INTEGRATING MONOMIALS OF TRIGONOMETRIC FUNCTIONS

In this handout we will see how to integrate any product of powers of the six basic trigonometric functions. Every such trigonometric monomial can be written in the form

$$f(x) = \sin^m x \cos^n x, \quad m,\, n \text{ integers}$$

simply by writing everything in terms of sines and cosines and then simplifying. We will consider the following six cases, which are exhaustive:

(1) $\sin^m x \cos^n x, \quad m, n \geq 0$
(2) $\sec^m x \csc^n x, \quad m, n \geq 0$
(3) $\tan^m x \sec^n x, \quad m > 0, n \geq 0$
(4) $\cot^m x \csc^n x, \quad m > 0, n \geq 0$
(5) $\tan^m x \sin^n x, \quad m > 0, n > 0$
(6) $\cot^m x \cos^n x, \quad m > 0, n > 0$

CASE 1: $\sin^m x \cos^n x$, with $m, n \geq 0$. If n is odd, save one copy of $\cos x$ and write the rest in terms of $\sin x$ using the identity $\cos^2 x = 1 - \sin^2 x$. If n is even and m is odd, save one copy of $\sin x$ and write the rest in terms of $\cos x$ using the identity $\sin^2 x = 1 - \cos^2 x$. If both n and m are even, simplify the integrand (repeatedly, if necessary), using the identities

$$\cos^2 x = \frac{1 + \cos 2x}{2}, \quad \sin^2 x = \frac{1 - \cos 2x}{2}.$$

Alternatively, use the reduction formulas

$$\int \sin^n x \, dx = -\frac{1}{n} \cos x \sin^{n-1} x + \frac{n-1}{n} \int \sin^{n-2} x \, dx,$$

or

$$\int \cos^n x \, dx = \frac{1}{n} \sin x \cos^{n-1} x + \frac{n-1}{n} \int \cos^{n-2} x \, dx,$$

which can be derived using integration by parts.

CASE 2: $\sec^m x \csc^n x$, with $m, n \geq 0$. If $m, n \geq 2$, simplify the integrand using the identity

$$\sec^m x \csc^n x = \sec^{m-2} x \csc^n x + \sec^m x \csc^{n-2} x.$$

Doing this repeatedly will ensure that at least one of m, n is 0 or 1, leaving the four cases:

$$\int \sec^k x \, dx, \quad \int \csc^k x \, dx, \quad \int \sec^k x \csc x \, dx, \quad \int \csc^k x \sec x \, dx.$$

Powers of secant:

- $\int \sec \theta \, d\theta = \ln |\sec \theta + \tan \theta| + C$
- $\int \sec^2 \theta \, d\theta = \tan \theta + C$

- If $k \geq 4$ is even, save two copies of secant and write the rest in terms of tangent using the identity $\sec^2\theta = \tan^2\theta + 1$ so that $\displaystyle\int \sec^k\theta\, d\theta = \int (u^2+1)^{\frac{k-2}{2}}\, du$, where $u = \tan\theta$.
- If $k \geq 3$ is odd, use the reduction formula
$$\int \sec^k\theta\, d\theta = \frac{1}{k-1}\left[\sec^{k-2}\theta\tan\theta + (k-2)\int \sec^{k-2}\theta\, d\theta\right],$$
which can be obtained by integrating $\int \sec^k\theta\, d\theta$ by parts ($u = \sec^{k-2}\theta$, $dv = \sec^2\theta\, d\theta$) and replacing $\tan^2\theta$ with $\sec^2\theta$ in the resulting integral.

Powers of cosecant: (analogous to powers of secant)

- $\displaystyle\int \csc\theta\, d\theta = -\ln|\csc\theta + \cot\theta| = \ln|\csc\theta - \cot\theta| + C$
- $\displaystyle\int \csc^2\theta\, d\theta = -\cot\theta + C$
- If $k \geq 4$ is even, save two copies of cosecant and write the rest in terms of cotangent using the identity $\csc^2\theta = \cot^2\theta + 1$ so that $\displaystyle\int \csc^k\theta\, d\theta = \int -(u^2+1)^{\frac{k-2}{2}}\, du$, where $u = \cot\theta$.
- If $k \geq 3$ is odd, use the reduction formula
$$\int \csc^k\theta\, d\theta = -\frac{1}{k-1}\left[\csc^{k-2}\theta\cot\theta + (k-2)\int \csc^{k-2}\theta\, d\theta\right],$$
which can be obtained by integrating $\int \csc^k\theta\, d\theta$ by parts ($u = \csc^{k-2}\theta$, $dv = \csc^2\theta\, d\theta$) and replacing $\cot^2\theta$ with $\csc^2\theta - 1$ in the resulting integral.

$\displaystyle\int \sec^k\theta\csc\theta\, d\theta$ can be solved either by using the reduction formula
$$\int \sec^k\theta\csc\theta\, d\theta = \int \sec^{k-2}\theta\csc\theta\, d\theta + \frac{\sec^{k-1}\theta}{k-1} + C$$
(which comes from replacing two copies of $\sec\theta$ with $\tan^2\theta + 1$), or by making the rationalizing substitution $u = \cos\theta$, $\dfrac{du}{u^2-1} = \dfrac{d\theta}{\sin\theta}$. For instance,
$$\int \sec\theta\csc\theta\, d\theta = \int \frac{1}{u(u-1)(u+1)}\, d\theta = \int \left(\frac{-1}{u} + \frac{1/2}{u+1} + \frac{1/2}{u-1}\right) du = \ln|\tan\theta| + C.$$

Similarly, $\displaystyle\int \sec\theta\csc^k\theta\, d\theta$ can be solved either by using the reduction formula
$$\int \sec\theta\csc^k\theta\, d\theta = \int \sec\theta\csc^{k-2}\theta\, d\theta - \frac{\csc^{k-1}\theta}{k-1} + C$$
or by making the rationalizing substitution $u = \sin x$, $\dfrac{du}{1-u^2} = \dfrac{d\theta}{\cos\theta}$.

CASE 3: $\tan^m\theta\sec^n\theta$, with $m > 0$, $n \geq 0$. If the power of secant is positive and even, save one copy of $\sec^2\theta$ and write the remaining copies of secant in terms of tangent by using the identity $\sec^2\theta = \tan^2\theta + 1$. If the power of tangent is odd and $n \geq 1$, save one copy of $\sec\theta\tan\theta$ and write the remaining copies of tangent in terms of secant using $\sec^2\theta = \tan^2\theta + 1$. For powers of tangent, use the following:

- $\displaystyle\int \tan\theta\, d\theta = \ln|\sec\theta| + C$
- for $k \geq 2$, $\displaystyle\int \tan^k\theta\, d\theta = \int (\sec^2\theta - 1)\tan^{k-2}\theta\, d\theta = \frac{1}{k-1}\tan^{k-1}\theta - \int \tan^{k-2}\theta\, d\theta.$

If m is even and n is odd (with both positive), then it is possible to express every copy of tangent in terms of secant using the identity $\sec^2\theta - 1 = \tan^2\theta$, thus putting us back in Case 2. Alternatively, it may help to express everything in terms of sine and cosine and then simplify using the identity $\sin^2\theta + \cos^2\theta = 1$.

CASE 4: $\cot^m x \csc^n x$, with $m > 0, n \geq 0$. Essentially the same as Case 3. If the power of cosecant is positive and even, save one copy of $\csc^2\theta$ and write the remaining copies of cosecant in terms of cotangent by using the identity $\csc^2\theta = \cot^2\theta + 1$. If the power of cotangent is odd and $n \geq 1$, save one copy of $\csc\theta\cot\theta$ and write the remaining copies of cotangent in terms of cosecant using $\csc^2\theta = \cot^2\theta + 1$. For powers of cotangent:

- $\displaystyle\int \cot\theta\, d\theta = \ln|\sin\theta| + C$
- for $k \geq 2$, $\displaystyle\int \cot^k\theta\, d\theta = \int (\csc^2\theta - 1)\cot^{k-2}\theta\, d\theta = \frac{1}{1-k}\cot^{k-1}\theta - \int \cot^{k-2}\theta\, d\theta.$

If m is even and n is odd (with both positive), then it is possible to express every copy of cotangent in terms of cosecant using the identity $\csc^2\theta - 1 = \cot^2\theta$, thus putting us back in Case 2. Alternatively, express everything in terms of sine and cosine and simplify using the identity $\sin^2\theta + \cos^2\theta = 1$.

CASE 5: $\tan^m x \sin^n x$, with $m > 0, n > 0$. If m is odd, then the rationalizing substitution $u = \sin\theta$ yields

$$\int \tan^m\theta \sin^n\theta\, d\theta = \int \frac{u^{m+n}}{(1-u^2)^{(m+1)/2}}\, du.$$

If $m + n$ is odd, then the rationalizing substitution $u = \cos\theta$ yields

$$\int \tan^m\theta \sin^n\theta\, d\theta = \int \frac{(1-u^2)^{\frac{m+n-1}{2}}}{-u^m}\, du.$$

If both m and n are even, then

$$\int \tan^m\theta \sin^n\theta\, d\theta = \int \frac{(1-\cos^2\theta)^{\frac{m+n}{2}}}{\cos^m\theta}\, d\theta,$$

which is a sum of powers of secants and cosines.

Alternatively, express everything in terms of sine and cosine and then simplify.

CASE 6: $\cot^m x \cos^n x$, with $m > 0, n > 0$. If m is odd, then the rationalizing substitution $u = \cos\theta$ yields

$$\int \cot^m\theta \cos^n\theta\, d\theta = \int \frac{-u^{m+n}}{(1-u^2)^{(m+1)/2}}\, du.$$

If $m + n$ is odd, then the rationalizing substitution $u = \sin\theta$ yields

$$\int \cot^m\theta \cos^n\theta\, d\theta = \int \frac{(1-u^2)^{\frac{m+n-1}{2}}}{u^m}\, du.$$

If both m and n are even, then

$$\int \cot^m\theta \cos^n\theta\, d\theta = \int \frac{(1-\sin^2\theta)^{\frac{m+n}{2}}}{\sin^m\theta}\, d\theta,$$

which is a sum of powers of sines and cosecants.

Alternatively, express everything in terms of sine and cosine and then simplify.