

PRACTICE PROBLEMS FOR THIRD MIDTERM

The third midterm will be a take-home exam, to be handed out Friday, April 30th, and due no later than noon on Monday, May 3rd. You may use your notes, class handouts, and class textbook during the exam, but nothing else; in particular, you may not use a calculator and you may not work with anyone else in the class or receive help from any other person while taking the exam. You should not spend more than 90 minutes on the exam, and you should attempt to recreate in-class testing conditions as closely as possible while taking it. You will sign an honor statement on the last page certifying that you followed these guidelines while taking the exam.

The midterm will cover Chapter 11 in the textbook. The majority of the questions will involve evaluating improper integrals and limits of sequences, and in particular determining whether such integrals and sequences converge or diverge. Many examples of this type can be found in the exercises in the textbook, and I have included some below in the first two questions to give you an idea of what you can expect to find on the exam. Theoretical issues involving sequences and convergence may also appear on the exam. Since questions on this material have not appeared as frequently in the homework, I have included below a number of “theoretical” questions of the sort you might expect to see on the midterm, and potentially on the final exam as well.

As a reminder, the final exam will be held Tuesday, May 12th, at 9am in the usual classroom.

(1) Evaluate the following limits:

- (1) $\lim_{n \rightarrow \infty} \frac{2^n}{n!}$
- (2) $\lim_{n \rightarrow \infty} \frac{n + (-1)^n}{n}$
- (3) $\lim_{n \rightarrow \infty} \arctan\left(\frac{e^n}{n^e}\right)$
- (4) $\lim_{x \rightarrow 0^+} x^2 \ln x$.

(2) Determine whether or not the following improper integrals converge, and evaluate any that do converge:

- (1) $\int_0^{\infty} \frac{1}{x^2 + 1} dx$
- (2) $\int_0^1 x^{-1/3} dx$
- (3) $\int_0^{\infty} \frac{1}{(x-2)^4} dx$

(3) State the Completeness Axiom for the real number line. (Note: the textbook calls this the “Least Upper Bound Axiom.”)

(4) Give a precise definition of the statement: $\lim_{n \rightarrow \infty} a_n = L$.

- (5) Suppose (a_n) is a sequence, and suppose there exists a positive integer $N \in \mathbb{N}$ such that

$$\frac{1}{n} < a_n < \frac{1}{\ln n}$$

for every $n \geq N$. Given only this information, answer the following:

- (1) Is it possible that (a_n) converges?
- (2) Must (a_n) converge?
- (3) If (a_n) converges, is it possible to say what its limit is? If so, what is $\lim_{n \rightarrow \infty} a_n$?

- (6) Suppose (a_n) is a sequence, and suppose that

$$a_n > \frac{1}{n} \quad \text{for every even number } n \in \mathbb{N}$$

and that

$$a_n < \frac{-1}{n} \quad \text{for every odd number } n \in \mathbb{N}.$$

Given only this information, answer the following:

- (1) Is it possible that (a_n) converges?
- (2) Must (a_n) converge?
- (3) If (a_n) converges, is it possible to say what its limit is? If so, what is $\lim_{n \rightarrow \infty} a_n$?

- (7) Suppose $(a_n)_{n=1}^{\infty}$ is a convergent decreasing sequence and that $\lim_{n \rightarrow \infty} a_n = L$. What is the greatest lower bound of (a_n) ? What is the least upper bound of (a_n) ?

- (8) Suppose (a_n) is a convergent sequence such that a_n is a rational number for each n , and such that $\lim_{n \rightarrow \infty} a_n = L$. Must L be a rational number? If not, give a counterexample.

- (9) Suppose $(a_n)_{n=1}^{\infty}$ is an increasing sequence, and for all $n \geq 1$ let $b_n = a_n - a_{n-1}$. Suppose there exists $\epsilon > 0$ such that $b_n \geq \epsilon$ for all n . Does (a_n) converge? What is $\lim_{n \rightarrow \infty} a_n$?

- (10) Find the least upper bound and greatest lower bound of the following sets, whenever they exist. Then state whether or not the lub or glb that you found is in the set.

- (1) $\{\frac{1}{n} \mid n \text{ is a nonzero integer}\}$
- (2) $\{\frac{1}{n^4} \mid n \text{ is an even integer}\}$
- (3) $\{x \in \mathbb{R} \mid x^4 \leq 5\}$
- (4) $\{\sec \frac{1}{n} \mid n \in \mathbb{N}\}$
- (5) $[0, 1)$

- (11) In each of the following, give a concrete example of a sequence with the given properties, or state that this is impossible:

- (1) unbounded and monotone;
- (2) bounded and divergent;
- (3) unbounded and convergent;
- (4) bounded and increasing;
- (5) divergent, bounded, and nonincreasing.