# An Easier Way to Show $\zeta(3) \notin \mathbf{Q}$ 

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In 1978 R. Apery proved that

$$
\zeta(3)=\sum_{n=1}^{\infty} \frac{1}{n^{3}}
$$

is an irrational number. His proof was shortened by F. Beukers, who translated it into equivalent statements about integrals of Legendre polynomials. Nevertheless, there is some "magic" in Beukers' proof involving a complicated change of variables. In this note we will use a different integral than Beukers which gives the same approximation as Beukers' (and Apery's), but which is much easier to maniuplate.

## 1 Some Formulas

For the sake of reference, let us state some key integral formulas

$$
\begin{equation*}
\int_{0}^{1} \int_{0}^{1} \frac{s^{a} t^{a}}{1-s t} d s d t=\sum_{n=1}^{\infty} \frac{1}{(n+a)^{2}} \tag{1}
\end{equation*}
$$

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$$
\begin{align*}
\int_{0}^{1} \int_{0}^{1} \frac{s^{a} t^{a} \ln (s t)}{1-s t} d s d t & =-2 \sum_{n=1}^{\infty} \frac{1}{(n+a)^{3}}  \tag{2}\\
\int_{0}^{1} \int_{0}^{1} \frac{s^{a} t^{a} \ln (t)}{1-s t} d s d t & =-\sum_{n=1}^{\infty} \frac{1}{(n+a)^{3}} . \tag{3}
\end{align*}
$$
\]

If $a \neq b$,

$$
\begin{gather*}
\int_{0}^{1} \int_{0}^{1} \frac{s^{a} t^{b}}{1-s t} d s d t=\frac{1}{b-a} \sum_{n=1}^{\infty}\left(\frac{1}{n+a}-\frac{1}{n+b}\right)  \tag{4}\\
\int_{0}^{1} \int_{0}^{1} \frac{s^{a} t^{b} \ln (s)}{1-s t} d s d t=\frac{1}{a-b} \sum_{n=1}^{\infty} \frac{1}{(n+a)^{2}}+\frac{1}{(a-b)^{2}} \sum_{n=1}^{\infty}\left(\frac{1}{n+a}-\frac{1}{n+b}\right)  \tag{5}\\
\int_{0}^{1} \int_{0}^{1} \frac{s^{a} t^{b} \ln (t)}{1-s t} d s d t=\frac{1}{b-a} \sum_{n=1}^{\infty} \frac{1}{(n+b)^{2}}+\frac{1}{(a-b)^{2}} \sum_{n=1}^{\infty}\left(\frac{1}{n+b}-\frac{1}{n+a}\right)  \tag{6}\\
\int_{0}^{1} \int_{0}^{1} \frac{s^{a} t^{b} \ln (s t)}{1-s t} d s d t=\frac{1}{a-b} \sum_{n=1}^{\infty}\left(\frac{1}{(n+a)^{2}}-\frac{1}{(n+b)^{2}}\right) . \tag{7}
\end{gather*}
$$

Formulas (1) and (4) are proven directly by expanding

$$
\frac{1}{1-s t}=\sum_{n=0}^{\infty} s^{n} t^{n}
$$

and integrating. The remaining formulas follow by differentiating (1) and (4) with respect to $a$ and $b$.

If $p(s, t) \in \mathbf{Z}[s, t]$ is a polynomial of degree $n$ with integral coefficients, then it follows from (2) and (7) that

$$
\begin{equation*}
\int_{0}^{1} \int_{0}^{1} \frac{p(s, t) \log (s t)}{1-s t} d s d t=\frac{a_{n}+b_{n} \zeta(3)}{d_{n}^{3}} \tag{8}
\end{equation*}
$$

where $a_{n}, b_{n}, d_{n} \in \mathbf{Z}$ and $d_{n}$ is the least common multiple of the integers $1,2, \ldots, n$. It is not hard to see (using the Prime Number Theorem) that for any fixed $\epsilon>0, d_{n} \leq e^{(1+\epsilon) n}$ for large $n$.

## Legendre Polynomials

Consider $P_{n}(s)=\frac{1}{n!} \frac{d^{n}}{d s^{n}}\left(s-s^{2}\right)^{n}$, a polynomial with integral coefficients. We will use the integrals

$$
\begin{equation*}
\int_{0}^{1} \int_{0}^{1} \frac{P_{n}(s) P_{n}(t) \log (s t)}{1-s t} d s d t=\int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \frac{P_{n}(s) P_{n}(t)}{1-(1-s t) u} d s d t d u . \tag{9}
\end{equation*}
$$

Since

$$
\begin{equation*}
\int_{0}^{1} \frac{1}{1-(1-x) t} d x=-\frac{\log (x)}{1-x} \tag{10}
\end{equation*}
$$

and $P_{n}(1-s)=(-1)^{n} P_{n}(s)$, this equals

$$
(-1)^{n} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \frac{P_{n}(s) P_{n}(t)}{1-(1-(1-s) t) u} d s d t d u
$$

Lemma 1.1 For $s, t \in(0,1)$ fixed,

$$
\begin{equation*}
\int_{0}^{1} \frac{1}{1-(1-(1-s) t) u} d u=\int_{0}^{1} \frac{1}{(1-(1-u) s)(1-(1-t) u)} d u \tag{11}
\end{equation*}
$$

Proof: Partial fractions give that

$$
\begin{equation*}
\frac{1}{(1-(1-u) s)(1-(1-t) u)}=\frac{1}{1-(1-s) t}\left(\frac{s}{1-(1-u) s}-\frac{1-t}{1-(1-t) u}\right) \tag{12}
\end{equation*}
$$

so

$$
\begin{gather*}
\int_{0}^{1} \frac{1}{(1-(1-u) s)(1-(1-t) u)} d u  \tag{13}\\
=\frac{1}{1-(1-s) t}\left(-s \frac{\log (1-s)}{s}+(1-t) \frac{\log (t)}{t-1}\right)=-\frac{\log (t(1-s))}{1-(1-s) t} . \tag{14}
\end{gather*}
$$

Using (10) with $x=1-(1-s) t$, we see that the two integrals are equal.

Beukers used the integral (9) and, by using integration by parts and changing variables, deduced from it that $\zeta(3)$ is irrational. We will do some easy integrations-by-parts, but avoid his change of variables.

The lemma implies that

$$
\begin{gather*}
(-1)^{n} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \frac{P_{n}(s) P_{n}(t)}{1-(1-(1-s) t) u} d s d t d u \\
=(-1)^{n} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \frac{P_{n}(s) P_{n}(t)}{(1-(1-u) s)(1-(1-t) u)} d s d t d u  \tag{15}\\
=\int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \frac{P_{n}(s) P_{n}(t)}{(1-(1-u) s)(1-t u)} d s d t d u .
\end{gather*}
$$

We already know that this is of the form $\frac{a_{n}+b_{n} \zeta(3)}{d_{n}^{3}}$. We will show it is very small, in fact $o\left(\frac{1}{d_{n}^{3}}\right)$, which shows that $\zeta(3) \notin \mathbf{Q}$.

By integrating by parts in the variables $s$ and $t$, each $n$ times, we get that

$$
\begin{align*}
& \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \frac{P_{n}(s) P_{n}(t)}{(1-(1-u) s)(1-t u)} d s d t d u  \tag{16}\\
= & \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \frac{\left(s-s^{2}\right)^{n}\left(t-t^{2}\right)^{n}\left(u-u^{2}\right)^{n}}{((1-(1-u) s)(1-t u))^{n+1}} d s d t d u \tag{17}
\end{align*}
$$

The function

$$
\begin{equation*}
f(s, t, u)=\frac{s(1-s) t(1-t) u(1-u)}{(1-(1-u) s)(1-t u)} \tag{18}
\end{equation*}
$$

vanishes on the boundary of $[0,1] \times[0,1] \times[0,1]$ and has its maximum at

$$
(s, t, u)=\left(2-\sqrt{2}, \sqrt{2}-1, \frac{1}{2}\right)
$$

where

$$
\begin{equation*}
f\left(2-\sqrt{2}, \sqrt{2}-1, \frac{1}{2}\right)=17-12 \sqrt{2} \approx .029437 \tag{19}
\end{equation*}
$$

Thus

$$
\begin{equation*}
\frac{a_{n}+b_{n} \zeta(3)}{d_{n}^{3}}=O\left((.029437)^{n}\right) . \tag{20}
\end{equation*}
$$

However, $d_{n}^{3}=O\left(e^{3.01 n}\right)$, and since $e^{3.01}(17-12 \sqrt{2}) \approx .597205$, we conclude that

$$
\begin{equation*}
\left|\frac{a_{n}+b_{n} \zeta(3)}{d_{n}^{3}}\right|=o\left(\frac{1}{d_{n}^{3}}\right), \tag{21}
\end{equation*}
$$

proving indeed that $\zeta(3) \neq \mathbf{Q}$.


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