quad (2)$$

where λ is an affine parameter along the geodesic. The right side here is evidently positive for $r > 3m$ and negative for $r: 2m < r < 3m$. It follows that any future endless null geodesic in the maximally extended Schwarzschild space–time starting at some point with $r > 3m$ and initially directed outwards, in the sense that $dr/d\lambda$ is initially positive, will continue outwards and escape to infinity. Any future endless null geodesic in the maximally extended Schwarzschild space–time starting at some point with $r: 2m < r < 3m$ and initially directed inwards, in the sense that $dr/d\lambda$ is initially negative, will continue inwards and fall into the black hole. The hypersurface $\{r = 3m\}$, known as the Schwarzschild photon sphere, thus distinguishes the borderline between these two types of behavior; any null geodesic starting at some point of the photon sphere and initially tangent to the photon sphere will remain in the photon sphere. (See Darwin^{1,2} for a detailed analysis of the behavior of null and timelike geodesics in Schwarzschild space–time.)

The Schwarzschild photon sphere also has physical significance for massive bodies. For any timelike geodesic in the exterior region the geodesic equations give

^{a)}Deceased.

^{b)}Electronic mail: shwetket@maths.uct.ac.za

^{c)}Electronic mail: ellis@maths.uct.ac.za

$$\frac{d^2r}{ds^2} = -\frac{m}{r^2} + (r-3m) \left\{ \left(\frac{d\theta}{ds} \right)^2 + \sin^2 \theta \left(\frac{d\phi}{ds} \right)^2 \right\}, \tag{3}$$

where s is the arc length along the geodesic. At any point with $r > 3m$ one may arrange for the two terms on the right of (3) to cancel and so obtain a timelike geodesic at constant r . For $r: 2m < r < 3m$ the right hand side of (3) is evidently negative. Thus any future endless timelike geodesic in the maximally extended Schwarzschild space–time starting at some point between the event horizon at $r = 2m$ and the photon sphere at $r = 3m$ and initially directed inwards, in the sense that dr/ds is initially negative, will continue inwards and fall into the black hole. Any observer who traverses a Schwarzschild photon sphere must therefore engage some form of propulsion or else be drawn in to the black hole to meet an inevitable fate.

A photon sphere has been defined by Virbhadra and Ellis³ as a timelike hypersurface of the form $\{r = r_0\}$ where r_0 is the closest distance of approach for which the Einstein bending angle of a light ray is unboundedly large. These authors subsequently⁴ considered the Einstein deflection angle for a general static spherically symmetric metric and obtained an equation for a photon sphere. The existence of a photon sphere in a space–time has important implications for gravitational lensing. In any space–time containing a photon sphere, gravitational lensing will give rise to relativistic images.^{3,4}

The Schwarzschild photon sphere may be usefully compared with the concept of a closed trapped surface. Any null geodesic originating from any point on a closed trapped surface in Schwarzschild space–time is drawn into the singularity at $r = 0$. By contrast, any null geodesic originating from any point on the photon sphere will be drawn into the singularity if and only if it is initially directed inwards.

Our main objectives in the present paper are to give a geometric definition of a photon surface in a general space–time and of a photon sphere in a general static spherically symmetric space–time.

An evolution equation is obtained for the cross-sectional area of a photon surface in a dynamic spherically symmetric space–time. It is shown, subject to suitable energy conditions, that in any static spherically symmetric space–time a black hole must be surrounded by a photon sphere, and a photon sphere must surround either a black hole, a naked singularity or more than a certain amount of matter. Many examples are given of photon spheres in static spherically symmetric space–times. Photon surface evolution is considered for the dynamic space–time example of Vaidya null dust collapse to a naked singularity.

II. PHOTON SURFACES

The hypersurface $S := \{r = 3m\}$ in Schwarzschild space–time has two main properties: first that any null geodesic initially tangent S will remain tangent to S , and second that S does not evolve with time. The following general definition of a photon surface is based on only the first of these properties. A more restrictive class of photon surfaces may be defined when the space–time admits a group of symmetries (see Definition II.2).

Definition II.1: A photon surface of (M, \mathbf{g}) is an immersed, nowhere-spacelike hypersurface S of (M, \mathbf{g}) such that, for every point $p \in S$ and every null vector $\mathbf{k} \in T_p S$, there exists a null geodesic $\gamma: (-\epsilon, \epsilon) \rightarrow M$ of (M, \mathbf{g}) such that $\dot{\gamma}(0) = \mathbf{k}$, $|\gamma| \subset S$.

Any null hypersurface is trivially a photon surface. Photon surfaces are conformally invariant structures. If S is a photon surface of (M, \mathbf{g}) then S is a photon surface of $(M, \Omega^2 \mathbf{g})$ for any smooth function $\Omega: M \rightarrow (0, \infty)$.

Note that Definition II.1 is entirely local. In particular, a photon surface S need contain no endless null geodesics of (M, \mathbf{g}) . Moreover, a photon surface need only be immersed, rather than embedded in M , and so may have self-intersections. If (M, \mathbf{g}) is of dimension $n + 1$ ($n \geq 2$) then, through each point p of a photon surface S in (M, \mathbf{g}) , there is an $(n - 2)$ -parameter family of null geodesics of (M, \mathbf{g}) that lie entirely in S .

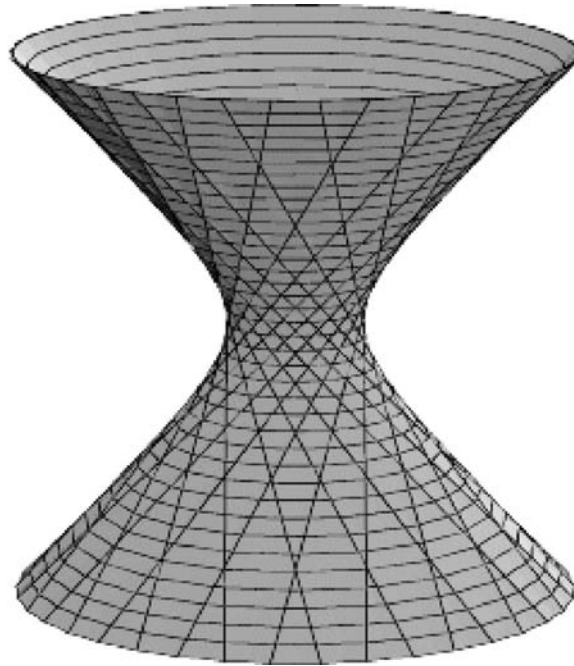


FIG. 1. A photon surface in Minkowski 3-space. The two families of ruling lines are null geodesics with respect to both the Minkowski 3-metric and the induced 2-metric. The lines may be regarded as the space–time paths of pulsed laser beams.

In this paper we will be principally concerned with photon surfaces in space–times of 3 + 1 dimensions. The exceptions are Examples 1 and 3 which give photon surfaces in space–times of dimension 2 + 1 and 4 + 1, respectively.

Example 1 (Minkowski 3-space): In Minkowski 3-space \mathbb{M}^3 , consider the single-sheeted hyperboloid S given by

$$-t^2 + x^2 + y^2 = a^2, \tag{4}$$

for some constant $a > 0$. This surface is doubly ruled, the rulings being given by

$$\gamma_{\theta}^{\pm}(t) := a(0, \cos \theta, \sin \theta) + at(1, \mp \sin \theta, \pm \cos \theta) \tag{5}$$

($-\infty < t < \infty, 0 \leq \theta < 2\pi$), where θ identifies the intersection points with $\{t = 0\}$ and t is the parameter along the ruling lines. The tangents $\dot{\gamma}_{\theta}^{\pm}(t)$ to the ruling lines are null with respect to the \mathbb{M}^3 metric. Clearly they are geodesics in \mathbb{M}^3 . At each point of S there can be just two null directions tangent to S . These must therefore be the directions of the two ruling lines through that point. Hence S is a photon surface in the sense of Definition II.1 (see Fig. 1).

Note that for any circle of the form

$$C = \{t_0, x_0 + r \cos \theta, y_0 + r \sin \theta\}, \quad r > 0, \tag{6}$$

and any future-directed timelike vector field \mathbf{X} along C that respects the symmetry of C , in the sense of

$$\mathbf{X} = (X^t, X^r \cos \theta, X^r \sin \theta), \tag{7}$$

for constant $X^t > 0, X^r$ such that $(X^t)^2 > (X^r)^2$, there is a unique single-sheeted hyperboloid S through C such that \mathbf{X} is tangent to S along C .

In the case $a=0$, Eq. (4) gives the null cone through the origin. The complement of p in this null cone is a null photon surface of \mathbb{M}^3 .

Example 2 (Minkowski 4-space): One may generalize Example 1 to Minkowski 4-space \mathbb{M}^4 as follows. Let S be a timelike hypersurface in \mathbb{M}^4 of the form

$$-t^2 + x^2 + y^2 + z^2 = a^2, \tag{8}$$

for some constant $a > 0$. The two-parameter family of lines,

$$\gamma_{\theta, \phi}^{\pm}(t) = a(0, \cos \theta, \sin \theta \sin \phi, \sin \theta \cos \phi) + at(1, \mp \sin \theta, \pm \cos \theta \sin \phi, \pm \cos \theta \cos \phi), \tag{9}$$

foliate S and are null geodesics with respect to the \mathbb{M}^4 metric. For each $p \in S$, the tangents at p to those $\gamma_{\theta, \phi}^{\pm}(t)$ that pass through p can be shown to generate the null cone of $T_p S$. Hence S is a photon surface in the sense of Definition II.1. In terms of the double null coordinates,

$$u := t + r, \tag{10}$$

$$v := t - r, \tag{11}$$

for $r := (x^2 + y^2 + z^2)^{1/2}$, Eq. (8) assumes the simple form

$$uv = -a^2. \tag{12}$$

In the future direction S tends asymptotically to the null hypersurface $\{v=0\}$, while in the past direction S tends asymptotically to the null hypersurface $\{u=0\}$.

Example 3 (de Sitter space): de Sitter space–time may be regarded⁵ as a single-sheeted hyperboloid in Minkowski 5-space \mathbb{M}^5 . By analogy with Examples 1 and 2, de Sitter space–time is thus realized as a photon surface in \mathbb{M}^5 .

Example 4 (The Robertson–Walker models): Every Robertson–Walker model is conformally flat and therefore locally conformally transformable to Minkowski space. The photon surfaces of any Robertson–Walker model may thus be obtained, at least locally, by conformal transformations of Minkowski space.

Theorem II.1: *Let S be a timelike hypersurface of (M, \mathbf{g}) . Let \mathbf{n} be a unit normal field to S and let h_{ab} be the induced metric on S . Let χ_{ab} be the second fundamental form on S and let σ_{ab} be the trace-free part of χ_{ab} . Then the following are equivalent:*

- (i) S is a photon surface;
- (ii) $\chi_{ab} k^a k^b = 0, \forall$ null $\mathbf{k} \in T_p S, \forall p \in S$;
- (iii) $\sigma_{ab} = 0$;
- (iv) every affine null geodesic of (S, \mathbf{h}) is an affine null geodesic of (M, \mathbf{g}) .

Proof: (i) \Rightarrow (ii). Suppose S is a photon surface. Let $p \in S$ and let $\mathbf{k} \in T_p S$ be null. There exists an affine null geodesic $\gamma: (-\epsilon, \epsilon) \rightarrow M$ of (M, \mathbf{g}) such that $\dot{\gamma}(0) = \mathbf{k}, |\gamma| \subset S$. One has

$$\chi_{ab} \dot{\gamma}^a \dot{\gamma}^b = n_{a;b} \dot{\gamma}^a \dot{\gamma}^b = (n_a \dot{\gamma}^a)_{;b} \dot{\gamma}^b = 0, \tag{13}$$

along γ . At p this gives $\chi_{ab} k^a k^b = 0$.

(ii) \Rightarrow (iii). Let $p \in S$. By (ii) one has $\sigma_{ab} k^a k^b = \chi_{ab} k^a k^b = 0, \forall$ null $\mathbf{k} \in T_p S$. Let $\{\mathbf{e}_{(0)}, \mathbf{e}_{(1)}, \mathbf{e}_{(2)}\}$ be an orthonormal basis for $T_p S$ with $\mathbf{e}_{(0)}$ timelike and $\mathbf{e}_{(1)}, \mathbf{e}_{(2)}$ spacelike. Any null $\mathbf{k} \in T_p S$, normalized such that $g(\mathbf{k}, \mathbf{e}_{(0)}) = -1$, has components $k^a = (1, \cos \psi, \sin \psi)$ with respect to $\{\mathbf{e}_{(0)}, \mathbf{e}_{(1)}, \mathbf{e}_{(2)}\}$ for some $\psi \in [0, 2\pi)$. A calculation gives

$$\sigma_{ab} k^a k^b = (\sigma_{00} + \frac{1}{2}\sigma_{11} + \frac{1}{2}\sigma_{22}) + 2\sigma_{01} \cos \psi + 2\sigma_{02} \sin \psi + \frac{1}{2}(\sigma_{11} - \sigma_{22}) \cos 2\psi + \sigma_{12} \sin 2\psi. \tag{14}$$

This must vanish for all $\psi \in [0, 2\pi)$. One thus has $\sigma_{01} = \sigma_{02} = \sigma_{12} = 0$ and $-\sigma_{00} = \sigma_{11} = \sigma_{22}$. Since σ_{ab} is trace-free one must also have $\sigma_{00} = \sigma_{11} + \sigma_{22}$. There follows $\sigma_{ab} = 0$.

(iii) \Rightarrow (iv). For any curve in S with null tangent \mathbf{k} one has

$$k^a \parallel_b k^b = h^a_c k^c ;_b k^b = k^a ;_b k^b + (\sigma_{bc} k^b k^c) n^a, \tag{15}$$

where \parallel denotes covariant differentiation in S with respect to \mathbf{h} . The second term on the right of (15) vanishes by hypothesis. If \mathbf{k} is tangent to an affine null geodesic of (S, \mathbf{h}) then the term on the left of (15) also vanishes and so \mathbf{k} is tangent to an affine null geodesic of (M, \mathbf{g}) .

(iv) \Rightarrow (i). Let $p \in S$ and let $\mathbf{k} \in T_p S$ be null. Let $\gamma: (-\epsilon, \epsilon) \rightarrow S$ be an affine null geodesic of (S, \mathbf{h}) such that $\dot{\gamma}(0) = \mathbf{k}$. Then, by (iv), γ is an affine null geodesic of (M, \mathbf{g}) such that $\dot{\gamma}(0) = \mathbf{k}$, $|\gamma| \subset S$. \square

Condition (iii) of Theorem II.1 is equivalent to a requirement that χ_{ab} is pure trace in the sense of

$$\chi_{ab} = \frac{1}{3} \Theta h_{ab}, \tag{16}$$

where $\Theta := h^{cd} \chi_{cd}$ is the expansion of the unit normal to S . For Example 1 one has $\Theta = 2/a$; for Example 2 one has $\Theta = 3/a$. [Note that, by a standard abuse of notation, h_{ab} denotes both the induced metric on S and the symmetric tensor field of rank (0,2) along S in M which satisfies $h_{ab} n^b = 0$ and pulls back to the induced metric on S .]

It is clear from condition (iii) of Theorem II.1 that a space–time must be specialized in some respect in order to admit any timelike photon surfaces in the sense of Definition II.1. For this reason it is helpful to restrict attention to space–times which admit groups of symmetries.

Definition II.2: Suppose (M, \mathbf{g}) admits a group G of isometries. A photon surface S of (M, \mathbf{g}) that is invariant under G , in the sense that each $g \in G$ maps S onto itself, will be called a G -invariant photon surface.

Clearly any G -invariant null hypersurface is a G -invariant photon surface. In particular, if $G = \mathbb{R}$ or $G = S$, then any Killing horizon^{6,7} is a G -invariant photon surface.

III. DYNAMIC SPHERICAL SYMMETRY: GENERAL THEORY

By definition, a general spherically symmetric space–time admits an $SO(3)$ isometry group for which the group orbits are spacelike 2-spheres. The following result describes the evolution of the cross-sectional area of an $SO(3)$ -invariant photon surface in a spherically symmetric space–time.

Theorem III.1: *Let (M, \mathbf{g}) be a spherically symmetric space–time. Let S be an $SO(3)$ -invariant timelike hypersurface of (M, \mathbf{g}) and let \mathbf{X} be the $SO(3)$ -invariant unit future-directed timelike tangent vector field along S orthogonal to the $SO(3)$ -invariant 2-spheres in S . Let \mathcal{T} be one such $SO(3)$ -invariant 2-sphere in S and let \mathcal{T}_s be the $SO(3)$ -invariant 2-sphere in S at arc length s from \mathcal{T} along the integral curves of \mathbf{X} . Then S is a photon surface of (M, \mathbf{g}) iff the area ${}^{(2)}A_s$ of \mathcal{T}_s satisfies*

$$\frac{d^2}{ds^2} {}^{(2)}A_s = \frac{1}{4} \frac{1}{{}^{(2)}A_s} \left(\frac{d}{ds} {}^{(2)}A_s \right)^2 + {}^{(2)}A_s \left(\frac{1}{3} \Theta^2 - G_{ab} n^a n^b \right) - 4\pi, \tag{17}$$

where n^a is the unit normal to S , Θ is the expansion of n^a and $G_{ab} := R_{ab} - \frac{1}{2} R g_{ab}$ is the Einstein tensor of (M, \mathbf{g}) .

Proof: Let h_{ab} be the induced Lorentzian 3-metric on S and, for each s , let ${}^{(2)}h_{ab}$ be the induced Riemannian 2-metric on \mathcal{T}_s . The expansion of \mathbf{X} in (S, \mathbf{h}) is given by ${}^{(2)}\Theta = {}^{(2)}h^{ab} X_{a;b}$ where the covariant derivative is that of (M, \mathbf{g}) . Since \mathbf{X} is both shear-free and vorticity-free in (S, \mathbf{h}) , the Raychaudhuri equation for \mathbf{X} in (S, \mathbf{h}) assumes the form

$$\frac{d}{ds} {}^{(2)}\Theta = -\frac{1}{2} ({}^{(2)}\Theta)^2 - {}^{(3)}R_{ab} X^a X^b, \tag{18}$$

where ${}^{(3)}R_{ab}$ is the Ricci tensor of (S, \mathbf{h}) .

From first principles one has

$${}^{(2)}R = {}^{(3)}R + 2 {}^{(3)}R_{ab} X^a X^b - ({}^{(2)}\chi^a_a)^2 + {}^{(2)}\chi^a_b {}^{(2)}\chi^b_a, \tag{19}$$

$${}^{(3)}R = R - 2R_{ab} n^a n^b + (\chi^a_a)^2 - \chi^a_b \chi^b_a, \tag{20}$$

where ${}^{(2)}\chi_{ab}$ is the second fundamental form of each \mathcal{T}_s in (S, \mathbf{h}) . Since X^a is shear-free and vorticity free in (S, \mathbf{h}) one has ${}^{(2)}\chi_{ab} = \frac{1}{2} {}^{(2)}\Theta h_{ab}$. The second fundamental form of S admits the canonical decomposition $\chi_{ab} = \frac{1}{3} \Theta h_{ab} + \sigma_{ab}$. Equations (19) and (20) therefore give

$${}^{(2)}R = {}^{(3)}R + 2 {}^{(3)}R_{ab} X^a X^b - \frac{1}{2} ({}^{(2)}\Theta)^2, \tag{21}$$

$${}^{(3)}R = R - 2R_{ab} n^a n^b + \frac{2}{3} \Theta^2 - \sigma^a_b \sigma^b_a, \tag{22}$$

which combine to yield

$$2 {}^{(3)}R_{ab} X^a X^b = {}^{(2)}R + 2G_{ab} n^a n^b - \frac{2}{3} \Theta^2 + \frac{1}{2} ({}^{(2)}\Theta)^2 + \sigma^a_b \sigma^b_a. \tag{23}$$

One may now substitute for the second term on the right of (18) to obtain

$$\frac{d}{ds} {}^{(2)}\Theta = -\frac{3}{4} ({}^{(2)}\Theta)^2 + \frac{1}{3} \Theta^2 - \frac{1}{2} {}^{(2)}R - G_{ab} n^a n^b - \frac{1}{2} \sigma^a_b \sigma^b_a. \tag{24}$$

From first principles one has ${}^{(2)}\Theta = (d/ds) \ln {}^{(2)}A$, and the Gauss–Bonnet theorem gives ${}^{(2)}R {}^{(2)}A = 8\pi$. Substituting for ${}^{(2)}\Theta$ and ${}^{(2)}R$ in (24) one obtains

$$\frac{d^2}{ds^2} {}^{(2)}A_s = \frac{1}{4} ({}^{(2)}A_s)^2 + ({}^{(2)}A_s) \left(\frac{1}{3} \Theta^2 - G_{ab} n^a n^b - \frac{1}{2} \sigma^a_b \sigma^b_a \right) - 4\pi. \tag{25}$$

This agrees with (17) iff $\sigma^a_b \sigma^b_a = 0$.

Construct, for the tangent bundle TS of S , an orthonormal basis field of the form $\{\mathbf{X}, \mathbf{e}_{(1)}, \mathbf{e}_{(2)}\}$, with $\mathbf{e}_{(1)}$ and $\mathbf{e}_{(2)}$ unit spacelike. With respect to this basis one has

$$\sigma^a_b \sigma^b_a = (\sigma^0_0)^2 + (\sigma^1_1)^2 + (\sigma^2_2)^2 + 2(\sigma^1_2)^2 - 2(\sigma^1_0)^2 - 2(\sigma^2_0)^2. \tag{26}$$

By spherical symmetry the vector field $\sigma^a_b X^b$ must be proportional to X^a . Hence one has $\sigma^1_0 = \sigma^2_0 = 0$. The vanishing of $\sigma^a_b \sigma^b_a$ is thus equivalent to the vanishing of σ_{ab} . One has $\sigma_{ab} = 0$ iff S is a photon surface. \square

A spherically symmetric metric is locally expressible in the form

$$g_{ab} = \begin{pmatrix} g_{00} & g_{01} & 0 & 0 \\ g_{10} & g_{11} & 0 & 0 \\ 0 & 0 & g_{\theta\theta} & 0 \\ 0 & 0 & 0 & g_{\theta\theta} \sin^2 \theta \end{pmatrix}, \tag{27}$$

with respect to coordinates (x^0, x^1, θ, ϕ) adapted to the spherical symmetry, where $g_{00}, g_{01} = g_{10}, g_{11}$ and $g_{\theta\theta} > 0$ depend only on x^0 and x^1 . It is often convenient to introduce a radial coordinate r , depending only on x^0 and x^1 , such that $g_{\theta\theta}$ is a function of r only. One is free to specify $g_{\theta\theta}$ as a function of r alone since to do so is, in effect, a definition of the coordinate r . This will be assumed to be done throughout this paper.

The following result is useful in the locating of $SO(3)$ -invariant photon surfaces in dynamic spherically symmetric space-times.

Lemma III.2: Let (M, \mathbf{g}) be a spherically symmetric space-time. Let S be an $SO(3)$ -invariant timelike hypersurface of (M, \mathbf{g}) and let \mathbf{X} be the $SO(3)$ -invariant unit future-directed timelike tangent vector field along S that is orthogonal to the $SO(3)$ -invariant 2-spheres in S . Then S is a photon surface of (M, \mathbf{g}) iff

$$X^a_{;b} X^b = \frac{1}{2} (g^{\theta\theta} n^b \partial_b g_{\theta\theta}) n^a, \quad (28)$$

holds along S , where n^a is the unit normal field to S in (M, \mathbf{g}) .

Proof: By spherical symmetry, and since \mathbf{X} is unit timelike, the vector field $\nabla_{\mathbf{X}} \mathbf{X}$ must be proportional to \mathbf{n} . Hence it suffices to show that S is a photon surface iff along S one has

$$\mathbf{n} \cdot \nabla_{\mathbf{X}} \mathbf{X} = \frac{1}{2} g^{\theta\theta} n^b \partial_b g_{\theta\theta}, \quad (29)$$

or equivalently,

$$\chi_{ab} X^a X^b = -\frac{1}{2} g^{\theta\theta} n^a \partial_a g_{\theta\theta}. \quad (30)$$

Construct for TS a local orthonormal basis field of the form $\{\mathbf{X}, \mathbf{e}_{(\theta)}, \mathbf{e}_{(\phi)}\}$. With respect to this basis field the components of χ^a_b form a diagonal matrix with

$$\chi^\theta_\theta = \chi^\phi_\phi = n^\theta_{;\theta} = \frac{1}{2} g^{\theta\theta} n^a \partial_a g_{\theta\theta}. \quad (31)$$

Equation (30) is thus equivalent to $\chi^0_0 = \chi^\theta_\theta = \chi^\phi_\phi$ which is in turn equivalent to $\sigma^0_0 = \sigma^\theta_\theta = \sigma^\phi_\phi$. In view of the trace-free property of σ^a_b , Eq. (30) is thus equivalent to $\sigma^a_b = 0$. From Theorem II.1 one has that $\sigma^a_b = 0$ holds along S iff S is a photon surface. \square

Let us continue to work with respect to the coordinate system $\{x^0, x^1, \theta, \phi\}$ employed in (27). Let $x^a(s)$ be an integral curve of the vector field \mathbf{X} in Lemma III.2. One has

$$\frac{dx^a}{ds} = X^a, \quad (32)$$

and Eq. (28) becomes

$$\frac{d^2 x^a}{ds^2} + \Gamma^a_{bc} \frac{dx^b}{ds} \frac{dx^c}{ds} = \frac{1}{2} (g^{\theta\theta} n^b \partial_b g_{\theta\theta}) n^a. \quad (33)$$

Since $\mathbf{X} = (dx^0/ds)(1, dx^1/dx^0, 0, 0)$ is unit timelike one has

$$\left(\frac{ds}{dx^0} \right)^2 = -g_{ab} \frac{dx^a}{dx^0} \frac{dx^b}{dx^0}. \quad (34)$$

The x^0 and x^1 components of Eq. (33) thus combine to give

$$\frac{d^2 x^1}{(dx^0)^2} = -\frac{1}{2} g^{\theta\theta} n^a \partial_a g_{\theta\theta} \left(n^1 - \frac{dx^1}{dx^0} n^0 \right) g_{bc} \frac{dx^b}{dx^0} \frac{dx^c}{dx^0} + \left(\frac{dx^1}{dx^0} \Gamma^0_{ab} - \Gamma^1_{ab} \right) \frac{dx^a}{dx^0} \frac{dx^b}{dx^0}, \quad (35)$$

where the components of \mathbf{n} are given by

$$n^0 = \psi g_{1a} \frac{dx^a}{dx^0}; \quad n^1 = -\psi g_{0a} \frac{dx^a}{dx^0}, \quad (36)$$

for

$$\psi := (-\Delta)^{-1/2} \left(-g_{ab} \frac{dx^a}{dx^0} \frac{dx^b}{dx^0} \right)^{-1/2}, \tag{37}$$

where

$$\Delta := g_{00}g_{11} - (g_{01})^2 \tag{38}$$

is the determinant of the time–space part of g_{ab} in (27). Equation (35) is the coordinate equivalent of (28) and provides for the easy determination of $SO(3)$ -invariant photon surfaces (see Example 10).

IV. STATIC SPHERICAL SYMMETRY: GENERAL THEORY

By definition, a spherically symmetric space–time is static if it admits an $SO(3) \times \mathbb{R}$ group of isometries such that the \mathbb{R} orbits are generated by a Killing field \mathbf{K} which is both hypersurface orthogonal and orthogonal to the $SO(3)$ orbits. In the present section we will be concerned with $SO(3) \times \mathbb{R}$ -invariant photon surfaces in static spherically symmetric space–times. Such surfaces may be termed photon spheres because, as will be seen, they are a natural generalization to general static spherically symmetric space–times of the Schwarzschild photon sphere concept. The term ‘‘photon sphere’’ will be regarded as applicable only in static spherically symmetric space–times. For clarity, the term ‘‘ $SO(3) \times \mathbb{R}$ -invariant photon surface’’ will usually be employed in preference to ‘‘photon sphere.’’

Although the space–times of Examples 1 and 2 are static and spherically symmetric, the photon surfaces in these space–times are not $SO(3) \times \mathbb{R}$ -invariant and so are not photon spheres. Of the Robertson–Walker space–times of Example 4, only the Einstein cylinder is both spherically symmetric and static. None of the photon surfaces of the Einstein cylinder are $SO(3) \times \mathbb{R}$ -invariant. Thus the Einstein cylinder has no photon spheres.

One may characterize an $SO(3) \times \mathbb{R}$ -invariant photon surface, or photon sphere, in a static spherically symmetric space–time by means of the following special case of Theorem III.1.

Theorem IV.1: *Let (M, \mathbf{g}) be a static spherically symmetric space–time with Killing field \mathbf{K} and let S be an $SO(3) \times \mathbb{R}$ -invariant timelike hypersurface of (M, \mathbf{g}) . Then S is an $SO(3) \times \mathbb{R}$ -invariant photon surface of (M, \mathbf{g}) if there exists an $SO(3)$ -invariant 2-sphere $\mathcal{T} \subset S$ satisfying*

$$A(\Theta^2 - 3G_{ab}n^an^b) = 12\pi, \tag{39}$$

where A is the area of \mathcal{T} , n^a is the unit normal to S and Θ is the trace of the second fundamental form of S . Conversely, if S is an $SO(3) \times \mathbb{R}$ -invariant photon surface of (M, \mathbf{g}) then (39) holds for every $SO(3)$ -invariant 2-sphere $\mathcal{T} \subset S$.

Proof: Note that the unit future-directed timelike tangent field \mathbf{X} along S in Theorem III.1 is proportional to the restriction to S of the Killing field \mathbf{K} .

Suppose first that there exists an $SO(3)$ -invariant 2-sphere $\mathcal{T} \subset S$ such that (39) holds. The quantities A , Θ and $G_{ab}n^an^b$ remain constant as \mathcal{T} is mapped along the flow lines of the Killing field \mathbf{K} . So they also remain constant as they are mapped along the flow lines of \mathbf{X} . Hence (17) holds with the term on the left and the first term on the right both zero. Thus, by Theorem III.1, S is a photon surface of (M, \mathbf{g}) . By hypothesis S is $SO(3) \times \mathbb{R}$ -invariant.

For the converse, suppose that S is an $SO(3) \times \mathbb{R}$ -invariant photon surface of (M, \mathbf{g}) . Then (17) holds for every $SO(3)$ -invariant 2-sphere $\mathcal{T}_s \subset S$. Since \mathbf{K} induces groups of local isometries, the area A_s of \mathcal{T}_s is independent of the parameter s . Hence the term on the left and the first term on the right of (17) both vanish and one obtains (39). \square

Corollary 1: *If one has $G_{ab}Y^aY^b \geq 0$, \forall vectors \mathbf{Y} and S is an $SO(3) \times \mathbb{R}$ -invariant timelike photon surface of (M, \mathbf{g}) , then for any $SO(3)$ -invariant 2-sphere $\mathcal{T} \subset S$ one has*

$$A\Theta^2 \geq 12\pi, \tag{40}$$

with equality holding iff $G_{ab}n^an^b=0$ along S . □

For Schwarzschild space–time where (see Example 5) the only timelike photon sphere is at $r=3m$, one has $A=4\pi(3m)^2$, $\Theta=1/(\sqrt{3}m)$ and $G_{ab}=0$ which verifies (39) and (40) for this case.

If the Einstein equations hold with a zero cosmological constant then, in the corollary to Theorem IV.1, the hypothesis $G_{ab}Y^aY^b\geq 0$ for all vectors \mathbf{Y} is equivalent to $T_{ab}Y^aY^b\geq 0$ for all vectors \mathbf{Y} . This is a physically reasonable energy condition. In particular, for a perfect fluid with density ρ and pressure p , it is equivalent to a condition that ρ and p are both non-negative. More generally, the condition holds for an energy tensor with a single timelike eigenvector (type I in the classification of Hawking and Ellis⁵) iff each energy tensor eigenvalue is non-negative.

The characterization of timelike photon surfaces provided by Theorem IV.1 involves derivatives of the metric components up to second order. The following result (Theorem IV.2) provides an entirely different characterization of $SO(3)\times\mathbb{R}$ -invariant photon surfaces in terms of derivatives of the metric components up to only first order.

Let a general static spherically symmetric metric \mathbf{g} be expressed in the form (27) with $g_{\theta\theta}$ a function of r only. The Killing equation $K_{(a;b)}=0$ and the orthogonality of \mathbf{K} to ∂_θ and ∂_ϕ gives $K^a\partial_a g_{\theta\theta}=0$ and hence $\nabla_{\mathbf{K}}r=0$ where r is to be regarded as a scalar field on M . Since r is independent of θ and ϕ it follows that any $SO(3)\times\mathbb{R}$ -invariant hypersurface S of (M, \mathbf{g}) must be of the form $\{r=\text{const}\}$. If S is also a timelike hypersurface then $r^{;a}$ is a spacelike vector field along S and \mathbf{K} is a timelike vector field in a neighborhood of S .

If the coordinates x^0 and x^1 implicit in (27) are chosen such that $\mathbf{K}=\partial_{x^0}$ then the Killing equation gives that all the metric components g_{ab} in (27) are independent of x^0 . Since r is constant along the integral curves of \mathbf{K} , the coordinate x^1 must be a function of r only. A natural choice is $x^1=r$. One may redefine x^0 according to $x^0\rightarrow x^0-\int(g_{01}/g_{00})dx^1$. This diagonalizes the time–space part of g_{ab} and leaves the components of g_{ab} independent of x^0 . Furthermore the curves $\{x^1, \theta, \phi=\text{const}\}$ are unchanged except that they are re-parametrized. The vector field ∂_{x^0} then becomes a conformal Killing field.

Define the tensor field,

$$\epsilon^{ab}:=(-\Delta)^{-1/2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{41}$$

on M , where the components are given with respect to the coordinate basis employed in (27) and Δ is the determinant of the time–space part of \mathbf{g} in (27), as in (38).

Theorem IV.2: *Let (M, \mathbf{g}) be a static spherically symmetric space–time with \mathbf{g} of the form (27), with $g_{\theta\theta}$ a function of the coordinate r only. Let S be an $SO(3)\times\mathbb{R}$ -invariant timelike hypersurface of (M, \mathbf{g}) and suppose that ∇r is nowhere-zero along S . Then S is an $SO(3)\times\mathbb{R}$ -invariant photon surface of (M, \mathbf{g}) iff*

$$2g_{\theta\theta}\epsilon^{ab}\epsilon^{cd}r_{;ac}r_{;b}r_{;d}+r^{;a}r_{;a}r^{;c}\partial_c g_{\theta\theta}=0, \tag{42}$$

holds along S .

Proof: Since (M, \mathbf{g}) is both spherically symmetric and static, the surface S is of the form $\{r=\text{const}\}$. The unit spacelike normal to S is therefore given by

$$n^a=\eta r^{;a} \tag{43}$$

for

$$\eta:=(r^{;a}r_{;a})^{-1/2}. \tag{44}$$

The second fundamental form of S is given by

$$\chi_{ab} := \eta h_a^c h_b^d r_{;cd}. \tag{45}$$

The vector fields,

$$\mathbf{X}^a := (-\Delta g^{bc} r_{;b} r_{;c})^{-1/2} (r_{;1}, -r_{;0}, 0, 0), \tag{46}$$

$$\mathbf{e}_{(\theta)} := (g_{\theta\theta})^{-1/2} \partial_\theta, \tag{47}$$

$$\mathbf{e}_{(\phi)} := (g_{\theta\theta} \sin^2 \theta)^{-1/2} \partial_\phi, \tag{48}$$

form an orthonormal frame field along S , with $\mathbf{e}_{(\theta)}$ and $\mathbf{e}_{(\phi)}$ unit spacelike and \mathbf{X} unit timelike. One has

$$\chi_{ab} X^a \mathbf{e}_{(\theta)}^b = \chi_{ab} X^a \mathbf{e}_{(\phi)}^b = \chi_{ab} \mathbf{e}_{(\theta)}^a \mathbf{e}_{(\phi)}^b = 0, \tag{49}$$

$$\chi_{ab} \mathbf{e}_{(\theta)}^a \mathbf{e}_{(\theta)}^b = \chi_{ab} \mathbf{e}_{(\phi)}^a \mathbf{e}_{(\phi)}^b = \frac{\eta r^{;a} \partial_a g_{\theta\theta}}{2 g_{\theta\theta}}, \tag{50}$$

$$\chi_{ab} X^a X^b = \eta^3 \epsilon^{ab} \epsilon^{cd} r_{;ac} r_{;b} r_{;d}. \tag{51}$$

Condition (iii) of Theorem II.1 holds iff χ_{ab} is proportional to h_{ab} and hence iff

$$-\chi_{ab} X^a X^b = \chi_{ab} \mathbf{e}_{(\theta)}^a \mathbf{e}_{(\theta)}^b = \chi_{ab} \mathbf{e}_{(\phi)}^a \mathbf{e}_{(\phi)}^b. \tag{52}$$

This is equivalent to

$$-\eta^2 \epsilon^{ab} \epsilon^{cd} r_{;ac} r_{;b} r_{;d} = \frac{r^{;a} \partial_a g_{\theta\theta}}{2 g_{\theta\theta}}, \tag{53}$$

which is in turn equivalent to (42). □

Equation (42) can have solutions such that $\{r = \text{const}\}$ is a spacelike hypersurface and therefore not a photon surface (see, e.g., Example 7). It is therefore always necessary in the use of Theorem IV.2 to check that the hypersurface $\{r = \text{const}\}$ is in fact timelike or null.

Note that in a region of space–time where the Killing field \mathbf{K} is spacelike, for example, behind the event horizon of Schwarzschild space–time, the hypersurfaces $\{r = \text{const}\}$ are necessarily spacelike and so cannot be photon spheres.

Case 1 ($x^1 := r$, components of g_{ab} independent of x^0): As discussed previously, for a static spherically symmetric metric it is possible to choose $x^1 := r$, with $g_{\theta\theta}$ depending only upon r and with all the components of g_{ab} independent of x^0 . In this case (42) reduces to

$$g_{00} \partial_r g_{\theta\theta} = g_{\theta\theta} \partial_r g_{00}. \tag{54}$$

This agrees with an equation obtained by Virbhadra and Ellis⁴ on the basis of a different definition³ of a photon sphere. Note that even though the components $g_{rr}, g_{0r} = g_{r0}$ do not appear in (54), they are not assumed to vanish. A particular sub-case of interest is that of time-space coordinates, with $t := x^0$ timelike in the sense of $g^{tt} < 0$. Another sub-case of interest is that of single null (radiation) coordinates, with $u := x^0$ null in the sense of $g^{uu} = 0$.

Case 2 (double null coordinates u, v): Let $x^0 := u$, $x^1 := v$ be double null coordinates in the sense of $g^{uu} = g^{vv} = 0$. The radial coordinate r is to be regarded as a function of u and v . Then (42) assumes the form

$$g_{\theta\theta} \{ r_{;uu} (r_{;v})^2 - 2 r_{;uv} r_{;u} r_{;v} + r_{;vv} (r_{;u})^2 \} + 2 (r_{;u} r_{;v})^2 \partial_r g_{\theta\theta} = 0. \tag{55}$$

The metric components $g_{uv} = g_{vu}$ enter here through the covariant derivatives of r .

Equations (42), (54) and (55) may be referred to as photon sphere equations since they give the location of timelike photon spheres in static spherically symmetric space-times.

In order to facilitate further progress, a general static spherically symmetric metric will be written in such a form as to cast the Einstein tensor in a particularly simple and convenient form.

One has from previous remarks that a general static spherically symmetric metric is locally expressible in the form

$$\mathbf{g} = g_{tt} dt^2 + g_{rr} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2), \quad (56)$$

where g_{tt} and g_{rr} are functions of r only. Let

$$m(r) := \frac{1}{2} r \left(1 - \frac{1}{g_{rr}} \right) \quad (57)$$

$$\mu(r) := \ln(-g_{tt}g_{rr}). \quad (58)$$

Then the metric assumes the form

$$\mathbf{g} = - \left(1 - \frac{2m(r)}{r} \right) e^{\mu(r)} dt^2 + \left(1 - \frac{2m(r)}{r} \right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2), \quad (59)$$

and the Einstein tensor is given by

$$G^a_b = 8\pi \begin{pmatrix} -\rho(r) & 0 & 0 & 0 \\ 0 & p_1(r) & 0 & 0 \\ 0 & 0 & p_2(r) & 0 \\ 0 & 0 & 0 & p_2(r) \end{pmatrix}, \quad (60)$$

for

$$8\pi\rho(r) := \frac{2m'(r)}{r^2}, \quad (61)$$

$$8\pi p_1(r) := \frac{1}{r^2} \{ (r - 2m(r))\mu'(r) - 2m'(r) \}, \quad (62)$$

$$8\pi p_2(r) := \frac{1}{4r^2} \{ (2(r + m(r)) - 3rm'(r))\mu'(r) + r(r - 2m(r))(\mu'(r))^2 - 4rm''(r) + 2r(r - 2m(r))\mu''(r) \}, \quad (63)$$

where a prime denotes differentiation with respect to r . For the right sides of (61), (62) and (63) to be defined at some radius $\hat{r} > 0$ one evidently needs $m(r)$ and $\mu(r)$ to be twice differentiable at $r = \hat{r}$. Equations (61) and (62) combine to give

$$\mu'(r) = \frac{8\pi r(\rho(r) + p_1(r))}{\left(1 - \frac{2m(r)}{r} \right)}, \quad (64)$$

whereby one may rewrite (63) in the more convenient form

$$\frac{2}{r}(p_2(r) - p_1(r)) = p_1'(r) + \frac{\left(\frac{m(r)}{r^2} + 4\pi r p_1(r)\right)(\rho(r) + p_1(r))}{\left(1 - \frac{2m(r)}{r}\right)}. \tag{65}$$

In the perfect fluid case $p_1(r) = p_2(r) =: p(r)$ Eq. (65) reduces to the Tolman–Oppenheimer–Volkoff equation,

$$p'(r) = - \frac{\left(\frac{m(r)}{r^2} + 4\pi r p(r)\right)(\rho(r) + p(r))}{\left(1 - \frac{2m(r)}{r}\right)}. \tag{66}$$

By means of Eqs. (61) and (64), the photon sphere equation (54) becomes

$$1 - \frac{3m(r)}{r} - 4\pi r^2 p_1(r) = 0. \tag{67}$$

For r such that $2m(r) < r$, and hence such that the hypersurface $\{r = \text{const}\}$ is timelike, Eq. (67) gives the location of the $SO(3) \times \mathbb{R}$ -invariant timelike photon surfaces for the metric (59).

Equation (67) is the basis for the following result which shows that, subject to a suitable energy condition, any black hole in a static spherically symmetric space–time must be surrounded by an $SO(3) \times \mathbb{R}$ -invariant photon surface. For the purpose of this and subsequent results, a function $f: \mathbb{R} \supset I \rightarrow \mathbb{R}$ on an interval I will be said to be piecewise C^r if I is the disjoint union of a locally finite collection of intervals I_i such that $f|_{I_i}$ is C^r . Each interval I_i may be open, closed or half-open.

Theorem IV.3: *Suppose the metric \mathbf{g} has the form (59) for $r_0 < r < \infty$, for some $r_0 > 0$, with $m(r)$ and $\mu(r)$ both C^0 , piecewise C^2 functions of $r \in (r_0, \infty)$. Suppose the following hold:*

- (1) $\rho(r)$ and $p_1(r)$ are bounded functions of $r \in (r_0, \infty)$;
- (2) $2m(r) < r, \forall r \in (r_0, \infty)$;
- (3) $\rho(r) \geq 0, p_1(r) \geq 0, \forall r \in (r_0, \infty)$;
- (4) $\lim_{r \rightarrow \infty} 4\pi r^2 p_1(r) = \lim_{r \rightarrow \infty} 4\pi r^2 \rho(r) = 0$;
- (5) for each value of t the 2-surfaces $\mathcal{T}_{t,r} := \{t = \text{const}\} \cap \{r = \text{const}\}, r_0 < r < \infty$, are such that $\mathcal{T}_t := \lim_{r \rightarrow r_0} \mathcal{T}_{t,r}$ exists as an embedded spacelike 2-sphere in (M, \mathbf{g}) and is marginally outer trapped.

Then (M, \mathbf{g}) admits an $SO(3) \times \mathbb{R}$ -invariant timelike photon surface of the form $\{r = r_1\}$ for some $r_1 \in (r_0, \infty)$.

Proof: Fix t and let \mathbf{k} be the outward future-directed null normal field along each $\mathcal{T}_{t,r}, r_0 < r < \infty$, normalized such that $\mathbf{g}(\mathbf{k}, \mathbf{n}) = 1$, where $\mathbf{n} = (1 - 2m(r)/r)^{1/2} \partial_r$ is the outward radial unit tangent to $\{t = \text{const}\}$. Since \mathbf{k} is parallelly propagated along each of the geodesic integral curves of \mathbf{n} , one has that $\lim_{r \rightarrow r_0} \mathbf{k}$ is a well-defined, nowhere-zero null vector field along $\mathcal{T}_t := \lim_{r \rightarrow r_0} \mathcal{T}_{t,r}$. For $r \in (r_0, \infty)$ the vector field \mathbf{k} has the form $k^a = (k^t, a(r)k^t, 0, 0)$ for

$$a(r) := (-g_{tt}/g_{rr})^{1/2} = \left(1 - \frac{2m(r)}{r}\right) e^{\mu(r)/2}. \tag{68}$$

The expansion of \mathbf{k} is given by

$$\Theta_{\text{out}} = {}^{(2)}h^b{}_a k^a{}_{;b} = \frac{2a(r)}{r} k^t. \tag{69}$$

The condition that \mathcal{T}_t is marginally outer trapped therefore implies

$$0 = \lim_{r \rightarrow r_0} a(r) = \lim_{r \rightarrow r_0} \left(1 - \frac{2m(r)}{r} \right) e^{\mu(r)/2}. \tag{70}$$

The non-negativity of $\rho(r)$ and $p_1(r)$ gives, by means of (61) and (64), that $m(r)$ and $\mu(r)$ are nondecreasing functions of $r \in (r_0, \infty)$. Thus (70) holds iff at least one of

$$\lim_{r \rightarrow r_0} \left(1 - \frac{2m(r)}{r} \right) = 0; \quad \lim_{r \rightarrow r_0} \mu(r) = -\infty \tag{71}$$

holds.

Suppose the first of (71) fails. Then the second must hold and one has

$$\lim_{r \rightarrow r_0} \left(1 - \frac{2m(r)}{r} \right)^{-1} < \infty. \tag{72}$$

From the boundedness of $\rho(r)$ and $p_1(r)$ on $r \in (0, \infty)$ one has, by means of (64) and (72), that $\limsup_{r \rightarrow r_0} \mu'(r)$ is finite. This is incompatible with the second of (71). Hence the first of (71) must hold.

Let $f: (r_0, \infty) \rightarrow \mathbb{R}$ be the left side of (67). By the non-negativity of $p_1(r)$ and the first of (71) one has $\lim_{r \rightarrow r_0} f(r) \leq -\frac{1}{2}$. By condition (4), Eq. (61) and l'Hôpital's rule one has $\lim_{r \rightarrow \infty} m(r)/r = \lim_{r \rightarrow \infty} r^2 p_1(r) = 0$ and hence $\lim_{r \rightarrow \infty} f(r) = 1$. Hence there exists some $r_1 \in (r_0, \infty)$ such that $f(r_1) = 0$. The hypersurface $\{r = r_1\}$ is an $SO(3) \times \mathbb{R}$ -invariant photon surface of (M, \mathbf{g}) . □

Condition (3) of Theorem IV.3 may be expressed more succinctly as $G_{ab} Y^a Y^b \geq 0, \forall$ vectors \mathbf{Y} . With regard to condition (5) of Theorem IV.3, to have required \mathcal{T}_t to be contained in the hypersurface $\{t = \text{const}\}$ would have been too strong since, for Schwarzschild space-time, no spacelike hypersurface of the form $\{t = \text{const}\}$ in the exterior region contains a marginally outer trapped 2-surface.

Theorem IV.3 may be interpreted to the effect that, subject to the energy conditions expressed in condition (3), any static spherically symmetric black hole must be surrounded by an $SO(3) \times \mathbb{R}$ -invariant timelike photon surface. The following result may then be regarded as a partial converse in that it shows, subject to a suitable energy condition, that if there exists an $SO(3) \times \mathbb{R}$ -invariant timelike photon surface then there must be a naked singularity or a black hole, or more than a certain amount of matter.

Proposition IV.4: *Suppose the metric \mathbf{g} has the form (59) for $0 < r < \infty$, with $m(r)$ and $\mu(r)$ both C^0 , piecewise C^2 functions of $r \in (0, \infty)$. If the following all hold:*

- (1) $\rho(r)$ is a nonincreasing, bounded function of $r \in (0, \infty)$;
- (2) $\lim_{r \rightarrow 0} m(r) = 0$;
- (3) $4m(r) < r, \forall r \in (0, \infty)$;
- (4) $p_1(r) \leq \rho(r)/3, \forall r \in (0, \infty)$,

then (M, \mathbf{g}) can contain no $SO(3) \times \mathbb{R}$ -invariant timelike photon surfaces.

Proof: By conditions (1) and (2) with Eq. (61) one has $m(r) \geq (4\pi/3)r^3\rho(r) \forall r > 0$. By condition (4) one therefore has $4\pi r^2 p_1(r) \leq (4\pi/3)r^2\rho(r) \leq m(r)/r, \forall r > 0$. The left side of (67) is thus bounded from below by $1 - 4m(r)/r, \forall r > 0$. This is positive by condition (3). The left side of (67) is therefore nonvanishing for all $r > 0$. □

Note that this result is valid even for negative $p_1(r)$ and $\rho(r)$. Condition (2) of Proposition IV.4 prohibits any curvature singularity at $r = 0$. Condition (3) may be interpreted as a requirement that there is no black hole and less than a certain amount of matter. The result shows that one of these two conditions must fail if there is an $SO(3) \times \mathbb{R}$ -invariant timelike photon surface and conditions (1) and (4) both hold.

When the matter is a perfect fluid it is possible to improve condition (3) of Proposition IV.4 to condition (3) of the following result.

Theorem IV.5: *Suppose the metric \mathbf{g} has the form (59) for $0 < r < \infty$, with $m(r)$ and $\mu(r)$ both C^1 , piecewise C^2 functions of $r \in (0, \infty)$. If the following all hold:*

- (1) *the matter is a perfect fluid with pressure $p(r)$ and density $\rho(r)$;*
- (2) $\lim_{r \rightarrow \infty} 4\pi r^2 p(r) = \lim_{r \rightarrow \infty} 4\pi r^2 \rho(r) = 0$;
- (3) $(24/7)m(r) < r, \forall r \in (0, \infty)$;
- (4) $p(r) \leq \rho(r)/3, \forall r \in (0, \infty)$,

then (M, \mathbf{g}) can contain no $SO(3) \times \mathbb{R}$ -invariant timelike photon surfaces.

Proof: Let $f: (0, \infty) \rightarrow \mathbb{R}$ be the left side of Eq. (67). Since $m(r)$ and $\mu(r)$ are C^1 , piecewise C^2 functions of $r \in (0, \infty)$, one has by Eq. (62) that $f(r)$ is a C^0 , piecewise C^1 function of $r \in (0, \infty)$. The function $f'(r)$ is then a piecewise C^0 function of $r \in (0, \infty)$ which, by means of the Tolman–Oppenheimer–Volkoff Eq. (66), is given by

$$r f'(r) = 3 \frac{m(r)}{r} - 12\pi r^2 \rho(r) - 8\pi r^2 p(r) + \frac{\left(\frac{m(r)}{r} + 4\pi r^2 p(r) \right)}{\left(1 - 2 \frac{m(r)}{r} \right)} 4\pi r^2 (\rho(r) + p(r)). \quad (73)$$

For $r = r_1 \in (0, \infty)$ such that

$$0 = f(r_1) = 1 - 3 \frac{m(r_1)}{r_1} - 4\pi r_1^2 p(r_1), \quad (74)$$

Eq. (73) reduces to

$$r_1 f'(r_1) = 1 - 8\pi r_1^2 (\rho(r_1) + p(r_1)). \quad (75)$$

From condition (3) and Eq. (74) one has $4\pi r_1^2 p(r_1) > 1/8$, whence by condition (4) one has $4\pi r_1^2 \rho(r_1) > 3/8$. Thus (75) gives $f'(r_1) < 0$.

By condition (2), Eq. (61) and l'Hôpital's rule one has $\lim_{r \rightarrow \infty} m(r)/r = \lim_{r \rightarrow \infty} 4\pi r^2 p(r) = 0$ and hence $\lim_{r \rightarrow \infty} f(r) = 1$. Since it has been established that $f'(r)$ is negative for all $r \in (0, \infty)$ such that $f(r) = 0$, one must therefore have $f(r) > 0$ for all $r \in (0, \infty)$. Hence the space–time can contain no $SO(3) \times \mathbb{R}$ -invariant timelike photon surfaces. \square

Note that, as for Proposition IV.4, Theorem IV.5 is valid even for negative pressure and density.

One would like to remove the insufficient matter parts of condition (3) of Proposition IV.4 and condition (3) of Theorem IV.5, in other words to weaken these to a no-black-hole condition $2m(r) < r, \forall r > 0$. But no result to this effect is forthcoming. On the other hand no counterexample is known.

To conclude this section it will be shown that the physical significance of the photon sphere in Schwarzschild space–time, as discussed in the Introduction, carries over to the general static spherically symmetric case. Suppose the metric has the form (59) for $r_0 < r < \infty$ and is asymptotically flat in the limit $r \rightarrow \infty$. Assume $p_1(r) \geq 0, m(r) \geq 0, \forall r > r_0$. The matter need not be a perfect fluid. Denote the left side of (67) by $f(r)$. The condition of asymptotic flatness gives $\lim_{r \rightarrow \infty} f(r) = 1$ so, if there are any $SO(3) \times \mathbb{R}$ -invariant timelike photon surfaces, there will be an outermost such surface S . For simplicity assume $f'(r) \neq 0$ at S . Let \mathcal{R}_{ext} be the connected component of $\{q \in M: f(q) > 0\}$ that has S as its inner boundary and extends to $r = \infty$. Let \mathcal{R}_{int} be the connected component of $\{q \in M: f(q) < 0\}$ that has S as its outer boundary.

Consider first the case of a future endless affine null geodesic $\gamma(\lambda)$. The null geodesic equations for the metric (59) give

$$\frac{d^2 r}{d\lambda^2} = rf(r) \left\{ \left(\frac{d\theta}{d\lambda} \right)^2 + \sin^2 \theta \left(\frac{d\phi}{d\lambda} \right)^2 \right\} - \frac{\mu'(r)}{2} \left(\frac{dr}{d\lambda} \right)^2. \quad (76)$$

At any point $p \in |\gamma| \cap \mathcal{R}_{\text{ext}}$ such that $dr/d\lambda = 0$ one has $d^2 r/d\lambda^2 > 0$. At any point $p \in |\gamma| \cap \mathcal{R}_{\text{int}}$ such that $dr/d\lambda = 0$ one has $d^2 r/d\lambda^2 < 0$. Thus if γ starts outside S (i.e., in \mathcal{R}_{ext}) and is initially directed outwards, in the sense that $dr/d\lambda$ is initially positive, then γ will continue outwards. If γ starts in \mathcal{R}_{int} and is initially directed inwards, in the sense that $dr/d\lambda$ is initially negative, then γ will continue inwards until it falls either into a singularity or through an $SO(3) \times \mathbb{R}$ -invariant photon surface other than S .

Consider now a unit speed timelike geodesic $\xi(s)$. The timelike geodesic equations give

$$\frac{d^2 r}{ds^2} = -\frac{m(r)}{r^2} - 4\pi r p_1(r) + rf(r) \left\{ \left(\frac{d\theta}{ds} \right)^2 + \sin^2 \theta \left(\frac{d\phi}{ds} \right)^2 \right\} - \frac{\mu'(r)}{2} \left(\frac{dr}{ds} \right)^2. \quad (77)$$

For any point $p \in \mathcal{R}_{\text{ext}}$ one can arrange for the first three terms on the right of (77) to cancel and so obtain a unit speed timelike geodesic $\xi(s)$ through $p \in \mathcal{R}_{\text{ext}}$ at constant r . For $p \in \mathcal{R}_{\text{int}}$ the first three terms on the right of (77) are evidently negative. For $p \in |\xi| \cap \mathcal{R}_{\text{int}}$ such that $dr/ds = 0$ one has $d^2 r/ds^2 < 0$. Thus if ξ starts in \mathcal{R}_{int} and is initially directed inwards, in the sense that dr/ds is initially negative, then ξ will continue inwards until it falls either into a singularity or through an $SO(3) \times \mathbb{R}$ -invariant photon surface other than S .

V. SPHERICAL SYMMETRY: EXAMPLES

The following are some examples of $SO(3) \times \mathbb{R}$ -invariant and $SO(3)$ -invariant photon surfaces in familiar space–times.

Example 5 (Schwarzschild space–time): The metric of Schwarzschild space–time in single null (radiation) coordinates has the form

$$\mathbf{g} = - \left(1 - \frac{2m}{r} \right) du^2 + 2 du dr + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \quad (78)$$

In this case Eq. (54) reduces to $r = 3m$. The timelike hypersurface $\{r = 3m\}$ is thus an $SO(3) \times \mathbb{R}$ -invariant photon surface, or photon sphere, as expected. There are no other $SO(3) \times \mathbb{R}$ -invariant timelike photon surfaces.

For a nonzero cosmological constant Λ , the Schwarzschild metric (78) generalizes to the Schwarzschild–de Sitter metric,

$$\mathbf{g} = - \left(1 - \frac{2m}{r} + \frac{\Lambda r^2}{3} \right) du^2 + 2 du dr + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \quad (79)$$

One finds that Eq. (54) reduces to $r = 3m$, independent of the value of Λ . This is surprising since Schwarzschild and Schwarzschild–de Sitter space–times are not conformally related.

Example 6 (Schwarzschild interior solution): The Schwarzschild interior solution describes a spherically symmetric distribution of perfect fluid of radius R , bounded pressure p and constant density $\rho_0 > 0$. The solution is to have a metric of the form (59) and is to be matched at $r = R$ to a Schwarzschild vacuum solution in such a way that the pressure is a continuous function of r . One thus has $p_1(r) = p_2(r) =: p(r)$ for all $r: 0 \leq r < \infty$, $\rho(r) = \rho_0$ for all $r: 0 \leq r \leq R$, $\rho(r) = 0$ for all $r: R < r < \infty$ and $p(r) = 0$ for all $r: R \leq r < \infty$. The pressure $p(r)$ for $r: 0 \leq r \leq R$ is to be obtained by an integration of the Tolman–Oppenheimer–Volkoff equation (66) subject to the boundary condition $p(R) = 0$. This yields

$$m(r) = \frac{4\pi}{3} \rho_0 r^3, \quad 0 \leq r \leq R, \quad (80)$$

$$e^{\mu(r)} = \frac{(3 - u(r))^2}{4u^2(r)}, \quad 0 \leq r \leq R, \tag{81}$$

$$p(r) = \frac{u(r) - 1}{3 - u(r)} \rho_0, \quad 0 \leq r \leq R, \tag{82}$$

for

$$u(r) := \left(\frac{3 - 8\pi\rho_0 r^2}{3 - 8\pi\rho_0 R^2} \right)^{1/2}, \quad 0 \leq r \leq R. \tag{83}$$

The spherically symmetric system described by the Schwarzschild interior solution can exist in a state of stable equilibrium iff $m(R)/R < 4/9$ (see Stephani⁸). This condition is equivalent to $8\pi\rho_0 R^2 < 8/3$, which implies $p(r) \geq 0, \forall r \geq 0$, and implies that the absence of a black hole is a general feature of the Schwarzschild interior solution.

The left side of Eq. (67) now assumes the form

$$1 - \frac{3m(r)}{r} - 4\pi r^2 p(r) = \begin{cases} 1 - \frac{8\pi\rho_0 r^2}{3 - u(r)} & : 0 < r \leq R, \\ 1 - \frac{3m(R)}{r} & : r \geq R. \end{cases} \tag{84}$$

For $m(R)/R < 1/3$ one has that (84) is positive for all $r > 0$, so there are no timelike photon spheres. For $m(R)/R = 1/3$ there is a single timelike photon sphere which lies at the boundary $r = R$ of the matter. For $1/3 < m(R)/R < 4/9$ there is one timelike photon sphere outside the matter at $r = 3m(R) > R$ and one timelike photon sphere inside the matter at

$$r = \left(\frac{1 - 3\pi\rho_0 R^2}{\pi\rho_0(3 - 8\pi\rho_0 R^2)} \right)^{1/2} = \frac{2R}{3} \left(\frac{1 - \frac{9m(R)}{4R}}{\left(1 - \frac{2m(R)}{R}\right) \frac{m(R)}{R}} \right)^{1/2} < R. \tag{85}$$

For fixed R the radius of the outer photon sphere is a strictly increasing function of $m(R)/R$ while the radius of the inner photon sphere is a strictly decreasing function of $m(R)/R$.

Thus a Schwarzschild interior solution matched to a Schwarzschild vacuum exterior solution contains no black hole, and contains one timelike photon sphere iff $1/3 = m(R)/R$ and two timelike photon spheres iff $m(R)/R$ lies in the range $1/3 < m(R)/R < 4/9$. However for such values of $m(R)/R$ the space–time is unphysical in that the pressure at the center is $p(0) \geq \rho_0/\sqrt{3} > \rho_0/3$. Therefore under the reasonable energy condition $p(r) \leq \rho_0/3, \forall r: 0 \leq r < \infty$ there are no photon spheres in this example.

The energy condition $0 \leq p(r) \leq \rho(r)/3, \forall r \in [0, \infty)$ is in fact satisfied iff $m(R)/R$ lies in the range $0 \leq m(R)/R \leq 5/18$. The corresponding space–times (in view of $5/18 < 7/24$) satisfy all of conditions (1) to (4) of Theorem IV.5, except the smoothness of $\rho(r)$ at $r = R$. Thus considering a smoothed family of solutions approximating the solution above and having it as a strict limit, but each with smooth $\rho(r)$ at $r = R$, we can apply Theorem IV.5 to show no photon spheres exist in all the cases discussed in this section where this energy condition is satisfied. On the other hand Proposition IV.4 is directly applicable without smoothing, but only applies to the subset of these cases with $0 \leq m(R)/R < 1/4$.

Example 7 (Reissner–Nordström space–time): The Reissner–Nordström solution comprises a metric \mathbf{g} and an electromagnetic field F_{ab} given by

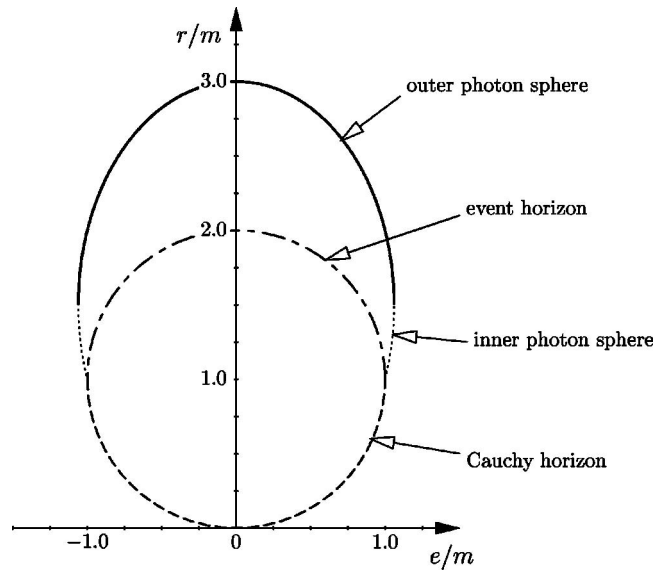


FIG. 2. The radii of the event horizon, Cauchy horizon and photon spheres for Reissner–Nordström space–time.

$$g = - \left(1 - \frac{2m}{r} + \frac{e^2}{r^2} \right) du^2 + 2 du dr + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \tag{86}$$

$$F_{tr} = -F_{rt} = \frac{e}{r^2}, \text{ all other components vanishing,} \tag{87}$$

where m is the ADM mass and e is the electric charge. We assume $m > 0$. There is an event horizon at $r = r_+ := m + \sqrt{m^2 - e^2}$ and a Cauchy horizon at $r = r_- := m - \sqrt{m^2 - e^2}$. The event horizon exists for $0 \leq (e/m)^2 \leq 1$, and the Cauchy horizon exists for $0 < (e/m)^2 \leq 1$. For $(e/m)^2 = 1$ they both lie at $r = m$. There can be no timelike photon spheres between the event horizon and the Cauchy horizon because the Killing field \mathbf{K} is spacelike there. Outside the event horizon the Killing field \mathbf{K} is timelike, so every hypersurface of the form $\{r = \text{const}\}$ which lies outside the event horizon and satisfies the photon sphere equation (54) is necessarily a timelike photon sphere.

Equation (54) assumes the form

$$r^2 - 3mr + 2e^2 = 0, \tag{88}$$

which has solutions r_{ps}^\pm given by

$$r_{ps}^\pm / m = \frac{3 \pm \sqrt{9 - 8(e/m)^2}}{2}. \tag{89}$$

The hypersurface $S^+ := \{r = r_{ps}^+\}$ exists for $0 \leq (e/m)^2 \leq 9/8$ and lies outside the event horizon and is therefore a timelike photon sphere. The hypersurface $S^- := \{r = r_{ps}^-\}$ exists for $0 < (e/m)^2 \leq 9/8$ but lies outside the event horizon only for $1 < (e/m)^2 \leq 9/8$, and so is a timelike photon sphere only then. The hypersurfaces S^+ and S^- coincide for $(e/m)^2 = 9/8$. The Cauchy horizon and event horizon are always null photon spheres.

For $0 \leq (e/m)^2 \leq 1$ the curvature singularity at $r = 0$ is locally naked but hidden behind an event horizon which lies strictly inside the only timelike photon sphere. For $1 < (e/m)^2$ the singularity at $r = 0$ is globally naked and is surrounded by two timelike photon spheres in the case $1 < (e/m)^2 < 9/8$, one timelike photon sphere in the case $(e/m)^2 = 9/8$ and by no photon spheres, either timelike or null, in the case $(e/m)^2 > 9/8$ (see Fig. 2).

Example 8 (Janis–Newman–Winicour space–time): The most general static spherically symmetric solution to the Einstein massless scalar field equations for a scalar field Φ satisfying $\square\Phi=0$ was obtained by Janis, Newman and Winicour.⁹ The Ricci tensor has the form $R_{ab} = 8\pi\Phi_{;a}\Phi_{;b}$. The solution is known¹⁰ to be expressible in the form

$$\mathbf{g} = -\left(1 - \frac{b}{r}\right)^\nu dt^2 + \left(1 - \frac{b}{r}\right)^{-\nu} dr^2 + \left(1 - \frac{b}{r}\right)^{1-\nu} r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{90}$$

$$\Phi = \frac{q}{b\sqrt{4\pi}} \ln\left(1 - \frac{b}{r}\right), \tag{91}$$

for $r:b < r < \infty$ where the constants b, ν are related to the ADM mass m and scalar charge q by

$$\nu = \frac{2m}{b}, \quad b = 2\sqrt{m^2 + q^2}. \tag{92}$$

We assume $b > 0$. There is a curvature singularity at $r = b$. In order to obtain $m \geq 0$ one must assume $0 \leq \nu \leq 1$. For $q = 0$ the solution reduces to the Schwarzschild solution.

Since all hypersurfaces of the form $\{r = \text{const}\}$ are timelike, one has from the photon sphere equation (54) that the only timelike photon sphere is at

$$r = \frac{b(2\nu + 1)}{2}, \tag{93}$$

which exists only for $\nu: \frac{1}{2} < \nu \leq 1$, i.e., for $0 \leq q^2 < 3m^2$. For $\frac{1}{2} < \nu \leq 1$ it is known¹¹ that a photon coming from infinity is deflected through an unboundedly large angle, i.e., the photon passes increasingly many times around the singularity as the closest distance of approach tends to the right side of (93).

Example 9 (Charged dilaton space–time): The static spherically symmetric charged dilaton solution¹² comprises a metric \mathbf{g} , a dilaton field Φ and an electromagnetic field F_{ab} given by

$$\mathbf{g} = -\left(1 - \frac{r_+}{r}\right)\left(1 - \frac{r_-}{r}\right)^\omega dt^2 + \left(1 - \frac{r_+}{r}\right)^{-1}\left(1 - \frac{r_-}{r}\right)^{-\omega} dr^2 + \left(1 - \frac{r_-}{r}\right)^{1-\omega} r^2 d\Omega^2, \tag{94}$$

$$e^{2\Phi} = \left(1 - \frac{r_-}{r}\right)^{(1-\omega)/\beta}, \tag{95}$$

$$F_{tr} = -F_{rt} = \frac{e}{r^2}, \quad \text{all other components vanishing}, \tag{96}$$

where r_+ and r_- are related to the ADM mass m and electric charge e by

$$r_+ + \omega r_- = 2m, \tag{97}$$

$$r_+ r_- = e^2(1 + \beta^2), \tag{98}$$

and β is a free parameter which controls the coupling strength between the dilaton and Maxwell fields, with ω defined in terms of β^2 by

$$\omega := \frac{1 - \beta^2}{1 + \beta^2}. \tag{99}$$

We assume $m > 0$.

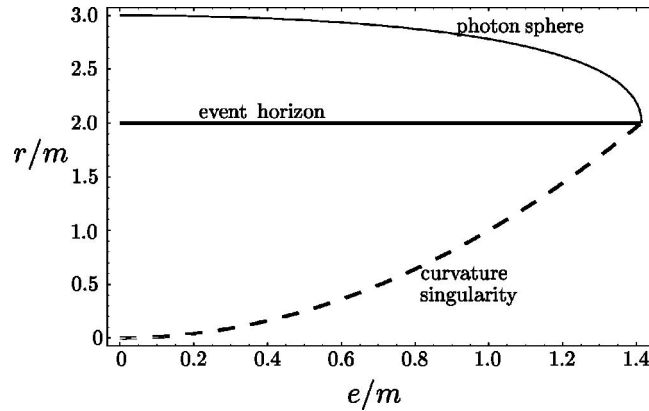


FIG. 3. The radii of the photon sphere, event horizon and curvature singularity are plotted against the electric charge for the charged dilaton solution.

For $\beta=0$ the solution reduces to the Reissner–Nordström solution considered in Example 7. For $\beta=0$ and $e=0$ the solution reduces to the Schwarzschild solution. The solution also reduces to the Schwarzschild solution for $e=r_-=0$ and arbitrary β . Here we shall consider the case $\beta^2=1$. In this case one has $(r_+, r_-)=(2m, e^2/m)$. There is an event horizon at $r=r_+=2m$ and a curvature singularity at $r=r_-=e^2/m$. For $0 \leq (e/m)^2 < 2$ the singularity at $r=r_-$ lies inside a black hole while for $(e/m)^2 > 2$ it is globally naked.

For $\beta^2=1$ the photon sphere equation (54) reduces to

$$r_{ps}^{\pm}/m = \frac{(6 + (e/m)^2) \pm \sqrt{36 + (e/m)^4 - 20(e/m)^2}}{4}. \tag{100}$$

For $0 \leq (e/m)^2 < 2$ one has $r_{ps}^- \leq r_- < r_+ < r_{ps}^+$ so there is a single timelike photon sphere. For $(e/m)^2 = 2$ one has $r_{ps}^- = r_- = r_+ = r_{ps}^+$ so there are no timelike photon spheres. For $2 < (e/m)^2 < 18$ both r_{ps}^+ and r_{ps}^- are complex so there are no timelike photon spheres. For $(e/m)^2 \geq 18$ one has $r_{ps}^- \leq r_{ps}^+ < r_-$ so there are again no timelike photon spheres. Thus in the black hole case $0 \leq (e/m)^2 < 2$ there is a single timelike photon sphere, while in the naked singularity case $(e/m)^2 > 2$ there are no timelike photon spheres. (See Fig. 3.)

Example 10 (Vaidya null dust collapse): The Vaidya null dust collapse model is a nonstatic, spherically symmetric space–time with a metric which, in terms of single null (radiation) coordinates (u, r, θ, ϕ) , assumes the form

$$g = - \left(1 - \frac{2m(u)}{r} \right) du^2 + 2 du dr + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \tag{101}$$

where $m(u)$ is a freely specifiable function of u . Setting $x^0 := u$, $x^1 := r$ in (35) one obtains

$$\frac{d^2 r}{du^2} = \frac{1}{r} \left(g_{uu} + \frac{dr}{du} \right) \left(g_{uu} + 2 \frac{dr}{du} \right) - \frac{3}{2} \frac{dr}{du} \partial_r g_{uu} - \frac{1}{2} (g_{uu} \partial_r g_{uu} + \partial_u g_{uu}), \tag{102}$$

$$= \frac{1}{r} \left\{ \left(1 - \frac{3m(u)}{r} \right) \left(1 - \frac{2m(u)}{r} - 3 \frac{dr}{du} \right) - \frac{dm(u)}{du} + 2 \left(\frac{dr}{du} \right)^2 \right\}. \tag{103}$$

This is the evolution equation for a spherically symmetric photon surface of the Vaidya collapse metric (101).

Consider the special case,

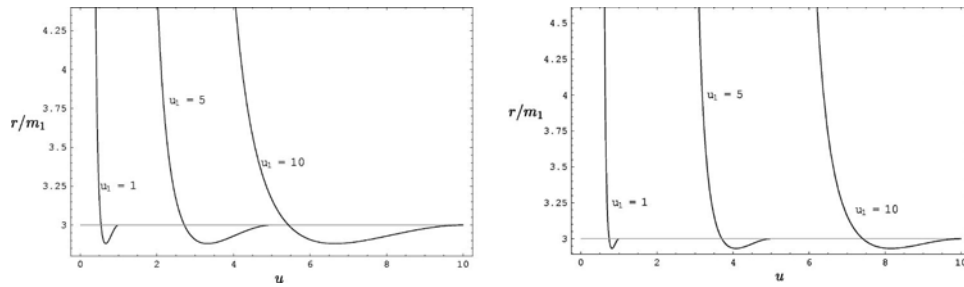


FIG. 4. The backwards evolution of the $r=3m_1$ Schwarzschild photon surface through collapsing Vaidya null dust. The space–time coordinates of the evolved surface are plotted for $m(u)=u/16$ (left) and $m(u)=u/32$ (right) for various values of the null time $u_1>0$ of the junction between the null dust and Schwarzschild regions. In each case the evolved photon surface fails to intersect the Minkowskian region $u<0$.

$$m(u) = \begin{cases} 0 & : -\infty < u < 0, \\ \lambda u & : 0 \leq u \leq u_1, \\ m_1 := \lambda u_1 & : u_1 < u < \infty, \end{cases} \quad (104)$$

for given constants $\lambda>0, u_1>0$. For $u<0$ the space–time is locally Minkowskian, for $u:0 \leq u \leq u_1$ there is inward falling null dust, and for $u>u_1$ the space–time is locally isometric to Schwarzschild space–time with ADM mass $m_1>0$. It is well-known there is a curvature singularity at $r=0$ and that for $\lambda:0 < \lambda \leq \frac{1}{16}$ the part of this singularity at $u=0$ is locally naked.

Fix $\lambda:0 < \lambda \leq \frac{1}{16}$. For $u>u_1$ Eq. (103) gives, as expected, that there is a photon surface at $r=3m_1$. We seek to evolve this photon surface backwards in time, though the in-falling null dust, to obtain a maximally extended photon surface S . The boundary conditions are

$$\left. \begin{aligned} r &= 3m_1 \\ \frac{dr}{du} &= 0 \end{aligned} \right\} \text{ at } u = u_1. \quad (105)$$

The results are shown in Fig. 4 for $\lambda=1/16$ and $\lambda=1/32$ for selected values of u_1 . One sees that in all cases S tends in the past direction to a null hypersurface of the form $\{u=\text{const}\}$. The conformal diagram must therefore be of the form sketched in Fig. 5.

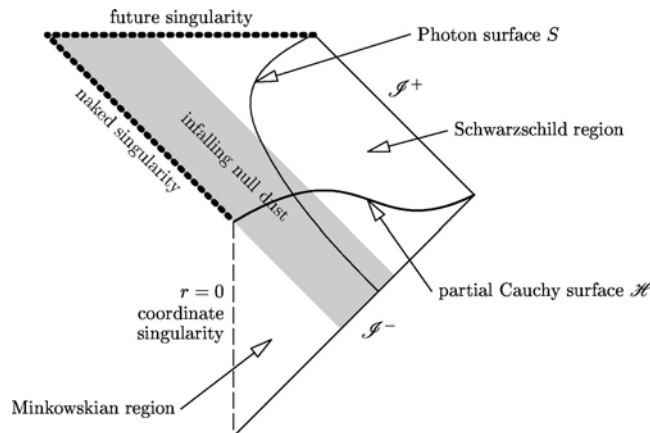


FIG. 5. The conformal diagram of the Vaidya null dust collapse model showing the photon surface S arising from the backwards evolution of the Schwarzschild photon sphere. Any partial Cauchy surface \mathcal{H} which extends to spatial infinity is cut into two components by S .

It is evident from Fig. 5 that the naked central singularity is enclosed within the photon surface S in the sense that any partial Cauchy surface \mathcal{H} extending to spatial infinity must intersect S in a 2-sphere. The physical significance of this may warrant further investigation.

VI. CONCLUDING REMARKS

The definition of a photon surface given in Sec. II is valid in an arbitrary space–time. However the result that a photon surface must have a second fundamental form which is pure trace indicates that a space–time must be specialized in some respect if it is to contain any photon surfaces. For spherically symmetric space–times there are always photon surfaces that respect the spherical symmetry. For space–times that are not spherically symmetric, the definitions of a photon surface and G -invariant photon surface may seem too restrictive. The problem is that, in general, one may not have orbiting null geodesics at a fixed radius. In Kerr space–time, for example, although there are orbiting null geodesics in the equatorial plane, those null geodesics which move in the direction of rotation do so at a different radius than those which move in the opposite direction. But it seems implausible that a concept which is physically important in the case of exact spherical symmetry should become invalid when even a small amount of angular momentum is introduced. A nontrivial generalization of the concepts of photon surface and G -invariant photon surface, at least to axially symmetric space–times, is thus required.

ACKNOWLEDGMENTS

This research was supported by the NFR of Sweden and NRF of South Africa.

¹C. Darwin, Proc. R. Soc. London A **249**, 180 (1958).

²C. Darwin, Proc. R. Soc. London A **263**, 39 (1961).

³K. S. Virbhadra and G. F. R. Ellis, Phys. Rev. D **62**, 084003 (2000).

⁴K. S. Virbhadra and G. F. R. Ellis, “Gravitational lensing by naked singularities.”

⁵S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space–time* (Cambridge University Press, Cambridge, 1973).

⁶B. Carter, Phys. Rev. **141**, 1242 (1966).

⁷R. H. Boyer, Proc. R. Soc. London A **311**, 245 (1969).

⁸H. Stephani, *General Relativity. An Introduction to the Theory of the Gravitational Field* (Cambridge University Press, Cambridge, 1982), p. 222.

⁹A. I. Janis, E. T. Newman, and J. Winicour, Phys. Rev. Lett. **20**, 878 (1968).

¹⁰K. S. Virbhadra, Int. J. Mod. Phys. A **12**, 4831 (1997).

¹¹K. S. Virbhadra, D. Narasimha, and S. M. Chitre, Astron. Astrophys. **337**, 1 (1998).

¹²J. H. Horne and G. T. Horowitz, Phys. Rev. D **46**, 1340 (1992).