

TECHNIQUES FOR GRAPHING FUNCTIONS

Given a function $y = f(x)$, our goal is to sketch a rough graph of f that reflects important features such as:

- intercepts
- asymptotes and end behavior
- local extrema
- intervals on which f increases and decreases
- inflection points
- intervals of concavity
- points of discontinuity or non-differentiability, including cusps and vertical tangents

Many of the techniques of calculus developed in Chapter 4 can help us do this. What follows is a general summary of graphing techniques, which can also be used as a checklist when trying to graph a particular given function.

- (1) Be able to graph (in some cases, only roughly) the following basic functions from memory:
 - lines (including y -intercept, point-slope, and general linear forms)
 - parabolas (vertex and general linear form, factoring, completing the square, determining orientation and degree of “narrowness / wideness”)
 - general shape of polynomials, given the degree and leading coefficients (end behavior, how many “humps,” behavior near a root depending on its degree, etc — in particular, you should be able to graph fairly accurately a polynomial that has been fully factored)
 - the power functions $y = x^a$, $a > 0$ (and especially a an integer)
 - the reciprocal function $y = x^{-1}$, and in general $y = x^a$ for $a < 0$
 - exponential functions $y = a^x$, for $a > 0$
 - logarithmic functions $y = \log_a x$, for $a > 0$
 - the absolute value function, and the function $y = |x|/x$
 - the six trig functions, but especially $y = \sin x$, $y = \cos x$, and $y = \tan x$
 - the inverse trig functions, but especially $y = \arctan x$
- (2) Translations, dilations and compressions. Specifically, given the graph of $y = f(x)$, know how to obtain from it the graph of:
 - $f(x) + c$, $c > 0$ (vertical shift up)
 - $f(x) - c$, $c > 0$ (vertical shift down)

- $f(x + c)$, $c > 0$ (horizontal shift left)
 - $f(x - c)$, $c > 0$ (horizontal shift right)
 - $cf(x)$, $c > 1$ (vertical stretch)
 - $f(cx)$, $c > 1$ (horizontal compression)
 - $cf(x)$, $0 < c < 1$ (vertical compression)
 - $f(cx)$, $0 < c < 1$ (horizontal stretch)
 - $-f(x)$ (reflection over x -axis)
 - $f(-x)$ (reflection over y -axis)
- (3) Identify the domain and range of the function you are trying to graph.
- (4) Find the intercepts of the function you are trying to graph.
- to find the y -intercept, just plug in $x = 0$
 - to find the x -intercept(s), solve $f(x) = 0$
- (5) If possible, use symmetry or periodicity to help you graph the function.
- if $f(x) = f(-x)$ for all x , then f is an even function
 - if $f(x) = -f(-x)$ for all x , then f is an odd function
 - the trig functions are periodic with period 2π
- (6) Find asymptotes and end behavior of the function you are trying to graph.
- f has a horizontal asymptote at $y = L$ if $\lim_{x \rightarrow \pm\infty} f(x) = L$
 - f has a vertical asymptote at $x = a$ if $\lim_{x \rightarrow a^\pm} f(x) = \pm\infty$ (these usually occur where a denominator is zero)
 - remember that all rational functions exhibit the same end behavior as a suitable monomial
- (7) Find the intervals on which the function increases and decreases
- a differentiable function f increases on an open interval I if $f' > 0$ on I
 - a differentiable function f decreases on an open interval I if $f' < 0$ on I
- (8) Find the local extrema of the function
- if f is differentiable, use the first or second derivative tests
 - remember that local extrema always occur at critical points
- (9) Find the intervals of concavity of the function
- a twice-differentiable function f is concave up on an open interval I if $f'' > 0$ on I
 - a twice-differentiable function f is concave down on an open interval I if $f'' < 0$ on I
- (10) Find the inflection points of the function
- test each of the critical points of f'' for a sign change in f''
- (11) If all else fails, construct a table of values and plot points!

For practice, try graphing the following functions, making sure to label intercepts, asymptotes, local extrema, inflection points, cusps, etc.

$$(1) y = x^3 + 6x^2 + 9x$$

$$(2) y = \frac{x-1}{x^2}$$

$$(3) y = \frac{x}{\sqrt{x^2+1}}$$

$$(4) y = x^{5/3} - 5x^{2/3}$$

$$(5) y = x \ln x$$

$$(6) y = xe^{-x^2}$$

$$(7) y = \sin x - x$$

$$(8) y = \ln(\sin x)$$

$$(9) y = x\sqrt{5-x}$$