STUDY AID FOR FINAL EXAM

The final exam will be held Wednesday, May 12th, from 9am to noon, in the usual lecture room. The exam will cover the entire course, with a slight emphasis on the more recent material from Chapters 11 and 12. Thus everything from the Fundamental Theorem of Calculus through alternating series (Section 12.5) will be covered on the exam. Taylor series will not appear on the exam. You will be allowed one full (8.5 by 11) double-sided sheet of notes for the exam. You may not use your calculator, textbook, or any other resource for the exam.

The following is a (not necessarily exhaustive) list of topics we have covered that may appear on the exam:

- The Fundamental Theorem of Calculus (parts 1 and 2)
- Areas of regions in the plane
- The average value of a function over an interval
- Volumes of solids of revolution (the methods of "disks," "washers," and "shells")
- One-to-one functions; inverse functions
  - derivatives of inverse functions (Theorem 7.1.8)
- Exponential and logarithmic functions
- Logarithmic differentiation
- Inverse trigonometric functions
- Exponential growth and decay
- Techniques of integration
- Sequences
  - limits of sequences; convergence / divergence of sequences
  - boundedness
  - the completeness axiom
  - important examples of sequences (Section 11.4)
  - Theorems 11.3.4 and 11.3.6
  - the squeeze theorem (ie, the "pinching theorem")
- Indeterminate forms and L'Hospital's rule
- Improper integrals
  - the comparison test (11.7.2)
  - the integrals $\int_0^1 \frac{1}{x^p}\,dx$ and $\int_1^\infty \frac{1}{x^p}\,dx$, $p > 0$
- Series
  - a series as a sequence of partial sums
  - convergence / divergence

- a necessary condition for convergence: Theorem 12.2.5
- telescoping series
- geometric series
- p-series
- the five convergence tests (Sections 12.3 – 12.4)
- alternating series
- absolute vs conditional convergence

As always, the best way to prepare for the exam is to work through practice problems. You already have a large supply of practice problems to draw upon, including homework problems, past exams, past exam study guides, the 100 integrals handout, and examples from lecture and from the textbook. Here is one final supply of practice problems, all drawn from the most recent material in Chapter 12.

(1) Determine whether the given series converges or diverges:

$$(a)\sum \frac{n}{n^3+1} \qquad (b)\sum \frac{\ln k}{k} \qquad (c)\sum \frac{\ln k}{k^3} \qquad (d)\sum \frac{n^{3/2}}{n^{5/2}+2n-1}$$

$$(e)\sum \frac{n\cdot 2^n}{3^{n+1}} \qquad (f)\sum \frac{\sqrt{n}}{n^2+1} \qquad (g)\sum \frac{\ln k}{e^k} \qquad (h)\sum \frac{\ln k}{k\sqrt{k}}$$

$$(i)\sum \frac{n!}{n^n} \qquad (j)\sum \frac{(n!)^2}{(2n)!} \qquad (k)\sum ke^{-k^2} \qquad (l)\sum \left(\sqrt{k}-\sqrt{k-1}\right)^k$$

(2) Find the sum of the series:

$$(a)\sum_{k=1}^{\infty} \frac{1}{3k(k+1)} \qquad (b)\sum_{n=0}^{\infty} \frac{(-1)^n}{5^n} \qquad (c)\sum_{k=0}^{\infty} \frac{1-2^k}{3^k} \qquad (d)\sum_{n=0}^{\infty} \frac{3^{n-1}}{4^{3n+1}}$$

(3) Determine whether the series is conditionally convergent, absolutely convergent, or divergent:

$$(a)\sum_{n=1}^{\infty}(-1)^{n-1}n^{-1/3} \qquad (b)\sum_{n=1}^{\infty}(-1)^{n-1}n^{-3} \qquad (c)\sum_{n=1}^{\infty} \frac{(-1)^n(n+1)3^n}{2^{2n+1}} \qquad (d)\sum_{n=2}^{\infty} \frac{(-1)^n\sqrt{n}}{\ln n}$$

(4) Find all positive values of r for which the series $\sum \frac{r^k}{k^r}$ converges.

(5) Find the values of p for which $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^p}$ converges.

(6) Find the values of p for which $\sum_{k=2}^{\infty} \frac{\ln k}{k^p}$ converges.

(7) Find the values of x for which $\sum_{n=1}^{\infty}(\ln x)^n$ converges.

(8) Give an example of a conditionally convergence series.

(9) Find a series expansion for the expression $\frac{x}{1+x}$, given that $|x|<1$.

SOLUTIONS

(1) (a) converges (comparison test with $\frac{1}{n^2}$)
(b) diverges (comparison test with $\frac{1}{k}$)
(c) converges (comparison test with $\frac{1}{k^2}$)
(d) diverges (limit comparison test with $\frac{1}{n}$)
(e) converges (ratio or root test)
(f) converges (limit comparison with $n^{-3/2}$
(g) converges (ratio test)
(h) converges (comparison test $k^{-5/4}$)
(i) converges (ratio test)
(j) converges (ratio test)
(k) converges (root test)
(l) converges (root test)

(2) (a) $\frac{1}{3}$ (b) $\frac{5}{6}$ (c) $-\frac{3}{2}$ (d) $\frac{16}{183}$

(3) (a) Conditionally convergent (p-series, $p \leq 1$); (b) Absolutely convergent (p-series, $p > 1$); (c) Absolutely convergent (root test); (d) Divergent ($\frac{\sqrt{n}}{\ln n} \not\to 0$).

(4) If $r = 1$, then $\sum \frac{r^k}{k^r}$ is the harmonic series, which diverges. If $r > 1$, then $\lim_{k\to\infty} \frac{r^k}{k^r} = \infty$, so the series $\sum \frac{r^k}{k^r}$ diverges. If $0 < r < 1$, then $\sum r^k$ is a convergent geometric series, and hence $\sum \frac{r^k}{k^r} \leq \sum r^k$ converges by the comparison test. Hence for r positive, $\sum \frac{r^k}{k^r}$ converges if and only if $r < 1$.

(5) Since $\int_1^\infty \frac{1}{x(\ln x)^p}\,dx$ converges if and only if $p > 1$, the same is true of $\sum \frac{1}{k(\ln k)^p}$ by the integral test.

(6) If $p \leq 1$, then $\sum \frac{\ln k}{k^p}$ diverges by comparison with the harmonic series. If $p > 1$, then there is α such that $1 < \alpha < p$. Then $\frac{\ln k}{k^p} = \frac{\ln k}{k^{p-\alpha} \cdot k^\alpha} \leq \frac{1}{k^\alpha}$ for k sufficiently large. Since $\alpha > 1$, $\sum \frac{1}{k^\alpha}$ converges, and hence so does $\sum \frac{\ln k}{k^p}$ by the comparison test. Thus $\sum_{k=2}^\infty \frac{\ln k}{k^p}$ converges if and only if $p > 1$. [Notice that part (h) of Problem (1) is a special case of this].

(7) This is a geometric series, so it will converge if and only if $|\ln x| < 1$. Hence $\sum_{n=1}^\infty (\ln x)^n$ converges if and only if $e^{-1} < x < e$.

(8) The alternating harmonic series, $\sum \frac{(-1)^k}{k}$.

(9) $\frac{x}{1+x} = x \cdot \frac{1}{1-(-x)} = x \cdot \sum_{k=0}^\infty (-x)^k = \sum_{k=0}^\infty (-1)^k x^{k+1}$.