UV^k-MAPPINGS ON HOMOLOGY MANIFOLDS

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1. Introduction

A deformation theorem of Bestvina and Walsh [2] states that, below middle and adjacent dimensions, $(k+1)$-connected mappings of manifolds to polyhedra can be deformed to UV^k-mappings. For example, if one has a map f from the n-sphere to the m-sphere, where $n \leq m$, one might expect a typical point inverse image to be a finite set (usually empty, if $n < m$), but the truth, however, may be rather the opposite: if $n > 4$, then f is homotopic to a map with (nonempty) simply connected point inverses. This is predicted by the high connectivity of the homotopy fiber of the map. It is sometimes more useful to consider approximations by maps that behave like these "space-filling curves," which are closer models of the underlying abstract homotopy theory, rather than adopt the usual strategy of approximating by smooth or piecewise linear maps. This phenomenon was an essential ingredient in the construction of nonresolvable homology manifolds in [4] and in the "desingularization" of higher dimensional homology manifolds in [5].

The goal of this paper is to establish results of this nature for maps from homology manifolds to polyhedra. The methods we develop here, which are new, even in the case of topological manifolds, are an adaptation to homology manifolds of a handle cancellation argument that has proved so useful in the classification theory for topological manifolds. These methods allow us to prove various controlled versions of the Bestvina-Walsh theorem for maps from a homology manifold to a polyhedron, provided the homology manifold possesses sufficient general position properties. In particular, we show how to take a map that is "ϵ-$(k+1)$-connected," a property we refer to as $AL^{k+1}(\epsilon)$, and "squeeze" it in a controlled fashion to be μ-$(k+1)$-connected, for arbitrarily small μ. The desired UV^k-map is obtained by taking a limit. The controls on the homotopies are strong enough to show that an ENR homology n-manifold, $n \geq 5$, with the disjoint disks property has the $UV^{[\frac{n-3}{2}]}$-approximation property, introduced in [5]. This is a considerable strengthening of the disjoint disks property and indicates yet another way in which the exotic homology manifolds constructed in [4] resemble topological manifolds.

A final observation is that the basic arguments apply to *any* ENR having sufficient general position properties. As an application we invoke a theorem of Krupski [11] to get the curious result that a 1-connected map from a connected, homogeneous, n-dimensional ANR, $n \geq 3$, to a connected ANR is homotopic to a surjection with connected point-inverses.

Here is our basic result.

Theorem 1. *Suppose X is a compact, connected,* ENR *homology n-manifold, $n \geq 3$, B is a connected finite polyhedron, and $f\colon X \to B$ is $AL^{k+1}(\epsilon)$, $2k+3 \leq n$ for some $\epsilon > 0$, where*

Date: October 22, 2007.
Key words and phrases. homology manifold, absolute neighborhood retract, UV^k-mappings.

X is assumed to have has the disjoint disks property if $k \geq 1$. Then f is $(C(k)\cdot\epsilon)$-homotopic to a UV^k-map, where $C(k)$ is a positive constant depending only on k.

In Section 5 we indicate how the proof of Theorem 1 can be modified to obtain the following controlled version.

Theorem 2. *Suppose X is a compact, connected,* ENR *homology n-manifold, B is a connected finite polyhedron, Y is a metric space, and $p\colon B \to Y$ is a map. If $f\colon X \to B$ is $AL^{k+1}(\epsilon)$ over Y, and X has the disjoint disks property if $k \geq 1$, then f is $(C(k)\cdot\epsilon)$-homotopic (over Y) to a UV^k-map, where $C(k)$ is a positive constant depending only on k.*

As Theorem 2 essentially defines the $UV^{[\frac{n-3}{2}]}$-approximation property [5], we get as a corollary

Theorem 3. *A compact, connected* ENR *homology n-manifold, $n \geq 5$, satisfying the disjoint disks property has the $UV^{[\frac{n-3}{2}]}$-approximation property.*

As a special case (Y = a point) we recover the analogue of the theorem of Bestvina and Walsh for "nice" homology manifolds.

Theorem 4. *Suppose X is a compact, connected,* ENR *homology n-manifold, and suppose B is a connected finite polyhedron. Suppose $f\colon X \to B$ is a $(k+1)$-connected map for some $k \geq 0$, $2k+3 \leq n$. If $k \geq 1$, we assume further X has the DDP. Then f is homotopic to a UV^k-map.*

Remark. By applying Theorem 2, one can easily generalize each of these results to allow B to be a compact ANR. If B is finite dimensional, it has a mapping cylinder neighborhood N in some euclidean space [14] with mapping cylinder projection $\pi\colon N \to B$. The composition of f with the inclusion $\iota\colon B \to N$ remains $AL^{k+1}(\epsilon)$ over B, so we can apply Theorem 2 to $\iota \circ f\colon X \to N$. Composing the result with π, which is cell-like, will then recover the desired homotopy of f to a UV^k-map. If B is infinite dimensional, cross with the Hilbert cube to get a Hilbert cube manifold (see [7]), which is triangulable, and proceed in much the same way.

Our methods provide an alternative proof of the Bestvina-Walsh theorem referred to above.

Theorem 5 (Bestvina and Walsh [2])**.** *Let M^m be a compact manifold and K a polyhedron. If $f\colon M \to K$ is a $(k+1)$-connected map and $f|\partial M$ is UV^k, then f is homotopic rel (∂M) to a UV^k map, provided that $k \leq \left[\frac{m-3}{2}\right]$.*

Other results of this type are due to Keldyš [10], Anderson [1], Wilson [18, 19], Walsh [17], Černavskii [6], and Ferry [8].

Remark. Lacher, ([12], §5 and §7) (see also [9]), has shown that a $UV^{[\frac{n-1}{2}]}$-map between compact n-manifolds must be cell-like and that a UV^{k-1}-map between $2k$-manifolds must be a *spine map*, in which spines of connected summands are collapsed to points. Thus, the result in Theorem 5 is best possible for maps from the n-sphere S^n to itself of degree $d \neq \pm 1$.

2. Definitions and Preliminary Results

Definition. (1) A **homology n-manifold** is a space X having the property that for each $x \in X$,

$$H_k(X, X-x;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & k=n \\ 0 & k \neq n. \end{cases}$$

(2) A **euclidean neighborhood retract** (ENR) is a space homeomorphic to a closed subset of euclidean space that is a retract of some neighborhood of itself. That is, a locally compact, finite dimensional ANR.

(3) A space X satisfies the **disjoint disks property** (DDP) if for every $\epsilon > 0$ and maps $f, g\colon D^2 \to X$, there are maps $f', g'\colon D^2 \to X$ so that $d(f, f') < \epsilon$, $d(g, g') < \epsilon$ and $f'(D^2) \cap g'(D^2) = \emptyset$.

(4) There are three notions of k-connectivity of a map, depending on the type of control one wishes to consider.

   (a) A map $f\colon A \to B$ between connected ENR's is **k-connected** if for every polyhedral pair (P, Q), $\dim P \leq k+1$, and pair of maps $\alpha\colon P \to B$ and $\alpha_0\colon Q \to A$ satisfying $f \circ \alpha_0 = \alpha|Q$, there is a map $\bar{\alpha}\colon P \to A$ extending α_0 such that $f \circ \bar{\alpha}$ is homotopic to α, rel α_0. This is equivalent to f inducing isomorphisms on homotopy groups through dimension $k-1$ and an epimorphism in dimension k.

   (b) Given $\epsilon > 0$, a map $f\colon A \to B$ is $AL^{k+1}(\epsilon)$ if it has the ϵ-homotopy lifting property for $(k+1)$-dimensional polyhedra. That is, (P, Q) is a polyhedral pair with $\dim P \leq k+1$, $\alpha_0\colon Q \to A$ and $\alpha\colon P \to B$, with $f \circ \alpha_0 = \alpha|Q$, then there is a map $\bar{\alpha}\colon P \to A$ extending α_0 such that $f \circ \bar{\alpha}$ is ϵ-homotopic to α in B, rel $\alpha|Q$. The lift $\bar{\alpha}$ of α will be called an **ϵ-lift of α, rel** α_0 (or, sometimes, rel Q).

   (c) Given $\epsilon > 0$ and a map $p\colon B \to C$, a map $f\colon A \to B$ is $AL^{k+1}(\epsilon)$ **over** C, if it has the ϵ-homotopy lifting property over C for $(k+1)$-dimensional polyhedra. That is, there are homotopy liftings as in (b), but the size of the homotopies are measured in C via p. This is the same as Quinn's notion of a relatively $(\epsilon, k+1)$-connected map over C (Definition 5.1 of [15]).
   Notice that (a) is the same as (c) when C is a point, and (b) is the same as (c) when $p = \mathrm{id}\colon B \to B$.

(5) A compact connected space C **has property** UV^k, $k \geq 0$, if for some (and, hence, any) embedding of C in an ANR X and every neighborhood U of C in X, there is a connected neighborhood V of C lying in U such that the inclusion $\pi_i(V) \to \pi_i(U)$ is 0 for $0 \leq i \leq k$.

(6) A surjection $f\colon A \to B$ between compact ENR's is UV^k, $k \geq 0$, if its point inverses have property UV^k. A UV^{-1}-map is merely a surjection.

The following basic result is due to Lacher [12, 13].

Theorem 6. *A map $f\colon A \to B$ between compact* ENR*'s is UV^k iff it is $AL^{k+1}(\epsilon)$ for every $\epsilon > 0$.*

We begin with a controlled UV^{-1} version.

Proposition 1. *Suppose A and B are compact, connected* ENR*'s of dimension ≥ 1, $\epsilon > 0$, and $\pi\colon B \to C$ is a map, where C is a metric space. If $f\colon A \to B$ is $AL^0(\epsilon)$ over C, then f is 2ϵ-homotopic (over C) to a surjection.*

Proof. Assume all measurements are made in C. Let P be a finite subset of B such that every point of B can be joined to a point of P by an arc of diameter $\leq \epsilon/2$. By hypothesis, there is a map $\alpha\colon P \to A$ whose composition with f is ϵ-homotopic to the inclusion. Since $\dim A \geq 1$, we may assume α is one-to-one. Let $P' = \alpha(P)$. Using the homotopy extension theorem on a small neighborhood of P' in A, we can get an extension of the ϵ-homotopy of $f|P'$ to α^{-1} to an ϵ-homotopy of f to a map that sends P' to P. Thus there is an ϵ-homotopy of f to a map that is $AL^0(\epsilon/2)$ over C. A sequence of such maps can be constructed so as to converge to a surjection that is 2ϵ-homotopic to f. □

The next lemma gives a criterion for determining when an extension of an $AL^{k+1}(\epsilon)$ is (almost) $AL^{k+1}(\epsilon)$.

Lemma 1. *Suppose $X_1 \subseteq X_2$ and B are compact* ENR*'s, $\delta > 0$, and $\epsilon > 0$, and suppose that for some integer $k \geq 0$, $f\colon X_2 \to B$ is a map such that*

(i) $f|X_1$ is $AL^{k+1}(\epsilon)$, and
(ii) if g is a map of a k-dimensional polyhedron R into X_2, then g is δ-homotopic over B to a map of R into X_1.

Then f is $AL^{k+1}(2\delta + \epsilon)$.

Proof. Suppose (P, Q) is a polyhedral pair, $\dim P \leq k + 1$, and suppose $\alpha\colon P \to B$, and $\alpha_0\colon Q \to X_2$ satisfy $f \circ \alpha_0 = \alpha|Q$. For any $\mu > 0$ there is a μ-homotopy of the identity on P to a map $r\colon P \to P$, which is fixed on Q and outside a neighborhood of Q, that deformation retracts a small regular neighborhood N of Q onto Q. By precomposing α with such a map, we can get an μ-homotopy of α to a map $\alpha_1\colon P \to B$, whose restriction to N can also be lifted by α_0.

Let $P_0 = \mathcal{C}\ell(P - N)$, and let $Q_0 = Q \cap P_0 = \mathrm{bd}N$. Since $\dim Q_0 \leq k$, there is a δ-homotopy (over B) of $\alpha_0|Q_0$ that takes Q_0 into X_1 . Since Q_0 is bicollared in N, this homotopy can be extended to a δ-homotopy of α_0 on N (over B) that is fixed on Q. Call the resulting map $\bar{\alpha}_0\colon N \to X_2$. Composing with f gives an δ-homotopy of $\alpha_1|N$ in B, which can be extended to an δ-homotopy of α_1 on P to $\alpha_2\colon P \to B$, since N is collared in P. By hypothesis, $f|X_1$ is $AL^{k+1}(\epsilon)$, and so $\alpha_2|P_0$ can be ϵ-lifted to X_1 (over B), rel $\bar{\alpha}_0|Q_0$. This map, in turn, extends to a map $\bar{\alpha}\colon P \to X_2$, whose restriction to Q is α_0, and, assuming $\mu < \delta$, $f \circ \bar{\alpha}$ is $(2\delta + \epsilon)$-homotopic to α rel Q. □

An argument virtually identical to the one just given also proves the following lemma.

Lemma 2. *Suppose X and B are compact* ENR*'s, $\delta > 0$, and $\epsilon > 0$. If $f\colon X \to B$ is $AL^{k+1}(\epsilon)$ and g is δ-homotopic to f, then g is $AL^{k+1}(2\delta + \epsilon)$.*

The proof of the next lemma is an easy application of the definition.

Lemma 3. *Suppose A, B, and C are compact metric spaces and $f\colon B \to C$ is an $AL^{k+1}(\epsilon)$-map for some $\epsilon > 0$. Then there exists $\delta > 0$ such that if $g\colon A \to B$ is $AL^{k+1}(\delta)$ over B, then $f \circ g\colon A \to C$ is $AL^{k+1}(2\epsilon)$ (over C).*

We shall separate the proof of Theorem 1 into two cases: $k = 0$ and $k \geq 1$. The intent is to present the main ideas first in a somewhat less cluttered setting, so that they may be a bit more transparent. This approach has, of course, introduced redundancies into the exposition, but we hope they prove to be of value to the reader.

3. UV^0

In this section we assume only that X is a compact ENR homology n-manifold, $n \geq 3$. The general position properties we require of X are summarized in the following lemma.

Lemma 4. *A connected* ENR *homology n-manifold X is arc-wise connected. Moreover, if $n \geq 3$, any map of a finite 1-complex into X can be approximated by an embedding.*

Proof. The first assertion is well-known, since for compact connected metric spaces the conditions, "locally connected," "locally path connected," and "locally arcwise connected" are equivalent. In particular, any continuous image of [0, 1] is arcwise connected and locally arcwise connected. The "moreover" part follows from the first statement and Alexander duality, which implies that if U is any connected open subset of X and A is a closed, 1-dimensional subset of U, then $H_1(U, U-A) \cong \check{H}^{n-1}(A) = 0$, which implies that $H_0(U-A) = 0$. That is, X has the "disjoint arcs property." □

We start by proving a simple homotopy analogue of our main result in the base case $k = 0$. Keep in mind that all measurements are made in B unless specifically indicated otherwise.

Proposition 2. *Suppose a surjection $f\colon X \to B$ is an $AL^1(\delta)$-map and $\mu > 0$. Then there is an* ENR *$\bar{X}$ obtained by adding 1- and 2-cells to X and an extension $\bar{f}\colon \bar{X} \to B$ such that $\bar{f}$ is $AL^1(\mu)$ and $\bar{X}$ 2δ-collapses to X.*

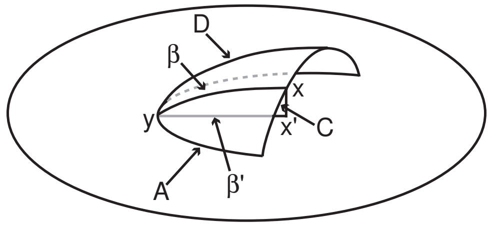
Proof. Triangulate B so that the diameter of the star of each simplex is less than $\mu' < \mu/3$, where μ' is chosen so that maps into B that are μ'-close are $\mu/3$-homotopic. The inverse image under f of each simplex $\sigma \in B$ is compact. If U is a small path-connected open neighborhood of σ in B, then $f^{-1}(\sigma)$ is contained in finitely many components of $f^{-1}(U)$. Attach finitely many 1-cells to X connecting the components of $f^{-1}(U)$ that contain points of $f^{-1}(\sigma)$. Do this for each $\sigma \in B$ and call the result X_1. The map f extends to $f_1\colon X_1 \to B$. If the neighborhood U of each $\sigma \in B$ is sufficiently small, f_1 is $AL^1(\mu/3)$. For each simplex σ in B, choose a neighborhood V of σ lying in U so that $f^{-1}(V)$ meets only components of $f^{-1}(U)$ which meet $f^{-1}(\sigma)$. A path in B can be broken into finitely many segments, each lying in one of these sets V. It suffices to μ'-lift one such segment relative to given lifts on the ends. But this is easily accomplished using the 1-cells of X_1.

Let C be a 1-cell in $\mathcal{C}\ell(X_1 - X)$. Since $f\colon X \to B$ is $AL^1(\delta)$, $f_1|C$ has a δ-lift to X, which we may assume is an embedding. Call the image arc A. Attach a 2-cell D to X_1 by identifying its boundary with $A \cup C$. Call the result X_2, and use the δ-homotopy from $f_1(C)$ to A to extend f_1 to $f_2\colon X_2 \to B$. Unfortunately, the map f_2 is no longer $AL^1(\mu/3)$, since all we know about the image of D is that it has size δ in B.

We remedy this as follows. Parameterize D as the quotient of $A \times I$ with the intervals over ∂A identified to points, and identify A with $A \times 0$ and C with $A \times 1$. Let A_0 be a finite subset of A such that every point of D is within $\mu/3$ (measured in B) of a point of $A_0 \times I \subseteq D$. Let y be a point of A_0, let $\beta = y \times I \subseteq D$, and let $x = y \times 1 \in C$. Since f is surjective, there is a point x' in X such that $f_2(x) = f(x')$. Since f_1 is $AL^1(\mu/3)$, there is a path β' in X_1 connecting y to x' such that $f_2 \circ \beta$ is $(\mu/3)$-homotopic to $f_1 \circ \beta'$ (rel $\{x, y\}$). We have a map from β to β' sending x to x' and y to y, so we can attach its mapping cylinder (rel y) to X_2. We can extend the map f_2 to this mapping cylinder, using the $(\mu/3)$-homotopy above, so that mapping cylinder fibers have size $< \mu/3$ in B. Thus, all points on the new 2-cell are $(\mu/3)$-close to X, as well. Performing this construction for all $y \in A_0$ produces a relative 2-complex X_3, and a map $f_3\colon X_3 \to B$, which, by Lemma 1, is $AL^1(\mu)$. X_3 δ-collapses to X_2, which, in turn, δ-collapses to $X_1 - \mathrm{int}C$.

Repeat this construction for every 1-cell, C, added to X to get X_1, making sure that the corresponding family of arcs A in X is mutually exclusive. The resulting space $\bar{X}$ 2δ-collapses to X and admits an $AL^1(\mu)$-map $\bar{f}\colon \bar{X} \to B$. □

The figure below illustrates a single 2-cell attached to X_2 and a single point $y \in A_0$. The placement of the path β' is misleading, however, since it can wind about the other 1-cells we attached to X when we formed X_1.



The next proposition demonstrates how to slide X onto $\bar{X}$, ending with a map with arbitrary preassigned AL^1 control. If X were a topological manifold, this could be accomplished fairly easily by adding canceling pairs of solid 1- and 2-handles to $X \times I$ along $X \times 1$. We begin by stating without proof a lemma which gives a useful criterion for a map to be an $AL^1(\epsilon)$-map.

Lemma 5. *Let B be a finite polyhedron or compact ANR. Then for each $\epsilon > 0$ there is a $\delta > 0$ so that if $f : X \to B$ is a surjective map from a compact locally path connected space to B and for each $x_0,\ x_1 \in X$ with $f(x_0) = f(x_1)$ there is a path ω in X with $\omega(0) = x_0$ and $\omega(1) = x_1$ such that $\mathrm{diam} f(\omega) < \delta$, then f is an $AL^1(\epsilon)$-map.*

Proposition 3. *Suppose X is a compact, connected,* ENR *homology n-manifold, $n \geq 3$, and suppose $\bar{X} = X \cup D$, where D is a 2-cell and $A = X \cap D = X \cap \partial D$ is an arc. Then for every neighborhood U of A in X and every $\eta > 0$, there is a homotopy $h\colon X \times I \to \bar{X}$ and a deformation retraction $d\colon \bar{X} \to X$ such that*

(i) each h_t is the identity outside U,
(ii) $d \circ h\colon X \times I \to X$ is an η-homotopy that deformation retracts a neighborhood of A onto A inside U,
(iii) $h_1\colon X \to \bar{X}$ is $AL^1(\eta)$.

Proof. Let $\delta > 0$ be given. We will construct a δ-homotopy $R : X \times I \to X$ so that

i. $R_t|(X - U) \cup A = id$ for all t.
ii. $R_1(V_0) \subset A$ for some neighborhood V_0 of A.
iii. There are neighborhoods V_i of A, $i = 1, 2, \ldots, k$, $k > 1/\delta$, so that $R_t(V_{i+1}) \subset V_i$ for all t. We set $V_i = \emptyset$ for $i > k$.
iv. If $z_0,\ z_1 \in R_1^{-1}(a) \cap \overline{U - V_0}$, then there is a path $\omega : [0,1] \to U - \bar{V}_1$ with $\omega(0) = z_0,\ \omega(1) = z_1$, and $\mathrm{diam}(\omega) < 2\delta$.
v. If $z_0,\ z_1 \in R_1^{-1}(a) \cap \overline{U - V_1}$, then there is a path $\omega : [0,1] \to U - \bar{V}_2$ with $\omega(0) = z_0,\ \omega(1) = z_1$, and $\mathrm{diam}(\omega) < 2\delta$.
vi. If $z_0,\ z_1 \in R_1^{-1}(a) \cap \overline{V_0 - V_2}$, then there is a path $\omega : [0,1] \to U - \bar{V}_3$ with $\omega(0) = z_0,\ \omega(1) = z_1$, and $\mathrm{diam}(\omega) < 2\delta$.
vii. If $z_0,\ z_1 \in R_1^{-1}(a) \cap \overline{V_i - V_{i+2}}$ for $i > 0$, then there is a path $\omega : [0,1] \to V_{i-1} - \bar{V}_{i+3}$ with $\omega(0) = z_0,\ \omega(1) = z_1$, and $\mathrm{diam}(\omega) < 2\delta$.

Given R_t as above, here is how we construct the desired homotopy h. Identify D with $A \times [0, 1]$ using the sup metric and construct a function $\lambda : X \to [0, 1]$ with $\lambda(\partial V_i) = \frac{i}{k}$ and $\lambda(\overline{V_i - V_{i+1}}) \subset [\frac{i}{k}, \frac{i+1}{k}]$ for each i. The map h_1 is defined by the formula

$$h_1(x) = \begin{cases} R_1(x) & \text{if } x \in X - V_0 \\ (R_1(x),\, \lambda(x)) & \text{if } x \in V_0. \end{cases}$$

Projection from $A \times [0, 1]$ to A gives a homotopy from h_1 to R_1 and the homotopy R_t gives a homotopy back to the identity. The deformation retraction d is given by projecting $A \times [0, 1]$ to A.

If $h_1(z_0) = h_1(z_1) = z$, then for $z \in U - A$, we must have $z_0, z_1 \in U - V_0$. Condition iv gives us a path of diameter $< 2\delta$ in $U - V_1$ connecting z_0 and z_1. The image of this path under R_1 has diameter $< 4\delta$ and its image under h_1 also has diameter $< 4\delta$.

If $h_1(z_0) = h_1(z_1) = z$, then for $z \in A = A \times 0$, we must have $z_0, z_1 \in U - V_1$. In this case, there is a path ω in $U - V_2$ with $\omega(0) = z_0$ and $\omega(1) = z_1$ such that $\text{diam}(\omega) < 2\delta$. The image of this path under R_1 has diameter $< 4d$ and its image under h_1 has diameter $< 4\delta$.

If $h_1(z_0) = h_1(z_1) = z$, then for $z \in A \times (0, 1]$, we must have $z_0, z_1 \in V_i - V_{i+2}$ for some $i \geq 0$. In this case, there is a path ω in $V_{i-1} - V_{i+3}$ or in $U - V_3$ in case $i = 0$, with $\omega(0) = z_0$ and $\omega(1) = z_1$ such that $\text{diam}(\omega) < 2\delta$. The image of this path under R_1 has diameter $< 4d$ and its image under h_1 has diameter $< 4\delta$.

By Lemma 5 above, choosing $\delta > 0$ sufficiently small will insure that h_1 is an $AL^1(\eta)$-map, as desired.

We now proceed with the construction of the map R_t and the neighborhoods U and $V_0, \ldots, V_k$. We start with a neighborhood strong deformation retraction from U to A. This is a map $R' : U \times I \to U$ such that $R'_0 = id$, $R'_1(U) \subset A$, and $R'_t|A = id$ for all t. Using the estimated homotopy extension theorem, we can find $R : U \times I \to U$ so that $R = R'$ on a neighborhood V_0 of A and so that R extends by id to all of X. We may assume that R is a τ-homotopy, where $\tau > 0$ is a constant as small as we like to be chosen later.

Let $z_0, z_1 \in R_1^{-1}(R_1(z_0)) \cap (U - V_0)$. There is an arc of diameter $< 2\tau$ connecting z_0 to z_1 in the complement of A. This is obtained by starting with the paths from z_0, z_1 to $R_1(z_0)$ given by the homotopy R_t and using Lemma 4 to push these paths into the complement of A by small moves.

Let $W_0 \subset \bar{W}_0 \subset V_0$ be an open neighborhood of A. For each $z \in R_1^{-1}(z_0) \cap (U - V_0)$, there is an $\epsilon_z > 0$ so that every point in $B_{\epsilon_z}(z)$ can be connected to z in $U - \bar{W}_0$ by a path of diameter $< \tau$. Choosing a finite cover of $R_1^{-1}(R_1(z_0)) \cap (U - V_0)$ by such balls, we can find finitely many paths α_i in the complement of A so that every pair of points in $R_1^{-1}(R_1(z_0)) \cap (U - V_0)$ can be connected by a path (or arc) of diameter $< 4\tau$ lying in $(U - \bar{W}_0)) \cup \bigcup_i \alpha_i([0, 1])$.

Moreover, we can choose $\sigma_{z_0} > 0$ so that $R_1^{-1}(B_{\sigma_{z_0}}) \cap (U - V_0)$ lies in the union of this finite collection of $B_{\epsilon_z}(z)$'s. Covering $R_1(U - V_0)$ by finitely many such $B_{\sigma_{z_0}}$'s, we see that there is a finite collection of arcs $\{\alpha_{ij}\}$ sp that if $z_0,\ z_1 \in U - V_0$ with $R_1(z_0) = R_1(z_1)$, then z_0 and z_1 are connected by a path (or arc) of diameter $< 4\tau$ lying in $(U - \bar{W}_0)) \cup \bigcup_{ij} \alpha_{ij}([0,\,1])$.

Choose V_1 to be a neighborhood of A such that $R(V_1 \times I) \cap (U - \bar{W}_0) \cup_{ij} \alpha_{ij}([0,\,1]) = \emptyset$. For $\tau < \delta/2$, this insures that conditions i-iv above are satisfied.

The construction of the remaining V_i's is similar to the construction of V_1 and the proof is omitted. □

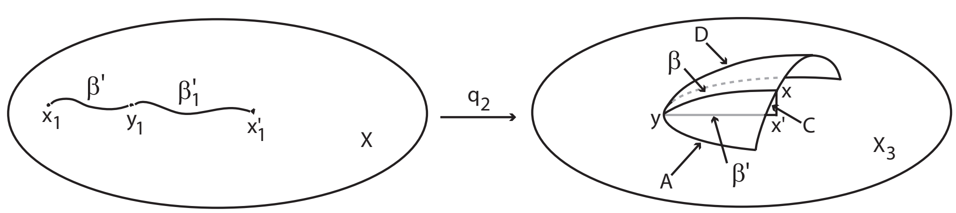
The following proposition provides the key to proving Theorem 1 for the case $k = 0$.

Proposition 4. *Suppose X is a compact, connected,* ENR *homology n-manifold, $n \geq 3$, B is a connected finite polyhedron, $f\colon X \to B$ is $AL^1(\epsilon)$, and $\mu > 0$. Then f is 10ϵ-homotopic to an $AL^1(\mu)$-map.*

Proof. Suppose X and B are given as in the hypothesis, and suppose $\mu > 0$. By Proposition 1, we can get a 2ϵ-homotopy of f to a surjection. The resulting map, which we shall still call f, is now, by Lemma 2, only $AL^1(5\epsilon)$. Set $\delta = 5\epsilon$.

Recall the construction in the proof of Proposition 2. We obtained $X_1 \subseteq X_2$ from X by attaching 1-cells to X to get X_1 and 2-cells to X_1 to get X_2, and extensions $f_1 \subset f_2$ of $f\colon X \to B$ to X_1 and X_2, respectively, such that f_1 is $AL^1(\mu')$ and f_2 is only $AL^1(\delta)$, where $\mu' > 0$ will be determined later. Since X is a homology n-manifold and $n \geq 3$, we may assume that the arcs in X along which the 2-cells are attached to form X_2 are mutually exclusive.

Enclose the attaching arcs in neighborhoods whose closures are mutually exclusive. Let D be a 2-cell of $X_2 - X_1$ attached to X along an arc A. (The complementary arc $C \subseteq \partial D$ was added when X_1 was constructed.) The arc $\beta \subseteq D$ and path $\beta' \subseteq X_1$ from points $x \in C$ and $x' \in X$, respectively, to a point y in A, were chosen so that $f_2(x) = f(x')$ and $f_2|\beta$ and $f_1|\beta'$ are μ'-homotopic in B.



Apply Proposition 3 to get a homotopy of the inclusion of X in X_2 to an $AL^1(\eta_2)$-map $q_2 : X \to X_2$ that is the identity on the complement of the union of the neighborhoods of the attaching arcs and is an η_2-homotopy on X after composing with the collapse $X_2 \searrow X$. Let y_1, x_1, x_1' be points of X that map to y, x, x', respectively. Then there are arcs β_1 and β_1' in X joining y_1 to x_1 and y_1 to x_1', respectively, such that $q_2(\beta_1)$ and $q_2(\beta_1')$ are η_2-close to β and β', respectively. We may assume that β_1 and β_1' are embedded and that $\beta_1 \cap \beta_1' = y_1$. We may also assume that the collection of all the arcs $\beta_1 \cup \beta_1'$ is mutually exclusive.

It is possible to arrange it so that $q_2(\beta_1) = \beta$ and $q_2(\beta_1') = \beta'$ at the expense of ending up with a map q_2 that is only $AL^1(6\eta_2)$. Given $\beta_1 \cup \beta_1'$ in X, let X' be the space obtained by attaching $(\beta_1 \cup \beta_1') \times I$ to X so that $(\beta_1 \cup \beta_1') \times 0$ is identified with $(\beta_1 \cup \beta_1')$ and the intervals over the endpoints of β_1 and β_1' are identified to points. Get a map $X' \to X_2$ extending q_2 using the η_2-homotopy from $q_2(\beta_1 \cup \beta_1')$ to $\beta \cup \beta'$, rel the endpoints of β and β'. Then, by Lemma 1, this map is $AL^1(3\eta_2)$. By Lemma 3 and Proposition 3, we can get a map from X to X' so that the composition $X \to X' \to X_2$ is $AL^1(6\eta_2)$. Thus, after rescaling, we may assume that q_2 is $AL^1(\eta_2)$, $q_2(\beta_1) = \beta$, and $q_2(\beta_1') = \beta'$.

In Proposition 2 this construction is performed a finite number of times for each of the 2-cells added to X to form X_2. Since the collection of arcs $\beta_1 \cup \beta_1'$ is mutually exclusive, we can perform this construction for all of the arcs simultaneously; hence, we can assume that we have an $AL^1(\eta_2)$-map $q_2 \colon X \to X_2$ that works as above for all of the (β, β') arc-path pairs.

The next step in the proof of Proposition 2 was to add mapping cylinders of the maps $\beta \to \beta'$ (rel y) to X_2, arriving at an ENR $\bar{X}$ obtained from X_2 by attaching 2-cells (the mapping cylinders) along $\beta \cup \beta'$, and an extension $\bar{f} \colon \bar{X} \to B$ of f_2 that is $AL^1(\mu')$ and δ-homotopic to the collapse from $\bar{X}$ to X_2 composed with f_2.

Form the space X_3 by attaching 2-cells to X along the arcs $\beta_1 \cup \beta_1'$, and get an $AL^1(\eta_2)$-map $q' \colon X_3 \to \bar{X}$ by combining $q_2 \colon X \to X_2$ with a map between corresponding attaching 2-cells that realizes the mapping cylinder identification. That is, the 2-cell attached along $\beta_1 \cup \beta_1'$ should be thought of as the product $\beta_1 \times I$, with $\beta_1 \times 0$ identified with β_1, $\beta_1 \times 1$ identified with β_1', and $y_1 \times I$ identified to the point y_1. Given an $\eta_3 > 0$ apply Proposition 3 to get an $AL^1(\eta_3)$-map $q_3 \colon X \to X_3$, along with accompanying homotopies.

Lemma 3 tells us that we can choose μ', η_2, and η_3 sequentially so that, after performing the constructions above, the composition

$$X \xrightarrow{q_3} X_3 \xrightarrow{q'} \bar{X} \xrightarrow{\bar{f}} B$$

is $AL^1(\mu)$. During this process, f has undergone two δ- or one 10ϵ-homotopy. $\square$

Proof of Theorem 1 in the case $k = 0$.
To get a UV^0-map from an $AL^1(\epsilon)$-map, simply apply Proposition 4 inductively to get a sequence of maps from X to B converging to a map that is $AL^1(\delta)$ for every $\delta > 0$. We may make the positive number μ in the Proposition 4 small enough so that the homotopy from the $AL^1(\mu)$-map to a UV^0-map has size $< \epsilon$; hence, f is 11ϵ-homotopic to a UV^0-map.

4. UV^k, $k \geq 1$

Throughout this section we will assume that X is an ENR homology n-manifold, $n \geq 5$, with the *DDP*. To proceed, we need the following finite generation lemma.

Lemma 6. *Suppose $f \colon X \to B$ is UV^{k-1}, where $k \geq 1$ and $2k+3 \leq n$. Given $\mu > 0$, we can attach finitely many $(k+1)$-cells of diameter $\leq \mu$ to X along their boundaries to obtain an* ENR *X' and extend f to an $AL^{k+1}(\mu)$-map $f' : X' \to B$.*

Proof. Triangulate B so that stars of vertices have diameter $\ll \mu$. Given $\alpha\colon I^{k+1} \to B$ with a lift $\alpha_0\colon \partial I^{k+1} \to X$, choose a subdivision of I^{k+1} so that the image of each simplex lies in a vertex star of the triangulation of B. Since f is UV^{k-1}, we can lift the k-skeleton of this subdivision. By [3, 16], we may assume the lifts to be embeddings. Attaching $(k+1)$-cells to allow us to extend the lift over the $(k+1)$-skeleton would produce the desired $AL^{k+1}(\mu)$-map, so we would like to know that $\pi_k(p^{-1}(U))$ is finitely generated for each such U. This is not necessarily true, but, since X is an ENR, it is true that $\mathrm{im}(\pi_k(p^{-1}(U)) \to \pi_k(p^{-1}(V)))$ is finitely generated whenever V is an open set such that $V \supset \mathcal{C}\ell(U) \supset U$. Choosing a finite set of generators for each such image and attaching $(k+1)$-cells to kill the images completes the construction. □

The next result is the analogue of Proposition 2 for $k \geq 1$.

Proposition 5. *Suppose $f\colon X \to B$ is UV^{k-1} and $AL^{k+1}(\delta)$, $k \geq 1$. For every $\mu > 0$ there is an* ENR *$\bar{X}$ obtained by adding cells of dimension $\leq k+2$ to X and an extension $\bar{f}\colon \bar{X} \to B$ so that $\bar{f}$ is $AL^{k+1}(\mu)$ and $\bar{X}$ 2δ-collapses to X.*

Proof. Since f is UV^{k-1}, Lemma 6 assures that we can attach finitely many $(k+1)$-cells to X along their boundaries, forming X_1, and extend f to $f_1\colon X_1 \to B$ so that f_1 is $AL^{k+1}(\mu')$, where $0 < \mu' \ll \mu$. Let C be one such $(k+1)$-cell. Since f is $AL^{k+1}(\delta)$, $f|\partial C$ has a δ-lift to X, which we may assume to be a 1-*LCC* embedding. Call the image A. A δ-homotopy of $f|A$ to $f_1|C$, rel $f|\partial A (= \partial C)$, allows us to attach a $(k+2)$-cell D to X_1 along $A \cup C$, obtaining X_2, and an extension f_1 to $f_2\colon X_2 \to B$ so that $f_2(D)$ has size δ in B.

Unfortunately, f_2 is only $AL^{k+1}(\delta)$. We generalize the proof of Proposition 2 so that we can systematically recover $AL^{k+1}(\mu')$.

Use the δ-homotopy of $f|A$ to $f_1|C$, rel $f|\partial A$ to parameterize D as the quotient of $A \times I$ with the intervals in $\partial A \times I$ identified to points. Here, A is identified with $A \times 0$ and C is identified with $A \times 1$. Suppose $0 < \eta \ll \mu'$. Introduce the following notation:

- J is the k-skeleton of a fine triangulation of A,
- $K \subseteq J$ is the $(k-1)$-skeleton of A,
- $R = J \times [0,1] \subseteq D$,
- $S = K \times [0,1] \subseteq R \subseteq D$,
- $L = S \cup (J \times \{0,1\}) \subseteq R \subseteq D$.

Choose the triangulation fine enough so that if P is an i-dimensional polyhedron, $0 \leq i \leq k$, then any map of P into D can be η-homotoped into R (over B).

By the inductive hypothesis we can η'-lift the map $f_2|L$ to X (rel $L \cap A$), for any preassigned $\eta' > 0$. This gives a map $\alpha_0\colon L \to X$, which is the identity on $L \cap A$, and which we may assume results in a 1-*LCC* embedding of $L \cup A$ into X [3, 16]. Let L' be the image of this map. Since η' can be made arbitrarily small, we may use the estimated homotopy extension theorem [4] to deform $f_2|D$ (rel A) slightly so that this lift is exact. Thus, we also have a map of the mapping cylinder M of the map $\alpha_0\colon L \to L'$ (rel $J \times 0$) into B with mapping cylinder fibers of size 0 in B. Attach M to X_2 along $L \cup L'$ to get X_2' and an extension $f_2'\colon X_2' \to B$ that is $AL^{k+1}(\eta)$. Observe that if M' is the portion of this mapping cylinder under S, then $X_2 \cup M'$ δ-collapses to X_2.

We now have a map $\alpha = f_2|R\colon (R, L) \to B$ and a lift α_0 of $\alpha|L$ to X. Thus, there is a μ'-lift $\bar{\alpha}\colon R \to X_1$, and the μ'-homotopy between $f_1 \circ \bar{\alpha}$ and α is fixed on L. This μ'-homotopy provides an extension of f_2' to the mapping cylinder $M_1 \supseteq M$ (rel $R \cap A$) of $\bar{\alpha}$ so that mapping cylinder fibers have size μ' in B. Attach this mapping cylinder to X_2' along $M \cup R \cup \bar{\alpha}(R)$ to get $\bar{X}$, which δ-collapses to X_2, and extend f_2' to $\bar{f}\colon \bar{X} \to B$.

The result of this construction is to produce a relative $(k+2)$-complex $(\bar{X}, X)$, which 2δ-collapses to X, such that every map of a k-dimensional polyhedron into $\bar{X}$ can be $(\eta + \mu')$-homotopied into X (over B). Lemma 1 guarantees that, if η and μ' are sufficiently small, then $\bar{f}$ is $AL^{k+1}(\mu)$.

One should observe that, although $\bar{f}$ is UV^{k-1} on X, it is *not* UV^{k-1} on $\bar{X}$. □

We now prove the analogue of Proposition 3.

Proposition 6. *Suppose X is a compact, connected,* ENR *homology n-manifold* ENR *with the DDP, $n \geq 5$, and suppose $\bar{X}$ is obtained by attaching a $(k+2)$-cell D to X along a $(k+1)$-cell $A \subseteq X$ in ∂D, $2k+3 \leq n$. Then for every neighborhood U of A in X and every $\eta > 0$, there is a homotopy $h\colon X \times I \to \bar{X}$ and a deformation retraction $d\colon \bar{X} \to X$ such that*

(*i*) *each h_t is the identity outside U,*
(*ii*) *$d \circ h\colon X \times I \to X$ is an η-homotopy that deformation retracts a neighborhood of A onto A inside U,*
(*iii*) *$h_1\colon X \to \bar{X}$ is $AL^{k+1}(\eta)$.*

Proof. By results of [3, 16] we may assume that A is 1-*LCC* embedded in X. (This requires $2k+3 \leq n$.) Assume also that X is 1-*LCC* embedded in $\mathbb{R}^m$ so as to have a mapping cylinder neighborhood N with mapping cylinder projection $\pi\colon N \to X$ [14]. If T is any triangulation of N, then, again by [3, 16], π restricted to the $(k+1)$-skeleton of T is can be approximated arbitrarily closely by a 1-*LCC* embedding whose image misses A ([3, 16]).

We proceed now very much as in the proof of Proposition 3. Given a neighborhood U of A and $\delta > 0$, there is a connected neighborhood V_0 of A whose closure lies in U and a δ-deformation retraction $r\colon V_0 \times I \to U$ of V_0 to A. Given $\epsilon > 0$ we can use the estimated homotopy extension theorem [4] to find a $\delta > 0$ so that the δ-deformation r can be extended to an ϵ-homotopy $R\colon X \times I \to X$ that is the identity outside U.

Let $f_0 = R_1|(X - V_0)$. Let W_0 be an open neighborhood of $X - V_0$ such that $\mathcal{C}\ell(W_0) \cap A = \emptyset$. Given $\eta_0 > 0$, let T_0 be a triangulation of N of mesh η_0 and let J_0 be (the polyhedron of) its $(k+1)$-skeleton. Assume $\pi|J_0$ is η_0-homotopic to a 1-*LCC* embedding $\pi_0\colon J_0 \to (X - A)$, as above, and set $L_0 = \pi_0(J_0)$. We can choose η_0 so that if (P, Q) is a polyhedral pair, $\dim P \leq k+1$, $\alpha\colon P \to U$, $\alpha_0\colon Q \to U - \mathcal{C}\ell(V_0)$, and $f_0 \circ \alpha_0 = \alpha|Q$, then there is an η_0-homotopy of α to a map $\alpha'\colon P \to L_0$ such that $\alpha'|Q$ is homotopic to α_0 in W_0. We may also assume that we can use the estimated homotopy extension theorem to get a homotopy of α' to $\bar{\alpha}$ such that $\bar{\alpha}|Q = \alpha_0$ and $\operatorname{im} \alpha \subseteq W_0 \cup L_0$.

Let V_1 be a neighborhood of A such that $\mathcal{C}\ell(V_1) \cap (\mathcal{C}\ell(W_0) \cup J_0) = \emptyset$ and such that $R(V_1 \times I) \subseteq V_0$. Repeat the construction above to $f_1 = R_1|(V_0 - V_1)$ to get a triangulation T_1 of N, with $(k+1)$-skeleton J_1, an approximation π_1 of $\pi|J_1$ taking J_1 to $L_1 \subseteq (X - A)$, and an open set W_1 containing $(V_0 - V_1)$ such that $(\mathcal{C}\ell(W_1) \cup L_1) \cap A = \emptyset$. Choices are made so that, replacing

U with V_0, etc., we get the appropriate lifting properties for maps of a $(k+1)$-dimensional polyhedron into V_0.

We can perform this construction any finite number of times.

Return now to $\bar{X} = X \cup_A D$. Let $X' = (X \times 0) \cup (A \times I) \subseteq X \times I$, $I = [0,1]$, and let $d' \colon X' \to X$ be projection to the first factor. Let $g \colon X' \to \bar{X}$ be the map that sends each of the vertical intervals in $\partial A \times I$ to a point, but is otherwise $1-1$, and let $d \colon \bar{X} \to X$ be the map induced by d'.

Let $\{0 = t_0 < t_1 < \cdots < t_k = 1\}$ be a subdivision of I of mesh less than η_1. Perform the construction above $k+1$ times producing open sets

$$U \supseteq V_0 \supseteq V_1 \supseteq \cdots \supseteq V_k \supseteq V_{k+1} \supseteq A.$$

For each $i = 1, \ldots, k$, let $\lambda_i \colon (\mathcal{C}\ell(V_{i-1}) - V_i) \to [t_{i-1}, t_i]$ be a Urysohn function that takes $\mathrm{bd}(V_{i-1})$ to t_{i-1} and $\mathrm{bd}(V_i)$ to t_i. Combine these maps to get a map $\lambda \colon X \to I$ that takes $X - V_0$ to 0 and V_{k+1} to 1.

A map $q' \colon X \to X'$ can then be defined by setting

(a) $q'(x) = (R_1(x), 0)$, if $x \in (X - V_0)$,
(b) $q'(x) = (R_1(x), \lambda_i(x))$, if $x \in (V_{i-1} - V_i)$, $i = 1, \ldots, k$, and
(c) $q'(x) = (R_1(x), 1)$, if $x \in V_{k+1}$.

Then $R_1 = d' \circ q'$, and the homotopy $\mathrm{id}_{X'} \simeq d'$ composed with q' gives a homotopy of q' to $d' \circ q' = R_1$. Piecing this homotopy together with R gives a homotopy $h' \colon X \times I \to X'$ from the inclusion $X \to X'$ to q'.

Let $h = g \circ h'$ and $q = g \circ q'$. If η_1 is sufficiently small, then $q = h_1$ will be $AL^{k+1}(\eta)$, and h and d will satisfy the conclusion of the proposition. □

Here is the key proposition for the general case.

Proposition 7. *Suppose X is a compact, connected,* ENR *homology n-manifold, $n \geq 5$, having the disjoint disks property, B is a connected finite polyhedron, and $f \colon X \to B$ is $AL^{k+1}(\epsilon)$, $2k+3 \leq n$. Then there is a constant $D(k)$, depending only on k, such that for every $\mu > 0$, f is $(D(k) \cdot \epsilon)$-homotopic to an $AL^{k+1}(\mu)$-map.*

Proof. We use induction on k, the case $k = 0$ having already been established. The proof of the inductive step follows closely the proof for the case $k = 0$. Keep in mind throughout that, unless otherwise indicated, all measurements are made in B, a fact that we will only recall when it seems necessary for the purpose of clarification.

Assume then that $k \geq 1$ and $f \colon X \to B$, and $AL^{k+1}(\epsilon)$ for some $\epsilon > 0$. Assume, inductively, that f is $(C(k-1) \cdot \epsilon)$-homotopic to a UV^{k-1}-map, which we shall still call f. Then, by Lemma 3 the "new" f is now only $AL^{k+1}((2C(k-1)+1)\epsilon)$. Set $\delta = (2C(k-1)+1)\epsilon$.

As in the proof of Proposition 5 build the relative $(k+2)$-complex $(\bar{X}, X)$, which 2δ-collapses to X and on which the map f extends to an $AL^{k+1}(\mu)$-map $\bar{f}\colon \bar{X} \to B$ for a given $\mu > 0$. As in the proof for $k = 0$ we need to recall the steps in the construction of $\bar{X}$.

We start by constructing $X_1 \subseteq X_2$ from X by attaching $(k+1)$-cells to X to get X_1 and $(k+2)$-cells to X_1 to get X_2, and extensions $f_1 \subset f_2$ of $f\colon X \to B$ to X_1 and X_2, respectively, such that f_1 is $AL^{k+1}(\mu')$ and f_2 is only $AL^{k+1}(\delta)$, where $0 < \mu' \ll \mu$, and X_2 δ-collapses to X. Each $(k+2)$-cell D is attached to X_1 along $\partial D = A \cup C$, where C is a $(k+1)$-cell attached to X while forming X_1, and $A \subseteq X$ is the complementary $(k+1)$-cell. We may assume, by [3, 16], that the collection of cells A is mutually exclusive in X.

In each $(k+2)$-cell D attached to X (along a $(k+1)$-cell A in its boundary) we identify a $(k+1)$-complex $R = J \times [0,1]$, where J is the k-skeleton of a fine triangulation of A. The next step is to attach the mapping cylinder M of a map $R \to R' \subseteq X_1 \subseteq X_2$ (rel $R \cap A$) to X_2, and, after doing this for each $(k+2)$-cell D, we obtain the space $\bar{X} \supseteq X_2$ and an extension of f_2 to $\bar{f}\colon \bar{X} \to B$ that is $AL^{k+1}(\mu)$. The space $\bar{X}$ 2δ-collapses to X: the first δ-collapse come from the collapses $M \searrow (R \cup R')$ of the relative mapping cylinders, and the second comes from the collapses $D \searrow A$.

For a given $\eta_2 > 0$ apply Proposition 6 to get a map $q_2\colon X \to X_2$ that is $AL^{k+1}(\eta_2)$ over X_2. We can η_2-lift each of the complexes $R \cup R'$ to $R_1 \cup R_1' \subseteq X$ and assume by [3, 16] that each of R_1 and R_1' is homeomorphic to R, that each $R_1 \cup R_1'$ is embedded, and that the collection of all such lifts is mutually exclusive. By an argument similar to the one in the proof for $k = 0$, we may assume that the lifts are exact. Thus, for each complex $R_1 \cup R_1'$, there is a homeomorphism $r\colon R_1 \to R_1'$, which is the identity on $R_1 \cap R_1'$, that commutes with q_2. For each (R_1, R_1')-pair attach the mapping cylinder M_1 of r to X forming X_3, and extend the map $q_2\colon X \to X_2$ to a map $q'\colon X_3 \to \bar{X}$, which is $AL^{k+1}(\eta_2)$ over $\bar{X}$, using the mapping cylinder identifications $M_1 \to M$.

For a given η_3 apply Proposition 6 again to get an $AL^{k+1}(\eta_3)$-map $q_3\colon X \to X_3$ over X_3. Lemma 3 tells us that we can choose μ', η_2, and η_3 sequentially so that, after performing the constructions above, the composition

$$X \xrightarrow{q_3} X_3 \xrightarrow{q'} \bar{X} \xrightarrow{\bar{f}} B$$

is $AL^{k+1}(\mu)$ over B.

During this process f has undergone two δ-homotopies, so that $D(k) = 2(2C(k-1)+1)$. Although the resulting map is $AL^{k+1}(\mu)$, it may no longer be UV^i for any $i = 0, ..., k$. ☐

Proof of Theorem 1 for $k \geq 1$.

Suppose $f\colon X \to B$ is $AL^{k+1}(\epsilon)$. Given arbitrary $\mu > 0$, Proposition 7 assures us that there is a $(2(2C(k-1)+1))$-homotopy of f to an $AL^{k+1}(\mu)$-map. If μ is sufficiently small, we can get an ϵ-homotopy of the resulting map to one that is UV^k. Thus, $C(k) = 4C(k-1)+3$. Since $C(-1) = 2$ (Proposotion 1), we get the explicit formula $C(k) = 3 \cdot 4^{k+1} - 1$.

5. Variations on the Main Theorems

We now show how to alter the proof of Theorem 1 to get the controlled version, Theorem 2. The key is in establishing an analogous simple homotopy version corresponding to Propositions 2 and 5.

Proposition 8. *Suppose Y is a metric space, $p\colon B \to Y$ is a map, and f is a UV^{k-1}- and an $AL^{k+1}(\delta)$-map over Y for some $\delta > 0$. Then for every $\mu > 0$, there is an* ENR $\bar{X}$ *obtained by adding cells of dimension $\leq k+2$ to X and an extension $\bar{f}\colon \bar{X} \to B$ so that $\bar{f}$ is $AL^{k+1}(\mu)$ over B and $\bar{X}$ 2δ-collapses to X over Y.*

Proof. Since f is UV^{k-1}, we can attach finitely many $(k+1)$-cells to X along their boundaries, forming X_1, and extend f to $f_1\colon X_1 \to B$ so that f_1 is $AL^{k+1}(\mu')$, where $0 < \mu' \ll \mu$ (Lemma 6). Let C be one such $(k+1)$-cell. Since f is $AL^{k+1}(\delta)$ over Y, the map $f_1|C\colon C \to B$ has a δ-lift $g\colon C \to X$ (over Y), rel $g|\partial C$. Let $A = g(C)$, and assume, by [3, 16], that A is 1-*LCC* embedded in X. Using the δ-homotopy of $f_1|C$ to $f \circ g$ (over Y), we may attach the mapping cylinder D of g, rel ∂C to X_1 and extend f_1 to $X_1 \cup D$. Then $X_1 \cup D$ δ-collapses to X over Y.

The rest of the proof now follows as in the proofs of Propositions 2 and 5. As in the proofs of these two propositions, the map f_2 is no longer $AL^{k+1}(\mu')$. The construction that remedies this defect, however, is exactly the same. □

Proof of Theorem 2. After constructing $\bar{X}$ using Proposition 8, we can apply Propositions 3 (for $k = 0$) and 6 (for $k \geq 1$) to get a homotopy of f, controlled over Y, to map that is $AL^{k+1}(\mu)$-map over B *and* over Y, for some preassigned $\mu > 0$. The resulting map satisfies the hypothesis of Theorem 1, which takes over to complete the proof. We need only ensure that subsequent homotopies are small enough in B so that their sizes add up to $< \epsilon$ in Y. □

6. A Final Observation

The reader may easily observe that the proofs we have given, particularly those of Propositions 3 and 6, do not actually require the space X be a homology manifold. All that is really required of X is that it be an ANR and possess the necessary general position properties. We say that a space X has the **disjoint k-cells property**, or DDP^k, if any two maps of a k-cell into X can be approximated by maps with disjoint images. It is easy to see that the DDP^k implies that maps $f\colon I^i \to X$ and $g\colon I^j \to X$ can be approximated by maps with disjoint images whenever $i, j \leq k$. The main results of [3] and [17] show that if an ANR X has the DDP^k, then any map of a k-dimensional polyhedron into X can be approximated by a k-*LCC* embedding (that is, an embedding whose complement is locally k-connected). The methods of this paper then apply to prove

Theorem 7. *Suppose X is a compact, connected* ANR *with the DDP^{k+1}. Then X has the UV^k-approximation property.*

Krupski has shown [11] that a homogeneous ANR of dimension ≥ 3 has the disjoint arcs property, or DDP^1. (Recall that a space X is **homogeneous** if, given points $x, y \in X$, there is a homeomorphism of X onto itself taking x to y.) Combining Krupski's result with Theorem 7 we obtain

Corollary 1. *Theorem 2 remains valid for $k = 0$, if X is assumed to be a compact, connected, homogeneous* ANR *of dimension ≥ 3. In particular, any 1-connected map of such an X to an* ANR *is homotopic to a map with connected point-inverses.*

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