Borel cardinal invariant properties of countable Borel equivalence relations

(Joint work with Samuel Coskey)

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Cardinal invariants (characteristics) of the continuum

Goal

To introduce "Borel cardinal invariant properties" of countable Borel equivalence relations, thus importing in a meaningful way the study of cardinal invariants of the continuum into the field of countable Borel equivalence relations.

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A cardinal invariant of the continuum is a cardinal number, typically between $\aleph_1$ and $\mathfrak{c} = 2^{\aleph_0}$ inclusive, that captures some combinatorial, topological, or analytic property of the real line $\mathbb{R}$ or a related space such as ω^ω or $\mathcal{P}(\omega)$.

Examples

- *The bounding number* $\mathfrak{b}$.
- *The pseudo-intersection number* $\mathfrak{p}$.
- *The splitting number* $\mathfrak{s}$.

The bounding number $\mathfrak{b}$

Definition

For $\alpha, \beta \in \omega^\omega$, we say that β bounds α, written $\alpha \leq^* \beta$, if $\alpha(n) \leq \beta(n)$ for all but finitely many n.

Definition

A family of functions $\mathcal{F} \subseteq \omega^\omega$ is bounded if there is a function $\beta \in \omega^\omega$ that bounds each $\alpha \in \mathcal{F}$. Otherwise $\mathcal{F}$ is unbounded.

Definition

The bounding number, $\mathfrak{b}$, is the minimal cardinality of an unbounded family $\mathcal{F} \subseteq \omega^\omega$.

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Obvious Fact

$\aleph_0 < \mathfrak{b}$.

The pseudo-intersection number $\mathfrak{p}$

Let $[\omega]^\omega = \{A \subseteq \omega \mid A \text{ is infinite}\}$, and for $A, B \in [\omega]^\omega$, write $A \subseteq^* B$ iff $B \setminus A$ is finite.

Definition

A family $\mathcal{F} \subseteq [\omega]^\omega$ is centered if every finite subset of $\mathcal{F}$ has infinite intersection. If $\mathcal{F} \subseteq [\omega]^\omega$ is centered, then $A \in [\omega]^\omega$ is a pseudo-intersection of $\mathcal{F}$ if $A \subseteq^* B$ for every $B \in \mathcal{F}$.

Definition

The pseudo-intersection number, $\mathfrak{p}$, is the minimal cardinality of a centered family $\mathcal{F} \subseteq [\omega]^\omega$ with no pseudo-intersection.

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$\aleph_0 < \mathfrak{p}$.

The splitting number $\mathfrak{s}$

Definition

For $A, B \in [\omega]^\omega$, A splits B if $|A \cap B| = |A^c \cap B| = \aleph_0$.

Definition

A family $\mathcal{F} \subseteq [\omega]^\omega$ is a splitting family if for every $B \in [\omega]^\omega$, there is $A \in \mathcal{F}$ such that A splits B.

Definition

The splitting number, $\mathfrak{s}$, is the minimal cardinality of a splitting family $\mathcal{F} \subseteq [\omega]^\omega$.

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A family $\mathcal{F} \subseteq [\omega]^\omega$ *is a* *splitting family* *if for every* $B \in [\omega]^\omega$, *there is* $A \in \mathcal{F}$ *such that A splits B.*

Definition

The *splitting number*, $\mathfrak{s}$, *is the minimal cardinality of a splitting family* $\mathcal{F} \subseteq [\omega]^\omega$.

(Slightly Less) Obvious Fact

$\aleph_0 < \mathfrak{s}$.

Diagonalizing countable families

(Slightly Less) Obvious Fact

$\aleph_0 < \mathfrak{s}$.

Proof.

Given $\{A_n \mid n \in \omega\} \subseteq [\omega]^\omega$, let $\mathcal{U}$ be a nonprincipal ultrafilter on ω, set $B_{-1} = \omega$, and then inductively define B_{n+1} to be whichever of $B_n \cap A_n$, $B_n \cap A_n^c$ is in $\mathcal{U}$. Then each B_n is infinite, so if we inductively choose $b_{n+1} \in B_{n+1}$ distinct from $b_0, \ldots, b_n$, then $\{b_n \mid n \in \omega\}$ is not split by any A_n. □

- Notice that this proof depends heavily on the ordering $\langle A_n \mid n \in \omega \rangle$.

Of course, the proofs that $\aleph_0 < \mathfrak{b}$ and that $\aleph_0 < \mathfrak{p}$ also depended on the ordering of the given countable family, but we will see that for $\mathfrak{s}$ there is a sense in which this dependence is less avoidable.

Enter countable Borel equivalence relations. . .

More generally, most cardinal invariants can be shown to be uncountable by a suitable diagonalization argument against a countable family.

- Thus if E is a countable Borel equivalence relation on the standard Borel space X, and if $\phi : X \to \omega^\omega$ is some function, then for each $x \in X$, the family $\phi([x]_E)$ cannot be unbounded (splitting, have no pseudo-intersection, etc).
- In particular, for each such function ϕ, there exists a function $\psi : X \to \omega^\omega$ such that for all $x \in X$ and for all $y \in [x]_E$, $\phi(y) \leq^* \psi(x)$.
- Moreover, ψ can clearly be chosen to be E-invariant.

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- In particular, for each such function ϕ, there exists a function $\psi : X \to \omega^\omega$ such that for all $x \in X$ and for all $y \in [x]_E$, $\phi(y) \leq^* \psi(x)$.
- Moreover, ψ can clearly be chosen to be E-invariant.
- But supposing ϕ is Borel, can ψ also be chosen to be Borel?

Perhaps surprisingly, this question is highly relevant to a major open problem in the theory of hyperfinite Borel equivalence relations.

Standard Borel spaces

Definition

- *A Polish space* $(X, \Im)$ *is a complete separable metric space.*
- *A standard Borel space* $(X, \mathbf{B}(\Im))$ *is a Polish space equipped only with its* σ*-algebra* $\mathbf{B}(\Im)$ *of Borel subsets.*

Examples of standard Borel spaces: $\mathbb{R}$, ω^ω, $[\omega]^\omega$, $\mathcal{P}(\omega) = 2^\omega$.

- Any two uncountable standard Borel spaces are isomorphic!

Definition

- *An equivalence relation* E *on the standard Borel space* X *is called Borel if* E *is Borel as a subset of* $X \times X$*, and countable (finite) if each* E*-class is countable (finite).*
- *A Borel equivalence relation* E *is hyperfinite if* $E = \cup_n E_n$ *is the countable increasing union of a sequence of finite Borel equivalence relations.*

Borel homomorphisms

Definition

Let E, F be Borel equivalence relations on the standard Borel spaces X, Y respectively. Then E is Borel reducible to F, written $E \leq_B F$, if there exists a Borel reduction from E to F, ie, a Borel function $f : X \to Y$ such that for all $x, y \in X$,

$$x \, E \, y \Leftrightarrow f(x) \, F \, f(y).$$

If $E \leq_B F$ and $F \leq_B E$, then we write $E \sim_B F$ and say that E and F are Borel bireducible

Definition

A Borel function $f : X \to Y$ satisfying the weaker condition

$$x \, E \, y \Rightarrow f(x) \, F \, f(y)$$

for all $x, y \in X$ is called a Borel homomorphism from E to F.

Smooth Borel equivalence relations

Definition

A Borel equivalence relation E on the standard Borel space X is smooth if there exists a standard Borel space Y and a Borel function

$$f : X \to Y$$

such that for all $x, y \in X$,

$$x \mathrel{E} y \Leftrightarrow f(x) = f(y).$$

- The relation $\leq_B$ defines a (pre-)partial order on the collection of all Borel equivalence relations on standard Borel spaces.
- The smooth relations comprise the $\leq_B$-least degree in the $\leq_B$-hierarchy of Borel equivalence relations with uncountably many classes.

Hyperfinite Borel equivalence relations

Fact

A countable Borel equivalence relation E on the standard Borel space X is smooth if and only if E admits a Borel transversal, ie, a Borel set $B \subseteq X$ such that for each $x \in X$, $|B \cap [x_E]| = 1$.

- Every smooth countable Borel equivalence relation is hyperfinite, but there exist non-smooth hyperfinite relations.

Example

The "almost equality" relation

$$\alpha =^* \beta \iff \alpha \leq^* \beta \text{ and } \beta \leq^* \alpha$$

on ω^ω, 2^ω, or $[\omega]^\omega$ is hyperfinite and nonsmooth. We call any of these (pairwise isomorphic) equivalence relations E_0.

Borel actions of countable groups

Definition

Let X be a standard Borel space, and Γ a countable (discrete) group. We say the action $\Gamma \curvearrowright X$ of Γ on X is **Borel** *if $x \mapsto \gamma \cdot x$ is Borel for each $\gamma \in \Gamma$. If $\Gamma \curvearrowright X$ is a Borel action of Γ on X, then we write E_Γ^X for the corresponding orbit equivalence relation on X*

Observation

If the countable group Γ acts in a Borel fashion on the standard Borel space X, then E_X^Γ is a countable Borel equivalence relation.

Theorem (Feldman-Moore)

If E is a countable Borel equivalence relation on a standard Borel space X, then there exists a countable group Γ and a Borel action of Γ on X such that $E = E_\Gamma^X$.

The unions problem

The basic open problem in the study of hyperfinite Borel equivalence relations:

The Unions Problem

Is the countable increasing union of a sequence of hyperfinite Borel equivalence relations hyperfinite?

In other words, does “hyper-hyperfinite" imply hyperfinite?

Naive attempt to solve the Unions Problem

- *Let* $E = \cup_n E_n$, $E_n = \cup_m E_n^m$ *be increasing unions, each* E_n^m *finite, so that each* E_n *is hyperfinite and* E *is hyper-hyperfinite.*
- *Let* $E = E_\Gamma^X$ *be the orbit equivalence relation arising from the Borel action of the countable group* $\Gamma = \{\gamma_i \mid i \in \omega\}$, *with* $\gamma_0 = id$.
- *Try to write* E *an an increasing union* $E = \cup_k F_k$ *of finite Borel equivalence relations* F_k.

The unions problem

Naive attempt to solve the Unions Problem

- *Let* $F_k = \bigcap_{n \geq k} E_n^{\psi(n)}$ *for some sequence of choices* $\psi \in \omega^\omega$.
- *Define the Borel function* $\chi_E : E \to \omega^\omega$ *by*

$$\chi_E(x,y)(n) = \begin{cases} \text{the least } m \text{ such that } x\ E_n^m\ y & \text{if } x\ E_n\ y \\ 0 & \text{otherwise.} \end{cases}$$

  Then χ_E *keeps track of exactly when a pair of elements* $(x,y) \in E$ *becomes equivalent in each union* $E_n = \cup E_n^m$ *for which* $x\ E_n\ y$.
- *Then clearly the* F_k *form an increasing sequence of finite Borel equivalence relations contained in* E*; so all that is left is to make sure that the union exhausts* E.
- *For this we need: for all* $(x,y) \in E$*, there exists* $n \in \omega$ *such that for every* $k \geq n$, $\psi(k) \geq \chi_E(x,y)(k)$.

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- *But we cannot ask* ψ *to dominate an entire family of size continuum; therefore, we should allow* ψ *to depend (in a Borel way) on the equivalence class, so that each* $\psi([x]_E)$ *only has to bound countably many functions at a time.*

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- *But we cannot ask* ψ *to dominate an entire family of size continuum; therefore, we should allow* ψ *to depend (in a Borel way) on the equivalence class, so that each* $\psi([x]_E)$ *only has to bound countably many functions at a time.*
- *Thus we require a Borel,* E*-invariant function* $\psi : X \to \omega^\omega$ *such that for each* $(x, y) \in E$*,*

$$\chi_E(x, y) \leq^* \psi(x) = \psi(y).$$

Naive attempt to solve the Unions Problem

- *Thus we require a Borel, E-invariant function $\psi : X \to \omega^\omega$ such that for each $(x, y) \in E$,*

$$\chi_E(x, y) \leq^* \psi(x) = \psi(y).$$

- *If we define $\phi_0 : X \to \omega^\omega$ by*

$$\phi_0(x)(n) = \max_{i \leq n} \chi_E(x, \gamma_i x),$$

then clearly $\chi_E(x, y) \leq^* \phi_0(x)$ for each $x \in X$, so it will suffice to ask that for each $x \in X$,

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Invariant Borel boundedness

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- *If we simply ask for ψ to bound every Borel function $\phi : X \to \omega^\omega$, we arrive at the definition of "invariant Borel boundedness."*

Invariant Borel boundedness

Definition

Let E be a countable Borel equivalence relation on the standard Borel space X. Then E is invariantly Borel bounded if for every Borel function $\phi : X \to \omega^\omega$, there exists an E-invariant Borel function $\psi : X \to \omega^\omega$ such that

$$\phi(x) \leq^* \psi(x) \quad \text{for all } x \in X.$$

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This property is too strong, however, as it turns out to be equivalent to smoothness.

Proposition

A countable Borel equivalence relation E on the standard Borel space X is invariantly Borel bounded if and only if it is smooth.

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Proposition

A countable Borel equivalence relation E on the standard Borel space X is invariantly Borel bounded if and only if it is smooth.

But since we only need $\phi(x) \leq^* \psi(x)$ for each x, there is no reason to ask ψ to be fully invariant.

Borel boundedness

Definition (Boykin-Jackson)

Let E be a countable Borel equivalence relation on the standard Borel space X. Then E is **Borel bounded** *if for every Borel function $\phi : X \to \omega^\omega$, there exists a Borel homomorphism $\psi : X \to \omega^\omega$ from E to $=^*$ such that for all $x \in X$, $\phi(x) \leq^* \psi(x)$.*

This definition is nontrivial, since every hyperfinite Borel equivalence relation is Borel bounded (the converse is open).

- Proof: If $E = \cup_n F_n$ is hyperfinite, then given any Borel $\phi : X \to \omega^\omega$, we can define

$$\psi(x)(n) = \max\{\phi(y)(n) \mid y \; F_n \; x\}.$$

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Theorem (Boykin-Jackson)

A hyper-hyperfinite countable Borel equivalence relation E is hyperfinite if and only if it is Borel bounded.

Generalizing Borel boundedness

Similar "Borel cardinal invariant properties" can be introduced for other combinatorial cardinal invariants of the continuum provided they are:

- uncountable; and
- defined on a space that carries a suitable notion of "almost equality," such as ω^ω, $[\omega]^\omega$, or 2^ω.

Definition (temporary)

E is Borel non-splitting if for every Borel function $\phi : X \to [\omega]^\omega$, there exists a Borel homomorphism $\psi : X \to [\omega]^\omega$ from E to $=^*$ such that for all $x \in X$, $\psi(x) \subseteq^* \phi(x)$ or $\psi(x) \subseteq^* \phi(x)^c$.

Definition (temporary)

E is Borel pseudo-intersecting if for every Borel function $\phi : X \to [\omega]^\omega$ such that the family $\{\phi(y) \mid y\ E\ x\}$ is centered for all $x \in X$, there exists a Borel homomorphism $\psi : X \to [\omega]^\omega$ from E to $=^*$ such that for all $x \in X$, $\psi(x) \subseteq^* \phi(x)$.

Downward closure

More generally, given a suitable cardinal invariant $\mathfrak{x}$, we might say that E has Borel invariant property x if:

> *For every Borel function $\phi : X \to Y$, there exists a Borel homomorphism $\psi : X \to Y$ from E to $=^*$ such that for all $x \in X$, $\psi(x)$ witness the fact that the countable family $\phi([x]_E)$ does not have the property that defines $\mathfrak{x}$.*

- Caveat: it would be highly desirable for these properties to be closed under Borel reducibility.

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Theorem (Boykin-Jackson)

If F is Borel bounded and $E \leq_B F$, then E is Borel bounded.

Proof.

- Let $f : X \to Y$ be a Borel reduction from E to F, and define the equivalence relation $E' \subseteq E$ on X by $x \; E' \; y \Leftrightarrow f(x) = f(y)$.

Downward closure of Borel boundedness

Proof.

- Let $\phi : X \to Y$ be any Borel function.
- E' is smooth, hence Borel bounded.
- Thus let $\phi' : X \to \omega^\omega$ be a Borel homomorphism from E' to $=^*$ such that for all $x \in X$, $\phi(x) \leq^* \phi'(x)$.
- Also let $B \subseteq X$ be a Borel transversal for E', with $\sigma : \mathrm{im}(f) \to X$ a Borel function such that $f \circ \sigma = \mathrm{id}_{\mathrm{im}(f)}$.
- Then define the Borel function $\tilde{\phi} : Y \to \omega^\omega$ by

$$\tilde{\phi}(y)(n) = \begin{cases} \phi'(\sigma(y))(n) & \text{if } y \in \mathrm{im}(f) \\ 0 & \text{otherwise} \end{cases}.$$

- Using the fact that F is Borel bounded, let $\tilde{\psi} : Y \to \omega^\omega$ be a Borel homomorphism from F to $=^*$ such that for all $y \in Y$, $\tilde{\phi}(y) \leq^* \tilde{\psi}(y)$.
- Finally, let $\psi = \tilde{\psi} \circ f$.

Properties s and p not (yet) closed under $\leq_B$

This argument involves two diagonalizations:

- First, $\phi'(\sigma(y))$ dominates the family enumerated (via ϕ) by the countable fiber $f^{-1}(\{y\})$.
- Then $\tilde{\psi}(y)$ dominates the countable family $\{\phi'(\sigma(z)) \mid z \in [y]_F\}$.

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Two problems arise if we attempt to use this argument for s and p.

Problems that arise with s and p

- *If each $B_n \subseteq \omega$ witness that $\{A_n^m \mid m \in \omega\}$ is not splitting, and if $C \subseteq \omega$ witness that $\{B_n \mid n \in \omega\}$ is not splitting, then nevertheless C may be split by some set A_n^m.*
- *If $\{A_n^m \mid m, n \in \omega\} \subseteq [\omega]^\omega$ is centered, and if each B_n is a pseudo-intersection of $\{A_n^m \mid m \in \omega\}$, there is no reason for $\{B_n \mid n \in \omega\}$ to be centered.*

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- *If $\{A_n^m \mid m, n \in \omega\} \subseteq [\omega]^\omega$ is centered, and if each B_n is a pseudo-intersection of $\{A_n^m \mid m \in \omega\}$, there is no reason for $\{B_n \mid n \in \omega\}$ to be centered.*

Consequently, we propose a slight adjustments in the definitions of b, s, and p (and other Borel invariant properties).

Adjusted definitions

Definition

If X is a standard Borel space, define the equivalence relation $E_{set}(X)$ on X^ω by

$$\langle x_n \rangle \; E_{set}(X) \; \langle y_n \rangle \Leftrightarrow \{x_n \mid n \in \omega\} = \{y_n \mid n \in \omega\}.$$

Definition

E has property b *(E is Borel bounded) if for every Borel homomorphism $\phi : X \to (\omega^\omega)^\omega$ from E to $E_{set}(\omega^\omega)$, there exists a Borel homomorphism $\psi : X \to \omega^\omega$ from E to $=^*$ such that for all $x \in X$ and for all $n \in \omega$, $\phi(x)(n) \leq^* \psi(x)$.*

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- In the original definition, ϕ assigns a countable family $\phi([x]_E)$ to each E-class by associating a single function to each $y \in [x]_E$.
- In the new definition, we give *each* element $x \in X$ knowledge of the entire family that ϕ associates to $[x]_E$.

The new definition justified

Proposition

The new definition of Borel boundedness is equivalent to the old one.

The new definition justified

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Proof.

- Given Borel $\phi : X \to \omega^\omega$, define the Borel homomorphism ϕ' from E to E_{set} by $\phi'(x)(n) = \phi(\gamma_n \cdot x)$, toward showing that the new definition implies the old.
- Conversely, let ϕ' be a Borel homomorphism from E to E_{set}, and define
$$\phi(x)(n) = \max\{\phi'(x)(k)(n) \mid k \leq n\},$$
so that $\phi : X \to \omega^\omega$ is a Borel function such that for each $x \in X$ and $n \in \omega$, $\phi'(x)(n) \leq^* \phi(x)$.

□

A zoo of Borel cardinal invariant properties

Definition

E has property $\mathfrak{s}$ *(E is Borel non-splitting) if for every Borel homomorphism* $\phi : X \to ([\omega]^\omega)^\omega$ *from E to* E_{set}*, there exists a Borel homomorphism* $\psi : X \to [\omega]^\omega$ *from E to* $=^*$ *such that for all* $x \in X$ *and for all* $n \in \omega$, $\psi(x) \subseteq^* \phi(x)(n)$ *or* $\psi(x) \subseteq^* \omega \setminus \phi(x)(n)$.

Definition

E has property $\mathfrak{p}$ *(E is Borel pseudo-intersecting) if for every Borel homomorphism* $\phi : X \to ([\omega]^\omega)^\omega$ *from E to* E_{set} *such that* $\{\phi(x)(n) \mid n \in \omega\}$ *is centered for each* $x \in X$*, there exists a Borel homomorphism* $\psi : X \to [\omega]^\omega$ *from E to* $=^*$ *such that for every* $x \in X$ *and* $n \in \omega$, $\psi(x) \subseteq^* \phi(x)(n)$.

A zoo of Borel cardinal invariant properties

Definition

E has property $\mathfrak{s}$ (E is Borel non-splitting) if for every Borel homomorphism $\phi : X \to ([\omega]^\omega)^\omega$ from E to E_{set}, there exists a Borel homomorphism $\psi : X \to [\omega]^\omega$ from E to $=^*$ such that for all $x \in X$ and for all $n \in \omega$, $\psi(x) \subseteq^* \phi(x)(n)$ or $\psi(x) \subseteq^* \omega \setminus \phi(x)(n)$.

Definition

E has property $\mathfrak{p}$ (E is Borel pseudo-intersecting) if for every Borel homomorphism $\phi : X \to ([\omega]^\omega)^\omega$ from E to E_{set} such that $\{\phi(x)(n) \mid n \in \omega\}$ is centered for each $x \in X$, there exists a Borel homomorphism $\psi : X \to [\omega]^\omega$ from E to $=^*$ such that for every $x \in X$ and $n \in \omega$, $\psi(x) \subseteq^* \phi(x)(n)$.

In a similar manner, we can define “Borel cardinal invariant properties” for each of the following cardinal invariants of the continuum:

$$\mathfrak{b}, \mathfrak{s}, \mathfrak{p}, \mathfrak{d}, \mathfrak{r}, \mathfrak{a}, \mathfrak{i}, \mathfrak{t}, \mathfrak{u}$$

Closure under $\leq_B$

Theorem

Each of the above Borel cardinal invariant properties is closed under Borel reducibility, $\leq_B$.

Proof for p

- *Let $f : X \to Y$ be a Borel reduction from E to F, $\sigma : im(f) \to X$ a Borel function such that $f \circ \sigma = id_{im(f)}$.*
- *Given a Borel homomorphism ϕ from E to E_{set} with each family $\{\phi(x)(n) \mid n \in \omega\}$ centered, define*

$$\tilde{\phi}(y)(n) = \phi(\gamma_{n_0}\sigma(y))(n_1),$$

where $n \mapsto \langle n_0, n_1 \rangle$ is a pairing function.

- *Then $\tilde{\phi}$ is a Borel homomorphism from F to E_{set}, so there exists a Borel homomorphism $\tilde{\psi}$ from F to $=^*$ such that $\tilde{\psi}(x) \subseteq^* \tilde{\phi}(x)(n)$ for all $x \in X$ and for all n. Define $\psi = \tilde{\psi} \circ f$.*

General remarks

- We now have a large new family of combinatorial properties of countable Borel equivalence relations.
- Each property holds of the smooth relations.
- In fact, each property (except s) holds of hyperfinite relations.
- It is completely unknown what properties hold of relations above hyperfinite.
- Goal: establish basic implications between the properties. [Diagram]
- Indeed, there is a rough correspondence between ZFC-provable inequalities among cardinal invariants of the continuum and implications among the Borel invariant properties.
- Many of these implications are obvious; b $\rightarrow$ r required a bit of work; p $\rightarrow$ b was actually hard.
- Some implications are “missing."

General nonsense

Implications for the Unions Problem

- *The most interesting properties* x *are those that do not imply hyperfiniteness, but additionally have the Boykin-Jackson property that hyper-hyperfiniteness plus* x *imply hyperfiniteness.*
- *It is unknown where there are any such properties, but if there are, they could provide a solution to the unions problem.*

Implications for Cardinal Invariants

- *Each cardinal invariant can be shown to be uncountable by a diagonalization argument against a countable family* $\mathcal{F}$*.*
- *This argument depends on a well-ordering of* $\mathcal{F}$*.*
- *The corresponding Borel invariant property helps to gauge the effectiveness of this diagonalization; the higher the property lies in the* $\leq_B$*-hierarchy, the easier it is to diagonalize in a way that does not explicitly depend on the well-order.*

The Borel non-splitting property, s

Theorem

If E is Borel non-splitting, then E is smooth.

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- Let μ denote the "coin-flipping" probability measure on 2^ω, so that μ is the ω-fold product of the $(\frac{1}{2}, \frac{1}{2})$-measure on $\{0, 1\}$.
- Then $X = [\omega]^\omega$ is a $=^*$-invariant, Borel, μ-measure 1 subset of 2^ω.
- The natural action of $\Gamma = \oplus_{i \in \omega} \mathbb{Z}/2\mathbb{Z}$ on X is ergodic, which means that every Γ-invariant subset of X has measure 0 or 1.
- Since $=^*$ is just E_Γ^X, this implies that every $=^*$-invariant subset of X has measure 0 or 1.
- Since s is closed under $\leq_B$, it will suffice to show that a single hyperfinite Borel equivalence relation fails to have property s.
- We show that the almost equality relation $=^*$ on X does not have property s, using the following lemma.

A lemma

Lemma

Suppose that $U \subseteq [\omega]^\omega$ has the following properties:

(a) *U is $=^*$-invariant;*

(b) *for any $\alpha \in [\omega]^\omega$, exactly one of α or α^c is in U.*

Then U is μ-non-measurable.

Proof.

Suppose U is measurable. Since U is $=^*$-invariant, U has measure 0 or 1 by ergodicity. But $\alpha \mapsto \alpha^c$ is a measure-preserving bijection that sends U to U^c, which implies that U has measure $\frac{1}{2}$, contradiction. □

Now we prove the theorem using the lemma.

- Let E be the hyperfinite Borel equivalence relation on X defined by

$$x \, E \, y \quad \Leftrightarrow \quad x =^* y \text{ or } x^c =^* y.$$

Proof that property s implies smoothness

- Suppose E is Borel non-splitting, and define the Borel homomorphism $\phi : X \to X^{\omega}$ from E to E_{set} by

$$\phi(x)(n) = \gamma_n \cdot x,$$

so that each $\phi(x)$ enumerates the family $[x]_E$.

- Let $\psi : X \to X$ be a Borel homomorphism from E to $=^*$ such that for all x and for all n,

$$\text{either} \quad \psi(x) \subseteq^* \phi(x)(n) \quad \text{or} \quad \psi(x) \subseteq^* \phi(x)(n)^c.$$

- Then in particular, for each x we have $\psi(x) \subseteq^* x$ or $\psi(x) \subseteq^* x^c$.
- Let $U = \{x \in X \mid \psi(x) \subseteq^* x\}$. It is easily seen that U satisfies properties (a) and (b) of the lemma, and hence U is non-measurable.
- On the other hand, U is clearly Borel, a contradiction.