INTEGRATING RATIONAL FUNCTIONS USING PARTIAL FRACTIONS

Recall that a *polynomial* is any function of the form

$$p(x) \quad = \quad a_0 + a_1 x_1 + \cdots + a_n x^n \quad = \quad \sum_{k=0}^{n} a_k x^k,$$

where $a_0, \ldots, a_n$ are real numbers and by convention $a_n \neq 0$ unless $n = 0$ and p is the zero function. Here a_n is called the *leading coefficient* of p and n is called the *degree* of p, sometimes written $n = \deg p$. A polynomial of degree zero is called *constant*, a polynomial of degree one is called *linear*, a polynomial of degree two is called *quadratic*, etc. A polynomial is called *monic* if its leading coefficient is 1. The collection of all polynomials is denoted $\mathbb{R}[x]$.

A real number α is called a *root*, or a *zero* of the polynomial $p(x)$ if and only if $p(\alpha) = 0$. A polynomial $p(x)$ is a *factor* of the polynomial $q(x)$, or is said to *divide* $q(x)$, if there is some polynomial $r(x)$ such that $p(x)r(x) = q(x)$. A number α is a root of $p(x)$ if and only if $x - \alpha$ divides $p(x)$.

If p is a polynomial of degree at least one, then p is called *reducible* if there is some polynomial q such that q divides p and $1 < \deg q < \deg p$. Otherwise p is called *irreducible*. For example, $2x^4+4x^3-14x^2-40x-24$ is reducible because

$$2x^4 + 4x^3 - 14x^2 - 40x - 24 = (x^2 - 2x - 3)(2x^2 + 8x + 8),$$

whereas $x^2 + 1$ is irreducible.

It is easy to determine whether or not a polynomial is irreducible:

- Every linear polynomial is irreducible.
- A quadratic polynomial $p(x) = ax^2 + bx + c$ is irreducible if and only if $b^2 - 4ac < 0$.
- All other (nonconstant) polynomials are reducible.

An important theorem of algebra guarantees *that every polynomial factors uniquely as a constant times a product of irreducible monic polynomials.* In other words, for every polynomial $p(x)$, there exists a unique real number α and uniquely determined irreducible monic polynomials $q_1(x), \ldots, q_k(x)$ such that

$$p(x) = \alpha\, q_1(x) \cdots q_k(x).$$

For example, the above polynomial $p(x) = 2x^4 + 4x^3 - 14x^2 - 40x - 24$ factors as

$$2x^4 + 4x^3 - 14x^2 - 40x - 24 \quad = \quad 2(x-3)(x+1)(x+2)(x+2).$$

Of course, the irreducible factors $q_1, \ldots, q_k$ do not have to be distinct; notice that in the above example there are two copies of $x + 2$. If we combine repeated factors, we obtain the following factorization of a polynomial $p(x)$:

Theorem 1 (Unique Factorization in $\mathbb{R}[x]$)**.** *Let $p(x)$ be a polynomial. Then there exists a real number α, distinct irreducible monic polynomials $q_1(x), \ldots, q_k(x)$, and positive integers $t_1, \ldots, t_k$ such that*

$$p = \alpha \cdot q_1^{t_1} \cdots q_k^{t_k}.$$

Moreover, the number α, the irreducible polynomials $q_1, \ldots, q_k$, and the integers $t_1, \ldots, t_k$ are all uniquely determined.

Now we are ready to discuss partial fractions. Recall that a *rational function* is simply a quotient of polynomials,

$$\frac{p(x)}{q(x)}.$$

The rational function p/q is said to be *proper* if $\deg p < \deg q$, and *improper* otherwise. An improper rational function can always be written as the sum of a polynomial and a proper rational function, just as an improper fraction can always be written as the sum of an integer and a proper fraction. For example,

$$\frac{x^5+2}{x^2-1} = x^3 + x + \frac{x+2}{x^2-1}.$$

The procedure for transforming an improper rational function into the sum of a polynomial and a proper rational function is called *long division*, and is identical to the long division of integers learned in grade school.

Now, suppose we are trying to integrate the rational function $\dfrac{P(x)}{Q(x)}$. The first step will be to use long division to rewrite P/Q as a sum

$$\frac{P(x)}{Q(x)} = p(x) + \frac{r(x)}{Q(x)}$$

of a polynomial and a proper rational function. Since it is easy to integrate a polynomial, this leaves only the problem of integrating a *proper* rational function.

Thus suppose $\dfrac{r(x)}{Q(x)}$ is a proper rational function. Our next step will be to apply Theorem 1 to $Q(x)$ to obtain its factorization,

$$\frac{r(x)}{Q(x)} = \frac{r(x)}{\alpha\, q_1(x)^{t_1} \cdots q_k(x)^{t_k}} = \frac{\alpha^{-1}\, r(x)}{q_1(x)^{t_1} \cdots q_k(x)^{t_k}}.$$

Now, finally, we come to the "partial fractions" step. Our goal will be to decompose

$$\frac{\alpha^{-1}\, r(x)}{q_1(x)^{t_1} \cdots q_k(x)^{t_k}}$$

as a sum of fractions with denominators that are certain powers of the irreducible polynomials $q_1, \ldots, q_k$. Namely, for each irreducible polynomial q_i, we will need a total of t_i fractions, having the denominators q_i, $q_i^2, \ldots, q_i^{t_i}$. The numerators of these fractions will be polynomials of degrees strictly less than $\deg q_i$.

Thus, for instance, if $k = 1$, we will attempt to write

$$\frac{\alpha^{-1}\, r(x)}{q_1(x)^{t_1}} = \frac{p_1(x)}{q_1(x)} + \frac{p_2(x)}{q_1(x)^2} + \cdots + \frac{p_{t_1}(x)}{q_1(x)^{t_1}},$$

where $p_1, \dots, p_{t_1}$ are polynomials of degree strictly less than deg q_1 for which we must solve. Since q_1 is irreducible, each $p_1, \dots, p_{t_1}$ has degree at most one, and hence is either constant or linear. If $k > 1$, we will obtain a similar sum of fractions for each q_i.

Each of these fractions is called a "partial fraction." The following theorem establishes that it is always possible to decompose a proper rational function into a sum of partial fractions. In its full abstraction it will probably be difficult to understand, but the examples that follow should make it clearer.

Theorem 2 (Decomposition into Partial Fractions)**.** *Suppose* $\frac{r(x)}{Q(x)}$ *is a proper rational function with Q monic, and suppose that $Q(x)$ has factorization*

$$Q(x) = \left(\prod_{i=1}^{m}(x-a_i)^{s_i}\right) \cdot \left(\prod_{j=1}^{n}(x^2+b_jx+c_j)^{t_j}\right)$$

as a product of irreducible linear and quadratic polynomials. Then there exist real numbers $A_{i,k}$, $B_{j,k}$ and $C_{j,k}$ such that

$$\frac{r(x)}{Q(x)} = \frac{r(x)}{\left(\prod_{i=1}^{m}(x-a_i)^{s_i}\right)\cdot\left(\prod_{j=1}^{n}(x^2+b_jx+c_j)^{t_j}\right)} = \sum_{i=1}^{m}\left(\sum_{1\le k\le s_i}\frac{A_{i,k}}{(x-a_i)^k}\right) + \sum_{j=1}^{n}\left(\sum_{1\le k\le t_j}\frac{B_{j,k}x+C_{j,k}}{(x^2+b_jx+c_j)^k}\right).$$

We may then solve for the numbers $A_{i,k}$, $B_{j,k}$ and $C_{j,k}$ using systems of equations to complete the procedure of writing a proper rational function as a sum of partial fractions. Each of the partial fractions is then "easy" to integrate, and so in this way we manage to integrate the original rational function.

The following example will probably make Theorem 2 much clearer. Suppose $Q(x)$ factors as

$$Q(x) = (x-2)(x-4)(x-7)^2(x-8)^4(x^2+1)(x^2+x+1)(x^2+3)^3.$$

Suppose also that $r(x)$ is some polynomial such that deg $r <$ deg Q. Then Theorem 2 says that there exist real numbers $A_1, \dots, A_{18}$ such that

$$\begin{aligned}\frac{r(x)}{Q(x)} &= \frac{A_1}{x-2}+\frac{A_2}{x-4}+\frac{A_3}{x-7}+\frac{A_4}{(x-7)^2}+\frac{A_5}{x-8}+\frac{A_6}{(x-8)^2}+\frac{A_7}{(x-8)^3}+\frac{A_8}{(x-8)^4}\\ &+ \frac{A_9x+A_{10}}{x^2+1}+\frac{A_{11}x+A_{12}}{x^2+x+1}+\frac{A_{13}x+A_{14}}{x^2+3}+\frac{A_{15}x+A_{16}}{(x^2+3)^2}+\frac{A_{17}x+A_{18}}{(x^2+3)^3}.\end{aligned}$$

Notice where each of the fractions in the above decomposition comes from, given the factorization of Q into irreducible polynomials. Of course, in any actual problems you are asked to work out there will be far fewer fractions involved; but the above example should provide a good understanding of the general case.

The final step now is to solve for the constants A_i. For a simpler example let's find the partial fraction decomposition of the rational function

$$\frac{-x^3+2x^2-x+1}{x(x^2+1)^2}.$$

Thus we write

$$\frac{-x^3+2x^2-x+1}{x(x^2+1)^2} = \frac{A}{x}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{(x^2+1)^2},$$

and attempt to solve for A, B, C, D, E. Upon finding a common denominator we obtain

$$A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x \quad = \quad -x^3+2x^2-x+1.$$

Multiplying out and collecting like terms, this gives us

$$(A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A \quad = \quad -x^3+2x^2-x+1.$$

Hence

$$\begin{cases} A+B &= 0 \\ C &= -1 \\ 2A+B+D &= 2 \\ C+E &= -1 \\ E &= 0 \end{cases}$$

A solution to this system is $A=1$, $B=-1$, $C=-1$, $D=1$, and $E=0$, so we conclude that

$$\frac{-x^3+2x^2-x+1}{x(x^2+1)^2} \quad = \quad \frac{1}{x} + \frac{-x-1}{x^2+1} + \frac{x}{(x^2+1)^2}.$$

Therefore

$$\begin{aligned} \int \frac{-x^3+2x^2-x+1}{x(x^2+1)^2}\,dx \quad &= \quad \int \frac{1}{x}\,dx - \frac{1}{2}\int \frac{2x}{x^2+1}\,dx - \int \frac{1}{x^2+1}\,dx + \frac{1}{2}\int \frac{2x}{(x^2+1)^2}\,dx \\ &= \quad \ln|x| - \frac{1}{2}\ln|x^2+1| - \arctan x - \frac{1}{2}(x^2+1)^{-1} + C. \end{aligned}$$

Any rational function can be integrated in this manner. In fact, decomposing a (proper) rational function into a sum of partial fractions will leave integrals of only four possible types to solve (at least after making an appropriate substitution):

$$\int \frac{1}{u}\,du, \quad \int \frac{1}{1+u^2}\,du, \quad \int \frac{1}{u^k}\,du, \quad \text{and} \quad \int \frac{1}{(1+u^2)^k}\,du, \quad \text{where } k>1.$$

These first three are easy to solve, and the last can be converted into an integral of an even positive power of cosine with the substitution $u = \tan\theta$. This integral can then be solved either by repeatedly using the identity

$$\cos^2 x = \frac{1+\cos 2x}{2},$$

or by using the reduction formula

$$\int \cos^n x\,dx = \frac{1}{n}\sin x\cos^{n-1} x + \frac{n-1}{n}\int \cos^{n-2} x\,dx,$$

which can be proved using integration by parts.